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On the Parametrization of Finite-Gap Solutions by Frequency and Wavenumber Vectors and a Theorem of I. Krichever

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Abstract. We discuss the parametrization of real finite-gap solutions of an integrable equation by frequency and wavenumber vectors. This parametrization underlies perturbation and averaging theories for the finite-gap solutions. Out of the framework of integrable equations, the parametrization gives a convenient coordinate system on the corresponding manifold of Riemann curves.

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The parametrization of finite-gap solutions of an integrable equation by the frequency and wavenumber vectors (and, possibly, by some extra scalar parameters), underlies, sometimes implicitly, perturbation theories for finite-gap solutions, in particular, the Whitham averaging theory [1–5], Bogoliubov–Krylov-like averaging [6, 7], and KAM-like theory [6, 8, 9] (see also in these references and in [10], the nonreasonance conditions which are important for the last two theories).

Here we discuss the parametrization for the Korteweg–de Vries equation

$$u_t(t, x) = 6uu_x + u_{xxx}. \tag{KdV}$$

A real n-gap solution of the equation depends on a hyperelliptic Riemannian surface Γ (the spectral curve of the solution) with $2n + 2$ real branching points

$$E_1 < E_2 < \cdots < E_{2n+1} < E_{2n+2} = \infty$$

which is given by the Its–Matveev formula

$$2\partial_x^2 \log \Theta(i(\mathbf{V}x + \mathbf{W}t + \mathbf{D})) + c. \tag{1}$$

Here Θ is the theta function with the period matrix $(2\pi i I, B)$ (B is the Riemann matrix of Γ), c is a real number, and $\mathbf{V}$, $\mathbf{W}$, $\mathbf{D} \in \mathbb{R}^n$. The vector $(\mathbf{V}, \mathbf{W}, c) \in \mathbb{R}^{2n+1}$ is defined by the curve Γ and the n-vector $\mathbf{D}$ (the phase-vector of the solution) is a free parameter, actually varying in the n-torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$ by the periodicity of the theta function.

For the KdV equation, the parametrization under discussion is justified by the following statement.

THEOREM. *The analytic map*

$$\mathbf{E} = (E_1 < E_2 < \cdots < E_{2n+1}) \mapsto (\mathbf{V}, \mathbf{W}, c) \in \mathbb{R}^{2n+1} \tag{2}$$

is nondegenerate everywhere.

This result was stated by I. Krichever in [7, pp. 26, 27], with a scheme of a proof given. Unfortunately, we were unable to restore the proof strictly within the framework of [7]. In this Letter, we present our restoration of the proof, which makes use of some new ideas in addition to ones of [7].

We note that a weaker form of the theorem's statement – *the map (2) is nondegenerate almost everywhere* – was known before [11]. This statement can be checked by direct calculations at the limiting point

$$\mathbf{E} = (E_1 = 0 < E_2 = E_3 < \cdots < E_{2n} = E_{2n+1}),$$

corresponding to the zero solution. It should also be noted that the natural question about whether or not the map (2) defines a *global* diffeomorphism is still open. The supposed answer is affirmative.

We finish the introduction with the remark that out of the framework of finite-gap solutions, the subject of this Letter may be treated as the parametrization of hyperelliptic curves by the b-periods of two normalized Abel differentials (and, possibly, by some additional constants expressed in terms of the Laurent coefficients of the corresponding Abel integrals in the singular points of the differentials). The theorem stated above gives the parametrization for the curves with an odd number of real branching points and the branching point at infinity.

1. Preliminaries

The Riemannian surface Γ is a hyperelliptic curve defined by the equation

$$z^2 = R(\lambda) = \prod_{i=1}^{2n+1} (\lambda - E_i), \quad (\lambda, z) = P \in \Gamma.$$

We denote by σ the hyperelliptic involution $(\lambda, z) \mapsto (\lambda, -z)$ and denote by (a_i, b_i), $i = 1, \ldots, n$ the canonical basis of cycles (the cycle a_i lies in Γ above the segment $[E_{2i}, E_{2i+1}]$ of λ-plane).

The vectors of 'frequencies' $\mathbf{W}$ and of 'wavenumbers' $\mathbf{V}$ are the b-periods of the standard Abel differentials $\mathrm{d}\Omega_1, \mathrm{d}\Omega_2$:

$$V_j = \oint_{b_j} \mathrm{d}\Omega_1, \qquad W_j = \oint_{b_j} \mathrm{d}\Omega_2. \tag{1.1}$$

The Abel differentials have the form

$$\begin{aligned} d\Omega_1 &= \frac{d\lambda}{2}\left(\lambda^n + \sum_{k=0}^{n-1} \alpha_k \lambda^k\right)\Big/\sqrt{R(\lambda)}, \\ d\Omega_2 &= \tfrac{3}{2}\, d\lambda\left(\lambda^{n+1} - \tfrac{1}{2} I \lambda^n + \sum_{k=0}^{n-1} \beta_k \lambda^k\right)\Big/\sqrt{R(\lambda)}, \end{aligned} \tag{1.2}$$

where

$$I = E_1 + \cdots + E_{2n+1}$$

and the real constants α_i, β_i are defined from the normalization conditions

$$\oint_{a_j} d\Omega_i = 0, \quad j = 1, \ldots, n, \quad i = 1, 2. \tag{1.3}$$

At infinity, the differentials have the asymptotics

$$d\Omega_1 = -(u^{-2} + Kc + O(|u|^2))\, du, \qquad d\Omega_2 = -3(u^{-4} + O(1))\, du, \tag{1.4}$$

where $u = \lambda^{-1/2}$, c is the same as in (1), (2) (i.e., the x-mean value of solution (1)) and K is a Γ-independent constant. For the first relation in (1.4), known as the *trace formula for the KdV equation,* see [7] and references therein.

Our proof makes use of the following properties of the zeroes of the differentials $d\Omega_1$, $d\Omega_2$.

PROPOSITION 1. (1) *All zeroes of the differential* $d\Omega_1$ *lie outside the branching points of* Γ; (2) *at least 2n zeroes of* $d\Omega_2$ *lie outside the branching points;* (3) *the zeroes of the differential* $d\Omega_1$ *lie outside the zeroes of* $d\Omega_2$.

We repeat below a simple proof of these statements given in [5, 12].

Proof. By (1.2), (1.3) each interval $\Delta_i =]E_{2i}, E_{2i+1}[$, $i = 1, \ldots, n$ contains a zero λ_i of $d\Omega_1(\lambda)$. Let $z_i^2 = R(\lambda_i)$. Then $(\lambda_i, \pm z_i) \in a_i$ are zeroes of $d\Omega_1(P)$. So all $2n$ zeroes of $d\Omega_1$ are localized and lie outside the branching points.

By the same reasons, each interval Δ_i contains a zero of $d\Omega_2(\lambda)$, thus proving the second assertion.

To prove the last one, suppose that some zero P_i of $d\Omega_1(P)$ coincides with one of $d\Omega_2(P)$. Then there exists a real constant ξ, such that the differential

$$d\tilde{\Omega}(P) = (\xi d\Omega_1 + d\Omega_2)(P)$$

has double zeroes at

$$P = (\lambda_i, z_i) \quad \text{and} \quad P = (\lambda_i - z_i), \quad \lambda_i \in \Delta_i.$$

Again, due to (1.2), (1.3), $d\tilde{\Omega}(\lambda)$ has zeroes in each interval $\Delta_j, j = 1, \dots, n$. So all $2n + 2$ zeroes of $d\tilde{\Omega}(P)$ in Γ are localized, and $d\tilde{\Omega}(\lambda)$ has no other zeroes (except λ_i) in Δ_i. But in such a case

$$\int_{\Delta_i} d\tilde{\Omega}(\lambda) \neq 0,$$

which contradicts to the normalization

$$\oint_{a_i} d\tilde{\Omega} = 0. \qquad \square$$

2. Proof of the Theorem

If the map M is degenerate at a point $\mathbf{E} = (E_1, \dots, E_{2n+1})$, then we can construct an analytic deformation $\Gamma(\tau)$ of the initial curve Γ (i.e. $\Gamma(0) = \Gamma$), such that for the vectors $\mathbf{V}(\tau)$, $\mathbf{W}(\tau)$, $c(\tau)$, we have

$$\begin{aligned} &\mathbf{V}(\tau) = \mathbf{V} + O(\tau^2), \qquad \mathbf{W}(\tau) = \mathbf{W} + O(\tau^2), \\ &c(\tau) = c + O(\tau^2), \end{aligned} \tag{2.1}$$

and the vector of the branching points $\mathbf{E}(\tau)$ has a nonzero τ-derivative at $\tau = 0$. Below, we prove that such a deformation $\Gamma(\tau)$ cannot exist: the relations (2.1) imply that $(\partial/\partial\tau)\mathbf{E}(0) = 0$.

We define Abel integrals $\Omega_j(P, \tau)$, $j = 1, 2$ as follows. Let γ_P be any path in $\Gamma(\tau)$ from σP to P. We set

$$\Omega_j(P, \tau) = \tfrac{1}{2} \int_{\gamma} d\Omega_j(P, \tau), \quad j = 1, 2.$$

Each integral Ω_j is multivalued, is defined up to half-periods of the differential $d\Omega_j$, and

$$\Omega_j(E_r(\tau), \tau) \ni 0 \quad \forall j = 1, 2, \forall r = 1, \dots, 2n + 1. \tag{2.2}$$

Let E_* be any finite branching point of $\Gamma(\tau)$ and γ_0 be a path from E_* to P. We can take $\gamma_P = -\sigma\gamma_0 \cup \gamma_0$. As the differentials $d\Omega_j$ are odd with respect to the involution σ, we have

$$\Omega_j(P, \tau) = \tfrac{1}{2}\left(\int_{\gamma_0} - \int_{\sigma\gamma_0}\right) d\Omega_j = \int_{\gamma_0} d\Omega_j. \tag{2.3}$$

In particular, the differential of Ω_j is really equal to $d\Omega_j$.

Suppose that a point $P = (\lambda, z) \in \Gamma$ lies outside the branching points. Then we can identify P with its projection λ. For τ small enough, the point λ also lies outside the set $\{E_1(\tau), \dots, E_{2n+1}(\tau)\}$. So we can define the functions

$$\partial_\tau \Omega_j(\lambda, \tau)|_{\tau=0}, \quad j = 1, 2.$$

LEMMA 1. *The functions*

$$\Gamma \ni P = (\lambda, z) \mapsto \partial_\tau \Omega_j(P) := \partial_\tau \Omega_j(\lambda, \tau)|_{\tau=0}, \quad j = 1, 2, \tag{2.4}$$

may be extended to meromorphic functions on the curve Γ. *These functions are regular out of the finite branching points* $E_1, \dots, E_{2n+1}$, *where they have first-order poles with*

$$\mathrm{Res}_{P=E_m} \partial_\tau \Omega_j(P) = x^j_{-1}(m) \partial_\tau E_m(0), \quad j = 1, 2, m = 1, \dots, 2n+1,$$

and $x^1_{-1}(m)$, $m = 1, \dots, 2n+1$, *are nonzero constants. The functions* (2.4) *are regular at infinity and vanish there. Moreover, for* $j = 1$, *the function* (2.4) *is* $O(|u|^3)$ *as* $u = \lambda^{-1/2}$ *tends to zero.*

Proof. Due to (1.3) and (2.1), the *a*- and *b*-periods of the differentials $d\Omega_j(P, \tau)$, $j = 1, 2$, are constant up to $O(\tau^2)$. So different branches of the Abel integral $\Omega_j(P, \tau)$ differ by const + $O(\tau^2)$, hence the functions (2.4) are well-defined and analytic out of the branching points.

Near a finite branching point E_m, we have

$$d\Omega_j(\lambda, \tau) = \sum_{k=-1}^{\infty} (\lambda - E_m)^{k/2} x^j_k(E_m, \tau)\, d\lambda, \quad j = 1, 2. \tag{2.5}$$

Note that, due to the first statement of Proposition 1, the coefficients $x^1_{-1}(E_m, 0)$, $m = 1, \dots, 2n+1$, are nonzero. From (2.3), (2.5), we obtain that in the vicinity of $P = E_m$

$$\partial_\tau \Omega_j(\lambda, 0)$$

$$= \sum_{k=1}^{\infty} \left(\frac{2}{k} \partial_\tau x^j_{k-2}(E_m, 0)(\lambda - E_m)^{k/2} + x^j_{k-2}(E_m, 0)(\lambda - E_m)^{(k-2)/2} \partial_\tau E_m \right). \tag{2.6}$$

The right-hand side of the last formula defines a meromorphic function in a neighborhood of E_m with a first-order pole at $\lambda = E_m$.

For $P = (\lambda, z)$ with λ large enough, for a contour γ_P we can take the lift to $\Gamma(\tau)$ of the circle in $\mathbb{C}_\lambda$ of the radius $|\lambda|$, cut at the point λ. At near infinity,

$$d\Omega_2 = -3u^{-4}\, du + d\Omega^0_2, \quad u = \lambda^{-1/2},$$

where the differential $d\Omega^0_2(u, \tau)$ is regular for u, τ small enough (see (1.2), (1.4)), then

$$\Omega_2(P, \tau) = u^{-3} + \tfrac{1}{2} \int_{\gamma_P} {}^d\Omega^0_2(u, \tau).$$

Hence, the function

$$\partial_\tau \Omega_2(P) = \tfrac{1}{2} \int_{\gamma_P} \partial_\tau d\Omega^0_2(u, 0)$$

is analytic in Γ near infinity and vanish at infinity.

For $j = 1$, we have by (1.4)

$$\Omega_1(P, \tau) = u^{-1} + Kcu + O(|u|^3),$$

so $\partial_\tau \Omega_1 = O(|u|^3)$ by (2.1) and the lemma is proven. □

As the numbers $x^1_{-1}(m)$ are nonzero, we have a consequence of the lemma:

COROLLARY 1. *To prove the theorem, it is enough to check that*

$$\partial_\tau \Omega_1(P) \equiv 0. \tag{2.7}$$

To prove (2.7), we construct, following [7], a function $\dot{\Omega}_2$ equal to the 'τ-derivative of Ω_2 and Ω_1 fixed'. To do it, fix a point $P \in \Gamma$ such that

$$d\Omega_1(P, 0) \neq 0, \tag{2.8}$$

and consider the following equation for a point $P(\tau) \in \Gamma(\tau)$:

$$\Omega_1(P(\tau), \tau) = \Omega_1(P, 0). \tag{2.9}$$

Due to (2.8) and the implicit function theorem, Equation (2.9) may be uniquely solved for small τ.

We define the function $\dot{\Omega}_2$:

$$\dot{\Omega}_2(P) := \frac{d}{d\tau} \Omega_2(P(\tau), \tau)|_{\tau=0}. \tag{2.10}$$

Due to the theorem's assumptions, replacement of the branch of the integral Ω_1, used in (2.9), will change the curve $P(\tau)$ by $O(\tau^2)$, and replacement of the branch of Ω_2 in (2.10) will change $\Omega_2(P(\tau), \tau)$ by const $+ O(\tau^2)$ and will not change the right-hand side in (2.10). So the function $\dot{\Omega}_2$ is single-valued.

LEMMA 2. *The function $\dot{\Omega}_2$ can be extended to a meromorphic function on Γ.*

Proof. If $P = (\lambda, z)$ lies outside the branching points of Γ, then for $P(\tau) = (\lambda(\tau), z(\tau))$, we have

$$\partial_\tau \lambda(0) = -\partial_\tau \Omega_1(\lambda, 0)/\partial_\lambda \Omega_1(\lambda, 0) \tag{2.11}$$

(we write here $d\Omega_1$ as $\partial_\lambda \Omega_1 \, d\lambda$). Hence, we find from (2.10) that

$$\dot{\Omega}_2(P) = \partial_\tau \Omega_2(P) - \partial_\tau \Omega_1(P) \frac{d\Omega_2(P, 0)}{d\Omega_1(P, 0)}. \tag{2.12}$$

Thus, by Lemma 1, $\dot{\Omega}_2(P)$ may be extended to a meromorphic function. □

By assertion 1 of Proposition 1, (2.8) holds at the points $E_j, j = 1, \ldots, 2n + 1$. By (2.2), the solution $P(\tau)$ of (2.9) with $P = E_j$ is $P(\tau) = E_j(\tau)$ and $\Omega_2(E_j(\tau), \tau) \equiv 0$. So we have

$$\dot{\Omega}_2(E_j, 0) = 0 \quad \forall j = 1, \ldots, 2n + 1, \tag{2.13}$$

and the function $\dot{\Omega}_2$ has the $2n + 1$ zeroes in the finite branching points of Γ.

By (2.12), (2.13), the only possible finite poles of $\dot{\Omega}_2$ lie in $2n$ zeroes of $d\Omega_1$. To study $\dot{\Omega}_2$ near infinity, let us observe that

$$\partial_\tau \Omega_2 = O(|\dot{u}|), \qquad \partial_\tau \Omega_1 = O(|u|^3),$$

by Lemma 1 and

$$d\Omega_2/d\Omega_1 = O(|u|^{-2})$$

by (1.4). So $\dot{\Omega}_2(\infty) = 0$. Altogether, the function $\dot{\Omega}_2$ has at least $2n + 2$ zeroes and no more than $2n$ poles. Hence, $\dot{\Omega}_2 \equiv 0$ and

$$\partial_\tau \Omega_2 \, d\Omega_1 = \partial_\tau \Omega_1 \, d\Omega_2. \tag{2.14}$$

All the poles of $\partial_\tau \Omega_1$ lie in the finite branching points. So by statement (2) of Proposition 1, the right-hand side of (2.14) has at least $2n$ zeroes outside the branching points. The differential $d\Omega_2(\lambda)$ has one more zero $\lambda_{n+1} \in \mathbb{C}$. To complete the proof, we should distinguish two cases:

(a) λ_{n+1} lies outside the branching points. Then the right-hand side in (2.14) has $2n + 2$ zeroes in $\Gamma \backslash \{E_1, \ldots, E_{2n+1}\}$. The zeroes of $d\Omega_1$ lie outside them by statement (3) of the proposition. Thus, the function $\partial_\tau \Omega_2$ vanishes at these points. So $\partial_\tau \Omega_2$ has $2n + 2$ finite zeroes, the zero at infinity and no more than $2n + 1$ poles. Hence, it vanishes identically, $\partial_\tau \Omega_1 \equiv 0$ by (2.14) and the theorem is proven.

(b) $\lambda_{n+1} = E_{j_*}$ for some $1 \leqslant j_* \leqslant 2n + 1$. Then the right-hand side is regular in E_{j_*}. As $d\Omega_1(E_{j_*}) \neq 0$, then the function $\partial_\tau \Omega_2$ also is regular in E_{j_*}. So it has no more than $2n$ poles. This function vanishes at the first $2n$ zeroes of $d\Omega_1$ and at infinity. Thus, $\partial_\tau \Omega_2 \equiv 0$, $\partial_\tau \Omega_1 \equiv 0$ by (2.14) and the proof is completed.

3. Final Remarks

The scheme to prove the parametrization theorem presented above is rather general*. We do not go into details but just mention that if, for a given integrable equation and its finite-gap solution, we take the statements of Proposition 1 for granted, we can proceed just as above to construct the functions $\partial_\tau \Omega_1$, $\partial_\tau \Omega_2$, and $\dot{\Omega}_2$ which are meromorphic on the spectral curve of the solution. If the vector of additional parameters $c(\tau)$ is chosen in such a way that the function $\dot{\Omega}_2$ vanishes at the infinite points of the spectral curve provided that (2.1) holds, then the vector $(\mathbf{V}, \mathbf{W}, c)$ gives the parametrization we are looking for. (Observe that in the given proof, the function $\dot{\Omega}_2$ vanishes at infinity due to the last statement of Lemma 1 and, finally, due to the 'clever' choice of the parameter c.)

The statements of Proposition 1 hold if the spectral curve has only real branching points (with the same proof), or if the branching points are complex but the spectral

*For example, in [13] some parametrization theorems for the nonlinear Schrödinger equation were obtained.

bands (equal to possible cuts of the curve) are small enough. In the latter case, the statements we need essentially results from the localization of zeroes of the Abel differentials, available via perturbation techniques.

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