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Some Remarks Concerning Unique Continuation for the Dirac Operator

NICULAE MANDACHE*
Université Paris 7, Laboratoire de Physique Mathématique et Géométrie, U.F.R. de Mathématiques, Tour 45-55, 5ème étage, case 7012, 2, Place Jussieu, F-75251 Paris Cedex 05, France

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Abstract. We show that the strong unique continuation property for the Dirac operator holds if the potential belongs to L^p with $p = \max(d, (3d-4)/2)$.

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1. Introduction

One says that a partial-differential equation defined in a connected domain has *the unique continuation property* (u.c.p.) if a solution of that equation that vanishes in a nonempty open set vanishes everywhere.

The u.c.p. was studied in connection with the uniqueness of the Cauchy problem to which it is equivalent in some sense. Nowadays, one of the main interests in studying the u.c.p. consists in proving that Schrödinger or Dirac operators do not have eigenvalues embedded in the continuous spectrum. The reader may consult [3, 6].

One says that a partial-differential equation has the *strong unique continuation property* (s.u.c.p.) in L^p if a solution u of that equation which vanishes of infinite order in L^p at a point x_0, i.e.,

$$\forall n \in \mathbb{N} \lim_{r \to 0} r^{-n} \int_{|x-x_0|<r} |u|^p \, dx = 0,$$

vanishes also in a neighborhood of x_0.

Much effort has been made to prove the (s.)u.c.p. for the Laplacian plus lower-order terms with singular coefficients. If the equation

$$\Delta u + W \cdot \nabla u + Vu = 0$$

has the s.u.c.p. for any $W \in L^{p_1}_{\mathrm{loc}}$ and any $V \in L^{p_0}_{\mathrm{loc}}$, then $p_1 \geqslant d$ and $p_0 \geqslant d/2$, where d is the dimension of the space. For $W = 0$ the s.u.c.p. with optimal p_0 (i.e., $p_0 = d/2$,

*On leave from Institute of Mathematics, Romanian Academy of Sciences, Bucharest, PO Box 1-764, Romania.

$d \geqslant 3$) was proved by Jerison and Kenig [9] and the u.c.p. with $p_0 = d/2$ and $p_1 = d$, was proved by T. H. Wolff [11]. Recently, T. H. Wolff [12] constructed in any dimension $d > 4$ a C^∞ function $u \neq 0$, flat at origin, and a function $W \in L^d$, such that $|\Delta u| \leqslant |W \nabla u|$, i.e., s.u.c.p. fails for $p_1 = d > 4$.

The method generally used in obtaining the unique continuation results is based on the so-called Carleman estimates, which take their name from T. Carleman who used them for the first time in proving uniqueness for a Cauchy problem [4].

Let P be a differential operator (in what follows, P will be the Dirac or Laplace operator) and let $\varphi: \Omega_1 \to \mathbb{R}$ be a smooth function, Ω_1 open, $\Omega_1 \subset \Omega$. The inequality

$$\| e^{\tau\psi} f \|_{L^q} \leqslant C \| e^{\tau\psi} P f \|_{L^p}, \tag{1}$$

where C remains bounded for a sequence of $\tau \to \infty$ and for all $f \in C_0^\infty(\Omega_1)$ is called a Carleman estimate. Suppose $V \in L^r$ with $1/p + 1/q = 1/r$. Then, if a Carleman estimate holds for a sufficiently general class of functions φ, then the equation $Pu + Vu = 0$ (or the inequality $|Pu| \leqslant |Vu|$) has the unique continuation property. The strong unique continuation property results in a similar way from some sharper Carleman estimates.

The present Letter is in line with the results obtained by A. Boutet de Monvel and V. Georgescu [2], A. Boutet de Monvel [1] and Jerison [8], concerning the unique continuation property for the Dirac operator, i.e. the u.c.p. for $(D + V)u = 0$, where D is the Dirac operator. In [2], the u.c.p. is proved in dimension 3 for a potential V in L^5. In [1], the exponent is improved to 3.5 and it is shown that, for no smaller exponent, a Carleman inequality can hold. In [8], the same results are proved for arbitrary dimension d, i.e., the u.c.p. for a potential in $L^{(3d-2)/2}$, and counterexamples are given showing that Carleman estimates cannot hold from L^p to L^q for $1/p - 1/q > 2/(3d - 2)$.

Our results are the first stating strong u.c.p. and are based on an improvement of Carleman's method due to T. H. Wolff [10]. We are able to lower the exponent to $\max(d, (3d - 4)/2)$, which gives optimal results in dimensions 3 and 4.

The problems of unique continuation for the inequalities $|\Delta u| \leqslant |W \nabla u|$ and $|Du| \leqslant |Vu|$ are closely related to each other. The main point is that we are able to pass from the former to the latter, using a property of the fundamental solution. This property is expressed in (2) and (3) and amounts to the commutativity of taking derivatives and Taylor expansions.

2. Conventions and Notations

Let us introduce some notations. $\{e_i\}_{i=1}^m$ will be the canonical basis in $\mathbb{R}^m$ (or $\mathbb{C}^m$). A general convention will be to deal with vector-valued distributions $\mathscr{D}'(\Omega, \mathbb{C}^m)$ with Ω open in $\mathbb{R}^d$, without explicitly mentioning the dimension m. $\mathscr{D}'$ and $\mathscr{E}'$ are the Schwartz spaces of distributions and of distributions with compact support, respectively. There is an abuse of notations which is often used, and which consists in using the same notations for distributions as for functions, e.g. in writing $\langle u, \varphi \rangle = \int u(x)\varphi(x)\,\mathrm{d}x$. For a kernel, $K(x, y) \in \mathscr{D}'(\Omega_x \times \Omega_y)$ and for the operator which it defines, the notation will be the same: $Ku(x) = \int K(x, y)u(y)\,\mathrm{d}y$. We will compose operators without explicitly mentioning that we are multiplying matrices. For example, $W\nabla u$ will have the meaning of a multiplication of a $1 \times d$ matrix with a $d \times 1$ matrix.

Let $P(-i\partial)$ be an elliptic constant coefficient differential operator (which, with the above convention, may also mean an elliptic system), $\partial = (\partial/\partial x_1, \dots, \partial/\partial x_d)$. Then let $E(x, y) = E(x - y)$ be a fundamental solution of P. We define

$$E_{x_0,n}(x, y) = E(x, y) - \sum_{|\alpha| < n} \partial_x^\alpha E(x_0, y)\,\frac{(x - x_0)^{|\alpha|}}{\alpha!}.$$

The Dirac operator in $\mathbb{R}^d$ is $D = \alpha \cdot \nabla = \Sigma_{k=1}^d \alpha_k\, \partial_k$, where α_k are $m \times m$ skew-Hermitian matrices satisfying the relations $\alpha_k\alpha_l + \alpha_l\alpha_k = -2\delta_{kl}$. We can choose $m = 2^{d/2}$ for d even and $m = 2^{(d+1)/2}$ for d odd.

Throughout the Letter, C and C' will denote generic positive constants, which will bear indexes showing on which data of the problem they depend.

3. Results

THEOREM. *Let $d \geqslant 3$, and let r, p, p' be positive numbers satisfying $1/p' + 1/p = 1$ and $1/p - 1/p' = 1/r$. If the (matrix-valued) potential V is in L^r, then the differential inequality in $\mathbb{R}^d$:*

$$|Du| \leqslant |Vu| \quad \text{with } u \in W^{1,p},$$

has

(a) *the strong unique continuation property in $L^{p'}$ for $r = \max(d, (3d - 4)/2)$,*
(b) *the unique continuation property for $r = d$, for $u \in W^{1,q}$ with $1/r + 1/2 = 1/q$.*

It is a simple fact that if unique continuation holds for $|Du| \leqslant |Vu|$, then the same will hold for $|\Delta u| \leqslant |W\nabla u|$ for W in the same regularity class as V.

In order to prove this, let u be a solution of $|\Delta u| \leqslant |W\nabla u|$ which vanishes on a nonempty open set. Let $\tilde{u} = uy_0$, where $y_0 \in \mathbb{R}^m$ is a vector of norm one. The vectors $\{\alpha_i y_0\}_{i=1}^d$ are linearly independent, because $\Sigma_{i=1}^d x_i\alpha_i$ is a unitary matrix for $\Sigma_{i=1}^d x_i^2 = 1$. We can therefore choose the $d \times m$ matrix A such that $A\alpha_j y_0 = e_j$. Then take $V = WA$ and the equality $VD\tilde{u} = W\nabla u$ holds. We know that $D^2 = -\Delta$, hence $|D^2\tilde{u}| = |\Delta u|$ and $D\tilde{u}$ satisfies the inequality $|DD\tilde{u}| \leqslant |VD\tilde{u}|$. As this inequality has u.c.p. by hypothesis, and $D\tilde{u}$ vanishes in a nonempty open set, we have $D\tilde{u} = 0$, which implies $\tilde{u}$ analytic and finally $\tilde{u} = 0$, so that $u = 0$. The proof is complete.

In particular, the counterexample from [12] (see Introduction) provides a counterexample for the Dirac operator; namely, in dimensions greater than 4, the s.u.c.p. does not hold for $|Du| \leqslant |Vu|$, for a certain $V \in L^d$.

Going in the opposite direction is not so easy because we obtain the unique continuation property only for functions in the image of the operator D. But in the case of the proof of T. H. Wolff [10] of the strong unique continuation property for the inequality $|\Delta u| \leqslant |W\nabla u|$, the strong unique continuation for $|Du| \leqslant |Vu|$ can be inferred due to the fact that the estimates are obtained for the integral operator which gives ∇u in terms of Δu.

Let us remark first that a property of the operation of dropping out the Taylor sum of order $< n$ from the kernel E can be viewed as a function of x. The Leibnitz rule gives

$$\partial_x^\beta(E_{0,n})(x, y) = (\partial^\beta E)_{0,n-|\beta|}(x, y), \tag{2}$$

$$\partial_y^\beta(E_{0,n})(x, y) = (-1)^{|\beta|}(\partial^\beta E)_{0,n}(x, y). \tag{3}$$

In particular, this implies that $E_{0,n}(x, y)$ is a two-sided inverse for our starting operator P:

$$E_{0,n}P\varphi = PE_{0,n}\varphi = \varphi, \quad \text{for any } \varphi \in \mathscr{E}'(\mathbb{R}^d\backslash\{0\}). \tag{4}$$

From now on, let us fix $P = \Delta$. Then $E(x, y) = c_d|x - y|^{2-d}$ will be the fundamental solution of Δ. Using $D^2 = -\Delta$, (4), and then (2) and (3), we obtain

$$\varphi(x) = -\int E_{0,n}(x, y)D^2\varphi(y)\,\mathrm{d}y = \int (D_y E_{0,n}(x, y))D\varphi(y)\,\mathrm{d}y$$

$$= -\int (D_x E_{0,n+1}(x, y))D\varphi(y)\,\mathrm{d}y \quad \text{for any } \varphi \in \mathscr{E}'(R^d\backslash\{0\}). \tag{5}$$

We will restate the results of [10], to make the exposition self-contained. The statements are slightly modified but the proofs are the same. Let r denote $|x|/|y|$, $\theta \in [0, \pi]$ denote the angle between the vectors x and y and $I_n = |x|^{-n}|y|^{n+d-2}E_{0,n}(x, y)$. Checking the symmetry properties of I_n, we see that it is a function of r and θ.

PROPOSITION 1 (Proposition 1.1 of [10]).

(i) $|\partial_x^\beta(I_n(x, y) - |x|^{-n}|y|^{n+d-2}E(x, y))|$
$\leqslant C_{d,\beta}n^{d-2+|\beta|}|x|^{-|\beta|}$ *for* $|x - y|/|y| \leqslant 1/n$.

(ii) *We can write* $I_n(x, y) = \mathrm{Re}(a(r, \theta)\,\mathrm{e}^{in\theta})$ *with a satisfying*
$|\partial_r^k \partial_\theta^l a(r, \theta)| \leqslant C_{d,k,l}n^{d/2-2}(|\sin\theta| + 1/n)^{1-d/2-l-k}(|\sin\theta| + |1 - r|)^{-1}$
in the region $|x - y|/|y| \geqslant 1/n$.

We want to express the kernel appearing in (5) in terms of I_n and its derivatives in order to use the above proposition for estimates for D:

$$D_x E_{0,n+1} = D_x(I_{n+1}|x|^{n+1}|y|^{-n-d+1})$$
$$= |x|^n|y|^{-n-d+1}(|x|D_x I_{n+1} + n\alpha\cdot x I_{n+1}).$$

Putting $J_n = -|x|D_x I_{n+1} - n\alpha\cdot x I_{n+1}$ we obtain the following proposition.

PROPOSITION 2 (Proposition 1.2 of [10]). *The above* J_n *satisfies*

$$|x|^{-n}\varphi(x) = \int J_n(x, y)|y|^{-n-d+1} D\varphi(y)$$

and

(i) *if* $|x - y| < 1|y|/n$, *then* $|J_n(x, y)| \leqslant C_d |y|^{d-1}|x - y|^{-d+1}$,
(ii) *if* $|x - y| > |y|/n$, *then we can write* $I_n(x, y) = \mathrm{Re}(q(r, \theta)\, \mathrm{e}^{in\theta})$ *with* q *satisfying*

$$|\partial_\theta^k q(r, \theta)| \leqslant C_{d,k} n^{d/2-1}(|\sin\theta| + 1/n)^{1-d/2-k}(|\sin\theta| + |1 - r|)^{-1}.$$

Proposition 2, which is derived from Proposition 1, is intended to retain the properties of the kernel J_n which ensure its continuity in the L^p spaces. We have cut its domain into two pieces: in the places where $|x - y|/|y| \leqslant 1/n$, the term which dominates in J_n is that which contains $DE(x, y)$ and the continuity can be obtained via the Hardy–Littlewood–Sobolev theorem. In the complementary set $|x - y|/|y| > 1/n$, the term which is polynomial in x is dominating in $E_{0,n}$ (i.e., in I_n) but it has an oscillating behaviour which allowed T. H. Wolff to obtain Proposition 3 below, using only the 'variable coefficients Plancherel', i.e., theorem 1.1 from [7]. We have to make first some additional notations.

We pass from the kernel J_n to the kernel

$$K_\nu = |x|^{-(d-1)/2+\rho}|y|^{-(d-1)/2-\rho} J_n(x, y)$$

which has the property

$$|x|^{-\nu}\varphi(x) = \int K_\nu(x, y)|y|^{-\nu} D\phi(y).$$

Here ρ is a number in $[0, 1)$ and $\nu = n + (d - 1)/2 - \rho$ is the analogous of the parameter τ used in (1).

In dimensions greater than 4, the weight $\psi(x) = -\log|x|$ does not give optimal results and, therefore, we choose a function $\psi\colon \mathbb{R}^+ \to \mathbb{R}$ in C^∞ which satisfies the following conditions: there is a $C > 0$ such that $C^{-1} < \psi'(t) < C$ for all t, and for any $\delta > 0$ there is a $C_\delta > 0$ such that $\psi''(t) \geqslant C_\delta \mathrm{e}^{-\delta t}$. Then the corresponding kernel is

$$L_\nu(x, y) = \exp(-\nu(\psi(t) - \psi(s) - \psi'(s)(t - s)))K_{\nu\psi'(s)}(x, y),$$

where s stands for $-\log|x|$ and t for $-\log|y|$. That is, L_ν satisfies

$$\mathrm{e}^{\nu\psi(s)}\varphi(x) = \int L_\nu(x, y)\, \mathrm{e}^{\nu\psi(t)} D\varphi(y)\, \mathrm{d}y \quad \text{for any } \varphi \in \mathscr{E}'(B(0, 1)\backslash\{0\}).$$

For γ subset of $\mathbb{R}^+$, denote

$$A(\gamma) = \{x \in \mathbb{R} \mid -\log|x| \in \gamma\}.$$

$\| K \|_{p \to q}$ will be the norm of the operator of kernel K from $L^p(B(0, 1))$ to $L^q(B(0, 1))$, and 1_A will be the operator of multiplication by the characteristic function of A. In what follows, we will suppose that v is a positive real number.

PROPOSITION 3 (Corollary 3.1 of [10]). *Let $r = \max\{d, (3d - 4)/2\}$ and γ an interval in $\mathbb{R}^+$ of length greater than $1/v$. If $d \geqslant 5$, then*

$$\| 1_{A(\psi^{-1}\gamma)} L_v \|_{p \to p'} \leqslant C_d (v \min(v^{-1/2}, |\gamma|))^{1/r}.$$

If $d = 3$ or 4 then

$$\| 1_{A(\gamma)} K_v \|_{p \to p'} \leqslant C_{d,\delta} (v \min(1, |\gamma|))^{(d-2)/2r}$$

where $\delta > 0$ is the distance from v to the set $\mathbb{Z} + (d - 1)/2$.

The main idea of [10] is to use the estimates of the parametrix truncated to the left as they appear above and to glue them to obtain a global assertion via the following lemma.

LEMMA 1 (Corollary 4.1 of [10]). *Suppose μ is a positive measure on $\mathbb{R}$ without atoms and such that*

$$\lim_{T \to \infty} \frac{1}{T} \log \mu(\{x : |x| > T\}) = -\infty. \tag{6}$$

Define, for $k \in \mathbb{R}$, $\mu_k(B) = \int_B e^{kx} \, d\mu(x)$. Then there exists a constant $C > 0$ such that for any $v > 0$ and any arithmetical progression of ratio $\lambda \in (0, v)$, namely $\{a + m\lambda\}_{m \in \mathbb{Z}}$, there exist a natural number n, n numbers $k_1, k_2, \ldots, k_n \in [v, 2v]$ belonging to the given progression, and n disjoint intervals $\{I_j\}_{j=1}^n$ such that

(i) $\mu_{k_j}(I_j) \geqslant \mu_{k_j}(\mathbb{R})/2$,
(ii) $\Sigma_{j=1}^n \max(|I_j|^{-1}, \lambda) \geqslant Cv$.

Proof of the Theorem. There are the two cases $d = 3, 4$ and $d \geqslant 5$ but we will give a single proof making several conventions which make it possible to state Proposition 3 above (and to make the proof) independently of the case which is being considered.

For the case $d \geqslant 5$, we choose ψ satisfying the conditions stated before Proposition 3. We choose $\lambda = v^{1/2}$ and arbitrary a (these two numbers refer to the arithmetic progression in Lemma 1). In the case $d = 3$ or 4, we let $\psi(s) = s$ and $\lambda = 1$, $a = (d - 1)/2 + 1/2$. Notice that, in this case, we have $L_v = K_v$, and the conclusion of Proposition 3 can be weakened to

$$\| 1_{A(\psi^{-1}\gamma)} L_v \|_{p \to p'} \leqslant C_d (v \min(\lambda^{-1}, |\gamma|))^{1/r} \tag{7}$$

independently of the case under study.

Let u be a solution of $|Du| \leqslant |Vu|$ with $V \in L^r, u \in W^{1,p}_{\text{loc}}$. Suppose that

$$\lim_{r \to 0} r^{-n} \int_{B(0,r)} |u|^{p'} = 0 \quad \text{for any } n \in \mathbb{Z} \tag{8}$$

and suppose that u does not vanish in any neighborhood of the origin.

Let us consider $\tilde{u}(x) = \chi(x)u(x)$, where $\chi \in C_0^\infty(B(0,1)\backslash\{0\})$ will be chosen later, and the continuous measure defined by

$$\mu(B) = \int_{A(\psi^{-1}B)} |V\tilde{u}|^p \,\mathrm{d}x.$$

Applying Lemma 1 to μ, which has compact support and therefore satisfies the decay hypothesis (6), we obtain that for every $v > 1$ there are real numbers $\{k_j\}_{j=1}^k$ belonging to the set $[v, 2v] \cap \{a + m\lambda : m \in \mathbb{Z}\}$ and the disjoint intervals $I_1, I_2, \ldots, I_n$ such that

$$\mu_{k_j}(I_j) = \| 1_{A(\psi^{-1}I_j)} \mathrm{e}^{k_j\psi(s)} V\tilde{u} \|_p^p \geqslant \mu_{k_j}(\mathbb{R})/2 = \| \mathrm{e}^{k_j\psi(s)} V\tilde{u} \|_p^p/2 \tag{9}$$

and

$$\sum_{j=1}^{n} \max(|I_j|^{-1}, \lambda) \geqslant Cv. \tag{10}$$

We can suppose that all the lengths of the I_j are greater than $1/v$ because otherwise we can drop all of them except one and enlarge it to the length $1/v$.

Then applying (7) for each of the operators $1_{A(\psi^{-1}I_j)} L_{k_j}$ and using the fact that $v < k_j < 2v$, we obtain

$$\| 1_{A(\psi^{-1}I_j)} \mathrm{e}^{k_j\psi(s)} \tilde{u} \|_{p'} \leqslant C_d(k_j \min(\lambda^{-1}, |I_j|))^{1/r} \| \mathrm{e}^{k_j\psi(s)} D\tilde{u} \|_p. \tag{11}$$

Let us suppose

$$\| \mathrm{e}^{k_j\psi(s)} D\tilde{u} \|_p = \| \mathrm{e}^{k_j\psi(s)}(\chi Du + (D\chi)u) \|_p \leqslant 2 \| \mathrm{e}^{k_j\psi(s)} V\tilde{u} \|_p \quad \forall j = 1, \ldots, n \tag{12}$$

and finish the proof, and turn later to the choice of χ as to ensure (12). Combining (9), (11) and (12), we obtain

$$\| 1_{A(\psi^{-1}I_j)} \mathrm{e}^{k_j\psi(s)} \tilde{u} \|_{p'} \leqslant C_d(v \min(\lambda^{-1}, |I_j|))^{1/r} \| 1_{A(\psi^{-1}I_j)} \mathrm{e}^{k_j\psi(s)} V\tilde{u} \|_p$$

from which we infer that

$$\| 1_{A(\psi^{-1}I_j) \cap \operatorname{supp} \tilde{u}} V \|_r \times C_d(v \min(\lambda^{-1}, |I_j|))^{1/r} \geqslant 1.$$

Raising to the rth power, summing, and using (10),

$$\int_{\operatorname{supp} \tilde{u}} |V|^r \geqslant 1/v \sum_{j=0}^{n} \max(\lambda, |I_j|^{-1}) \geqslant C_d. \tag{13}$$

To conclude, we will choose the truncation function χ in the definition of $\tilde{u}$ and the number v so that (12) be satisfied and the relation (13) be impossible.

To this end we choose $\chi_1 \in C_0^\infty(B(0,1))$, $\chi_1 = 1$ in the neighbourhood of 0, with support so small that $\int_{\operatorname{supp}\chi_1} |V|^r$ is smaller than the constant appearing in (13).

We can suppose that Vu vanishes in no neighbourhood of 0. If it would not be so, $Du = 0$ in the neighbourhood of 0, and this implies that u is analytic there, and having a zero of infinite multiplicity, it must vanish in the neighbourhood of 0. Let $r_0 > 0$ be such that $\chi_1 = 1$ in $B(0, r_0)$ and $0 < r_1 < r_0$. Then

$$\| \mathrm{e}^{k\psi(s)}(D\chi_1)u \|_p < C\,\mathrm{e}^{C'v(\log r_0 - \log r_1)} \| \mathrm{e}^{k\psi(s)} V\chi_1 u \|_p,$$

because $\int_{B(0,r_1)} |Vu|^p > 0$. We can choose v so that

$$\| e^{k\psi(s)}(D\chi_1)u \|_p \leqslant \tfrac{1}{2} \| e^{k\psi(s)} V\chi_1 u \|_p \quad \text{for all } k \in [v, 2v]. \tag{14}$$

Now we have to choose $\chi_2 \in C^\infty$, equal to zero in the neighbourhood of the origin, and equal to 1 outside a small ball $B(0, r_2), r_2 < r_0$, and to set $\chi = \chi_1\chi_2 = \chi_1 + \chi_2 - 1$. Using Hölder inequality, we obtain that Vu and u vanish of infinite order in L^p at 0 (i.e., there are relations analogous to (9)). From the fact that χ_2 can be chosen so that $|D\chi_2| \leqslant Cr_2^{-1}$ it results that for r_2 small enough we will have

$$\| e^{k\psi(s)}(D\chi_2)u \|_p + \| e^{k\psi(s)}(1 - \chi_2)Vu \|_p \leqslant \tfrac{1}{2} \| e^{k\psi(s)} V\chi_1 u \|_p \quad \text{for all } k \in [v, 2v]. \tag{15}$$

Summing (14) and (15), and using the triangle inequality and $|Du| \leqslant |Vu|$, we obtain (12) (notice that $D\chi = D\chi_1 - D\chi_2$). The proof is complete.

The u.c.p. stated in (b) is a consequence of theorem 2 in [11] which we state below

THEOREM. *Suppose M is a d-dimensional manifold, $d \geqslant 3$, $1/q - \frac{1}{2} = 1/d$* and *ω is a differential form on M with $W^{1,q}_{\mathrm{loc}}$ coefficients such that $|d\omega| + |d^*\omega| \leqslant V|\omega|$ with $V \in L^d_{\mathrm{loc}}$. Then if $|\omega|$ vanishes in an open set, it vanishes identically.*

We may apply the theorem in our case, since Dirac operators can be represented as $d + d^*$, where d is the exterior differentiation of forms in $\mathbb{R}^d$ and d^* its adjoint with respect to the usual metric. This is done in [5], chapter 12.

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