Quasi-Classical Limit of Toda Hierarchy and *W*-Infinity Symmetries

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Abstract. Previous results on quasi-classical limit of the KP hierarchy and its *W*-infinity symmetries are extended to the Toda hierarchy. The Planck constant $\hbar$ now emerges as the spacing unit of difference operators in the Lax formalism. Basic notions, such as dressing operators, Baker–Akhiezer functions, and tau function, are redefined. $W_{1+\infty}$ symmetries of the Toda hierarchy are realized by suitable rescaling of the Date–Jimbo–Kashiara–Miwa vertex operators. These symmetries are contracted to $w_{1+\infty}$ symmetries of the dispersionless hierarchy through their action on the tau function.

1. Introduction

Dispersionless analogues of integrable systems of KP and Toda type [1] provide an interesting family of integrable 'contractions'. In the context of field theory, the dispersionless Toda equation is studied as continuous (or large-N) limit of the ordinary Toda field theory [2] as well as a dimensional reduction of four-dimensional self-dual gravity [3]. Recently, a hierarchy of higher flows has been constructed [4] and applied to two-dimensional string theory [5].

Dispersionless (or long wavelength) limit can also be understood as quasi-classical limit. In a previous paper [6], we considered the KP hierarchy and its dispersionless version from this point of view, and could show a direct connection between *W*-infinity symmetries of the two hierarchies, i.e., $W_{1+\infty}$ symmetries of the KP hierarchy and $w_{1+\infty}$ symmetries of the dispersionless KP hierarchy. In this Letter, we present similar results on the Toda hierarchy and its dispersionless version.

2. Lax Formalism of Dispersionless Toda Hierarchy

To begin with, let us briefly review the Lax formalism of the dispersionless Toda hierarchy [4]. The dispersionless Toda hierachy consists of an infinite number of

commuting flows with 'time variables' $z = (z_1, z_2, ...)$ and $\hat{z} = (\hat{z}_1, \hat{z}_2, ...)$. The Lax equations can be written

$$
\frac{\partial \mathscr{L}}{\partial z_n} = \{\mathscr{B}_n, \mathscr{L}\}, \qquad \frac{\partial \mathscr{L}}{\partial \hat{z}_n} = \{\hat{\mathscr{B}}_n, \mathscr{L}\},
$$

$$
\frac{\partial \hat{\mathscr{L}}}{\partial z_n} = \{\mathscr{B}_n, \hat{\mathscr{L}}\}, \qquad \frac{\partial \hat{\mathscr{L}}}{\partial \hat{z}_n} = \{\hat{\mathscr{B}}_n, \hat{\mathscr{L}}\}, \quad n = 1, 2, ...,
$$

(1)

where $\mathscr{L}$ and $\hat{\mathscr{L}}$ are Laurent series

$$
\mathscr{L} = p + \sum_{n=0}^{\infty} u_n(z, \hat{z}, s)p^{-n},
$$

$$
\hat{\mathscr{L}} = \sum_{n=1}^{\infty} \hat{u}_n(z, \hat{z}, s)p^n
$$

(2)

of a variable p, and $\mathscr{B}_n$ and $\hat{\mathscr{B}}_n$ are given by

$$
\mathscr{B}_n = (\mathscr{L}^n)_{\geq 0}, \qquad \hat{\mathscr{B}}_n = (\hat{\mathscr{L}}^{-n})_{\leq -1}.
$$

(3)

Here $(\ )_{\geq 0}$ and $(\ )_{\leq -1}$ denote the projection of Laurent series onto a linear combination of p^n with $n \geq 0$ and ≤ -1, respectively. The Poisson bracket $\{\ ,\ \}$ is defined by

$$
\{A(p, s), B(p, s)\} = p \frac{\partial A(p, s)}{\partial p} \frac{\partial B(p, s)}{\partial s} - \frac{\partial A(p, s)}{\partial s} p \frac{\partial B(p, s)}{\partial s}
$$

(4)

on the two-dimensional 'phase space' with coordinates (p, s). This Lax system can be extended to a larger system. The extended Lax representation possesses, in addition to the above equations, another set of dispersionless Lax equations

$$
\frac{\partial \mathscr{M}}{\partial z_n} = \{\mathscr{B}_n, \mathscr{M}\}, \qquad \frac{\partial \mathscr{M}}{\partial \hat{z}_n} = \{\hat{\mathscr{B}}_n, \mathscr{M}\},
$$

$$
\frac{\partial \hat{\mathscr{M}}}{\partial z_n} = \{\mathscr{B}_n, \hat{\mathscr{M}}\}, \qquad \frac{\partial \hat{\mathscr{M}}}{\partial \hat{z}_n} = \{\hat{\mathscr{B}}_n, \hat{\mathscr{M}}\}, \quad n = 1, 2, ...,
$$

(5)

and the canonical Poisson relations

$$
\{\mathscr{L}, \mathscr{M}\} = \mathscr{L}, \qquad \{\hat{\mathscr{L}}, \hat{\mathscr{M}}\} = \hat{\mathscr{L}}
$$

(6)

for a second set of Laurent series

$$
\mathscr{M} = \sum_{n=1}^{\infty} n z_n \mathscr{L}^n + s + \sum_{n=1}^{\infty} v_n(z, \hat{z}, s)\mathscr{L}^{-n},
$$

$$
\hat{\mathscr{M}} = - \sum_{n=1}^{\infty} n \hat{z}_n \hat{\mathscr{L}}^{-n} + s + \sum_{n=1}^{\infty} \hat{v}_n(z, \hat{z}, s)\hat{\mathscr{L}}^n.
$$

(7)

These somewhat complicated equations can actually be cast into a simple, compact 2-form equation:

$$\frac{d\mathcal{L} \wedge d\mathcal{M}}{\mathcal{L}} = \omega = \frac{d\hat{\mathcal{L}} \wedge d\hat{\mathcal{M}}}{\hat{\mathcal{L}}}, \tag{8}$$

where

$$\omega = \frac{dp}{p} \wedge ds + \sum_{n=1}^{\infty} d\mathcal{B}_n \wedge dz_n + \sum_{n=1}^{\infty} d\hat{\mathcal{B}}_n \wedge d\hat{z}_n. \tag{9}$$

3. Lax Formalism of Toda Hierarchy with Planck Constant

To interpret this hierarchy as quasi-classical limit, we reformulate the ordinary Toda hierarchy [7] in the language of difference operators in a continuous variable s with spacing unit $\hbar$. The Lax equations are then given by

$$\hbar \frac{\partial L}{\partial z_n} = [B_n, L], \qquad \hbar \frac{\partial L}{\partial \hat{z}_n} = [\hat{B}_n, L],$$

$$\hbar \frac{\partial \hat{L}}{\partial z_n} = [B_n, \hat{L}], \qquad \hbar \frac{\partial \hat{L}}{\partial \hat{z}_n} = [\hat{B}_n, \hat{L}], \quad n = 1, 2, \dots, \tag{10}$$

where the Lax operators L and $\hat{L}$ are difference operators of the form

$$L = e^{\hbar \partial/\partial s} + \sum_{n=0}^{\infty} u_n(\hbar, z, \hat{z}, s)\, e^{-n\hbar\partial/\partial s},$$

$$\hat{L} = \sum_{n=1}^{\infty} \hat{u}_n(\hbar, z, \hat{z}, s)\, e^{n\hbar\partial/\partial s} \tag{11}$$

and B_n and $\hat{B}_n$ are given by

$$B_n = (L^n)_{\geqslant 0}, \qquad \hat{B}_n = (\hat{L}^{-n})_{\leqslant -1}. \tag{12}$$

Here $(\ )_{\geqslant 0}$ and $(\ )_{\leqslant -1}$ denote the projection onto a linear combination of $e^{n\hbar\partial/\partial s}$ with $n \geqslant 0$ and $\leqslant -1$, respectively.★ This Lax representation, too, can be extended into a larger system. Besides the above Lax equations, the extended Lax representation contains the second set of Lax equations

$$\hbar \frac{\partial M}{\partial z_n} = [B_n, M], \qquad \hbar \frac{\partial M}{\partial \hat{z}_n} = [\hat{B}_n, M],$$

$$\hbar \frac{\partial \hat{M}}{\partial z_n} = [B_n, \hat{M}], \qquad \hbar \frac{\partial \hat{M}}{\partial \hat{z}_n} = [\hat{B}_n, \hat{M}], \quad n = 1, 2, \dots, \tag{13}$$

★These notations are quite different from earlier papers [7]. In particular, L and $\hat{L}$ correspond to L and M therein, whereas M and $\hat{M}$ introduced here have no counterpart. Furthermore, W and $\hat{W}$ introduced below correspond to $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ in those papers, whereas Ψ and $\hat{\Psi}$ to $W^{(\infty)}$ and $W^{(0)}$ therein. We would like to apologize for this notational inconsistency.

and the canonical commutation relations

$$[L, M] = \hbar L, \qquad [\hat{L}, \hat{M}] = \hbar \hat{L} \tag{14}$$

for a second set of difference operators

$$M = \sum_{n=1}^{\infty} nz_n L^n + s + \sum_{n=1}^{\infty} v_n(\hbar, z, \hat{z}, s)L^{-n},$$

$$\hat{M} = -\sum_{n=1}^{\infty} n\hat{z}_n \hat{L}^{-n} + s + \sum_{n=1}^{\infty} \hat{v}_n(\hbar, z, \hat{z}, s)\hat{L}^n. \tag{15}$$

These equations obviously bear a close resemblance to ordinary quantum mechanics. To be consistent with ordinary quantum mechanics, $\hbar$ in these formulas should be replaced by $i\hbar$.

The dispersionless hierarchy emerges from the ordinary hierarchy as quasi-classical limit as follows. We assume smooth asymptotic behavior of the coefficients of the Lax operators as $\hbar \to 0$:

$$u_n(\hbar, z, \hat{z}, s) = u_n^{(0)}(z, \hat{z}, s) + \mathrm{O}(\hbar),$$

$$v_n(\hbar, z, \hat{z}, s) = v_n^{(0)}(z, \hat{z}, s) + \mathrm{O}(\hbar),$$

$$\hat{u}_n(\hbar, z, \hat{z}, s) = \hat{u}_n^{(0)}(z, \hat{z}, s) + \mathrm{O}(\hbar), \tag{16}$$

$$\hat{v}_n(\hbar, z, \hat{z}, s) = \hat{v}_n^{(0)}(z, \hat{z}, s) + \mathrm{O}(\hbar).$$

One can then define Laurent series $\mathscr{L}$, $\mathscr{M}$, $\hat{\mathscr{L}}$ and $\hat{\mathscr{M}}$ from these coefficients as

$$\mathscr{L} = p + \sum_{n=1}^{\infty} u_{n+1}^{(0)}(z, \hat{z}, s)p^{-n}, \quad \text{etc.,} \tag{17}$$

and Laurent polynomials $\mathscr{B}_n(p)$ and $\hat{\mathscr{B}}_n(p)$, similarly, from the coefficients

$$b_{n,i}(\hbar, z, \hat{z}, s) = b_{n,i}^{(0)}(z, \hat{z}, s) + \mathrm{O}(\hbar),$$

$$\hat{b}_{n,i}(\hbar, z, \hat{z}, s) = \hat{b}_{n,i}^{(0)}(z, \hat{z}, s) + \mathrm{O}(\hbar) \tag{18}$$

of the difference operators

$$B_n(p) = e^{n\hbar\partial/\partial s} + \sum_{i=0}^{n-2} b_{n,i}(\hbar, z, \hat{z}, s)\, e^{i\hbar\partial/\partial s},$$

$$\hat{B}_n(p) = \sum_{i=1}^{n} \hat{b}_{n,i}(\hbar, z, \hat{z}, s)\, e^{-i\hbar\partial/\partial s} \tag{19}$$

as

$$\mathscr{B}_n(p) = p^n + \sum_{i=0}^{n-2} b_{n,i}^{(0)}(z, \hat{z}, s)p^i,$$

$$\hat{\mathscr{B}}_n(p) = \sum_{i=1}^{n} \hat{b}_{n,i}^{(0)}(z, \hat{z}, s)p^{-i}. \tag{20}$$

In quasi-classical $(\hbar \to 0)$ limit, commutators of difference operators turn into Poisson brackets as

$$[e^{\hbar \partial/\partial s}, s] = \hbar \, e^{\hbar \partial/\partial s} \to \{p, s\} = s. \tag{21}$$

The Lax equations of $L, M, \hat{L}$ and $\hat{M}$ can thereby be reduced to the dispersionless Lax equations of $\mathscr{L}, \mathscr{M}, \hat{\mathscr{L}}$, and $\hat{\mathscr{M}}$.

4. Dressing Operators, Baker–Akhiezer Functions and Tau Function

The notions of dressing operators, Baker–Akhiezer functions and tau function [7] can be reformulated so as to fit into the above setting.

The dressing operators are now given by difference operators of the form

$$W = 1 + \sum_{n=1}^{\infty} w_n(\hbar, z, \hat{z}, s) \, e^{-n\hbar \partial/\partial s},$$

$$\hat{W} = \sum_{n=0}^{\infty} \hat{w}_n(\hbar, z, \hat{z}, s) \, e^{n\hbar \partial/\partial s}. \tag{22}$$

The Lax operators are then given by the 'dressing' relations

$$L = W e^{\hbar \partial/\partial s} W^{-1}, \qquad \hat{L} = \hat{W} e^{\hbar \partial/\partial s} \hat{W}^{-1},$$

$$M = W \left(\sum_{n=1}^{\infty} n z_n \, e^{n\hbar \partial/\partial s} + s \right) W^{-1}, \tag{23}$$

$$\hat{M} = \hat{W} \left(- \sum_{n=1}^{\infty} n \hat{z}_n \, e^{-n\hbar \partial/\partial s} + s \right) \hat{W}^{-1}.$$

The Lax equations can be converted into the evolution equations

$$\hbar \frac{\partial W}{\partial z_n} = B_n W - W e^{n\hbar \partial/\partial s}, \qquad \hbar \frac{\partial W}{\partial \hat{z}_n} = \hat{B}_n W,$$

$$\hbar \frac{\partial \hat{W}}{\partial z_n} = B_n \hat{W}, \qquad \hbar \frac{\partial \hat{W}}{\partial \hat{z}_n} = \hat{B}_n \hat{W} - \hat{W} e^{-n\hbar \partial/\partial s}. \tag{24}$$

The coefficients w_n and $\hat{w}_n$, unlike u_n, etc., are singular as $\hbar \to 0$, as we shall see in the following analysis of Baker–Akhiezer functions.

Baker–Akhiezer functions are given by (formal) Laurent series of a 'spectral parameter' λ:

$$\Psi = \left(1 + \sum_{n=1}^{\infty} w_n(\hbar, z, \hat{z}, s) \lambda^{-n} \right) \exp \hbar^{-1}[z(\lambda) + s \log \lambda], \tag{25}$$

$$\hat{\Psi} = \left(\sum_{n=0}^{\infty} \hat{w}_n(\hbar, z, \hat{z}, s) \lambda^{n} \right) \exp \hbar^{-1}[\hat{z}(\lambda^{-1}) + s \log \lambda],$$

where

$$z(\lambda) = \sum_{n=1}^{\infty} z_n \lambda^n, \qquad \hat{z}(\lambda^{-1}) = \sum_{n=1}^{\infty} \hat{z}_n \lambda^{-n}. \tag{26}$$

Dressing relations (23) can now be transformed into linear equations of Ψ and $\hat{\Psi}$:

$$\lambda\Psi = L\Psi, \qquad \hbar\lambda\frac{\partial\Psi}{\partial\lambda} = M\Psi,$$

$$\lambda\hat{\Psi} = \hat{L}\hat{\Psi}, \qquad \hbar\lambda\frac{\partial\hat{\Psi}}{\partial\lambda} = \hat{M}\hat{\Psi}. \tag{27}$$

In particular, the coefficients v_n and $\hat{v}_n$ of M and $\hat{M}$ can be read off from logarithmic derivatives of Ψ and $\hat{\Psi}$:

$$\hbar\lambda\frac{\partial\log\Psi}{\partial\lambda} = \sum_{n=1}^{\infty} nz_n\lambda^n + s + \sum_{n=1}^{\infty} v_n(\hbar, z, \hat{z}, s)\lambda^{-n},$$

$$\hbar\lambda\frac{\partial\log\hat{\Psi}}{\partial\lambda} = -\sum_{n=1}^{\infty} n\hat{z}_n\lambda^{-n} + s + \sum_{n=1}^{\infty} \hat{v}_n(\hbar, z, \hat{z}, s)\lambda^n. \tag{28}$$

Evolution equations (24) of the dressing operators, too, can be converted into linear equations of Ψ and $\hat{\Psi}$:

$$\hbar\frac{\partial\Psi}{\partial z_n} = B_n\Psi, \qquad \hbar\frac{\partial\Psi}{\partial\hat{z}} = \hat{B}_n\Psi,$$

$$\hbar\frac{\partial\hat{\Psi}}{\partial z_n} = B_n\hat{\Psi}, \qquad \hbar\frac{\partial\hat{\Psi}}{\partial\hat{z}} = \hat{B}_n\hat{\Psi}. \tag{29}$$

These equations resemble time-dependent Schrödinger equations in ordinary quantum mechanics. This implies that Ψ and $\hat{\Psi}$ takes a WKB asymptotic form as $\hbar \to 0$:

$$\Psi = \exp[\hbar^{-1}S(z, \hat{z}, s, \lambda) + O(\hbar^0)],$$

$$\hat{\Psi} = \exp[\hbar^{-1}\hat{S}(z, \hat{z}, s, \lambda) + O(\hbar^0)], \tag{30}$$

where $S(z, \hat{z}, s, \lambda)$ and $\hat{S}(z, \hat{z}, s, \lambda)$ have Laurent expansion of the form

$$S(z, \hat{z}, s, \lambda) = z(\lambda) + s\log\lambda + \sum_{n=1}^{\infty} S_n(z, \hat{z}, s)\lambda^{-n},$$

$$\hat{S}(z, \hat{z}, s, \lambda) = \hat{z}(\lambda^{-1}) + s\log\lambda + \sum_{n=0}^{\infty} \hat{S}_n(z, \hat{z}, s)\lambda^n. \tag{31}$$

In particular, w_n and $\hat{w}_n$ are singular as $\hbar \to 0$.

The tau function, too, exhibits characteristic singular behavior as $\hbar \to 0$. In the presence of $\hbar$, we define the tau function $\tau(\hbar, z, \hat{z}, s)$ as a function that reproduces the

Baker–Akhiezer functions as

$$\frac{\tau(\hbar, z - \hbar\varepsilon(\lambda^{-1}), \hat{z}, s)}{\tau(\hbar, z, \hat{z}, s)} \exp \hbar^{-1}[z(\lambda) + s \log \lambda] = \Psi(\hbar, z, \hat{z}, \lambda),$$

$$\frac{\tau(\hbar, z, \hat{z} - \hbar\varepsilon(\lambda), s + \hbar)}{\tau(\hbar, z, \hat{z}, s)} \exp \hbar^{-1}[\hat{z}(\lambda^{-1}) + s \log \lambda] = \hat{\Psi}(\hbar, z, \hat{z}, \lambda),$$

(32)

where

$$\varepsilon(\lambda) = \left(\lambda, \frac{\lambda^2}{2}, \dots, \frac{\lambda^n}{n}, \dots\right).$$

(33)

In the case of $\hbar = 1$, this reduces to the ordinary definition. Taking the logarithm of (32) and comparing them with the WKB asymptotic form (30) of Ψ and $\hat{\Psi}$, one can easily find that $\log \tau(\hbar, z, \hat{z}, s)$ should behave as

$$\log \tau(\hbar, z, \hat{z}, s) = \hbar^{-2} F(z, \hat{z}, s) + O(\hbar^{-1}) \quad (\hbar \to 0)$$

(34)

with an appropriate scaling function $F(z, \hat{z}, s)$. The Laurent coefficients S_n and $\hat{S}_n$ of $S(z, \hat{z}, s, \lambda)$ and $\hat{S}(z, \hat{z}, s, \lambda)$ can be written

$$S_n = -\frac{1}{n}\frac{\partial F}{\partial z_n}, \qquad \hat{S}_n = -\frac{1}{n}\frac{\partial F}{\partial \hat{z}_n}, \qquad \hat{S}_0 = \frac{\partial F}{\partial s}.$$

(35)

This implies that $F(z, \hat{z})$ is nothing but the logarithm of the tau function of the dispersionless Toda hierarchy [4]:

$$F = \log \tau_{\mathrm{dToda}}.$$

(36)

The function F may be called the 'free energy' in analogy with matrix models of two dimensional quantum gravity [6].

5. Hamilton–Jacobi Equations and Legendre Transformation

The 'phase functions' S and $\hat{S}$ satisfy a set of Hamilton–Jacobi equations, which turn out to reproduce the Lax formalism of the dispersionless Toda hierarchy after a Legendre transformation. To see this, let us gather up the Hamilton–Jacobi equations into 1-form equations:

$$dS(z, \hat{z}, s, \lambda) = \mathcal{M}(\lambda)\frac{d\lambda}{\lambda} + \frac{\partial S}{\partial s}\,ds + \sum_{n=1}^{\infty} \mathcal{B}_n(e^{\partial S/\partial s})\,dz_n + \sum_{n=1}^{\infty} \hat{\mathcal{B}}_n(e^{\partial S/\partial s})\,d\hat{z}_n,$$

$$d\hat{S}(z, \hat{z}, s, \lambda) = \hat{\mathcal{M}}(\lambda)\frac{d\lambda}{\lambda} + \frac{\partial \hat{S}}{\partial s}\,ds + \sum_{n=1}^{\infty} \mathcal{B}_n(e^{\partial \hat{S}/\partial s})\,dz_n + \sum_{n=1}^{\infty} \hat{\mathcal{B}}_n(e^{\partial \hat{S}/\partial s})\,d\hat{z}_n,$$

(37)

where

$$\mathscr{M}(\lambda) = \sum_{n=1}^{\infty} nz_n\lambda^n + s + \sum_{n=1}^{\infty} v_n^{(0)}(z, \hat{z}, s)\lambda^{-n},$$

$$\hat{\mathscr{M}}(\lambda) = -\sum_{n=1}^{\infty} n\hat{z}_n\lambda^{-n} + s + \sum_{n=1}^{\infty} \hat{v}_n^{(0)}(z, \hat{z}, s)\lambda^n. \tag{38}$$

Exterior differentiation of these equations give a 2-form equation of the form

$$\frac{d\lambda}{\lambda} \wedge d\mathscr{M}(\lambda) = d\left(\frac{\partial S}{\partial s}\right) \wedge ds + \sum_{n=1}^{\infty} d\mathscr{B}_n(e^{\partial S/\partial s}) \wedge dz_n + \sum_{n=1}^{\infty} d\hat{\mathscr{B}}_n(e^{\partial S/\partial s}) \wedge d\hat{z}_n \tag{39}$$

and a similar 2-form equation including $\hat{\mathscr{M}}(\lambda)$ and $\hat{S}$ in place of $\mathscr{M}(\lambda)$ and S. We now distinguish between the two λ's in S and $\hat{S}$ as $S(z, \hat{z}, s, \lambda)$ and $\hat{S}(z, \hat{z}, s, \hat{\lambda})$, and solve the equations

$$\exp \partial S(z, \hat{z}, s, \lambda)/\partial s = \exp \partial \hat{S}(z, \hat{z}, s, \hat{\lambda})/\partial s = p \tag{40}$$

with respect to λ and $\hat{\lambda}$. Obviously, this is a kind of Legendre transformation. Let us write the solutions

$$\lambda = \mathscr{L}(z, \hat{z}, s, p), \qquad \hat{\lambda} = \hat{\mathscr{L}}(z, \hat{z}, s, p) \tag{41}$$

and define

$$\mathscr{M} = \mathscr{M}(\lambda)|_{\lambda = \mathscr{L}(z, \hat{z}, s, p)}, \qquad \hat{\mathscr{M}} = \hat{\mathscr{M}}(\hat{\lambda})|_{\hat{\lambda} = \hat{\mathscr{L}}(z, \hat{z}, s, p)}. \tag{42}$$

Then the above 2-form equations coincide with (8), hence these $\mathscr{L}, \mathscr{M}, \hat{\mathscr{L}}$ and $\hat{\mathscr{M}}$ indeed satisfy the dispersionless Toda hierarchy.

6. W-Infinity Symmetries

We now turn to the issue of W-infinity symmetries. W-infinity symmetries of the Toda hierarchy can be formulated in two different ways, i.e. in bosonic and fermionic languages. For the analysis of quasi-classical limit, the bosonic language is more convenient.

The bosonic description is based on the so-called 'vertex operators' [8]. Actually, the Toda hierarchy has two copies of $W_{1+\infty}$ symmetries, which mutually commute. They are realized by the infinitesimal action $\tau \to \tau + \varepsilon Z\tau$, $\tau \to \tau + \varepsilon \hat{Z}\tau$ of the vertex operators

$$Z(\hbar, \tilde{\lambda}, \lambda) = \frac{\exp\left(\hbar^{-1}[z(\tilde{\lambda}) - z(\lambda)]\right)(\tilde{\lambda}/\lambda)^{s/\hbar} \exp(\hbar[-\tilde{\partial}_z(\tilde{\lambda}^{-1}) + \tilde{\partial}_z(\lambda^{-1})]) - 1}{\tilde{\lambda} - \lambda},$$

$$\hat{Z}(\hbar, \tilde{\lambda}, \lambda) = \frac{\exp\left(\hbar^{-1}[\hat{z}(\tilde{\lambda}^{-1}) - \hat{z}(\lambda^{-1})]\right)(\tilde{\lambda}/\lambda)^{s/\hbar} \exp(\hbar[-\tilde{\partial}_{\hat{z}}(\tilde{\lambda}) + \tilde{\partial}_{\hat{z}}(\lambda)]) - 1}{\tilde{\lambda}^{-1} - \lambda^{-1}}, \tag{43}$$

where

$$\tilde{\partial}_z(\lambda^{-1}) = \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \frac{\partial}{\partial z_n}, \qquad \tilde{\partial}_{\hat{z}}(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{\partial}{\partial \hat{z}_n}. \tag{44}$$

If one expands these two-parameter families of symmetries into Fourier modes along the double loop $|\tilde{\lambda}| = |\lambda| = $ const., the outcome are two copies of gl(∞) symmetries [8]. If one first expands these vertex operators into Taylor series along the diagonal $\tilde{\lambda} = \lambda$,

$$Z(\hbar, \tilde{\lambda}, \lambda) = \sum_{l=1}^{\infty} \frac{(\tilde{\lambda} - \lambda)^{l-1}}{(l-1)!} W^{(l)}(\hbar, \lambda), \quad \text{etc.,} \tag{45}$$

and further into Fourier modes along the loop $|\lambda| = $ const.,

$$W^{(l)}(\hbar, \lambda) = \sum_{n=-\infty}^{\infty} W_n^{(l)}(\hbar)\lambda^{-n-l}, \quad \text{etc.,} \tag{46}$$

the coefficients $W_n^{(l)}(\hbar)$ and $\hat{W}_n^{(l)}(\hbar)$ ($n \in \mathbb{Z}, l \geq 1$) become generators of $W_{1+\infty}$ symmetries. These symmetry generators are differential operators of finite order in z and $\hat{z}$, respectively, and differ from the ordinary ($\hbar = 1$) definition [8] by the simple rescaling

$$z_n \to \hbar^{-1}z_n, \qquad \frac{\partial}{\partial z_n} \to \hbar \frac{\partial}{\partial z_n},$$

$$\hat{z}_n \to \hbar^{-1}\hat{z}_n, \qquad \frac{\partial}{\partial \hat{z}_n} \to \hbar \frac{\partial}{\partial \hat{z}_n}. \tag{47}$$

We have thus essentially the same $W_{1+\infty}$ symmetries as the KP hierarchy, but now in duplicate.

Let us show how to contract these $W_{1+\infty}$ symmetries into $w_{1+\infty}$ symmetries of the dispersionless hierarchy. The essence is the same as in the case of the KP hierarchy [6]. First, with the aid of basic relations (34) and (35), we write the action of $W^{(l)}(\hbar, \lambda)$ on the tau function in terms of $\mathcal{M}(\lambda)$:

$$\frac{W^{(l)}(\hbar, \lambda)\tau(\hbar, z, \hat{z}, s)}{\tau(\hbar, z, \hat{z}, s)}$$

$$= \frac{1}{l}\left(\frac{\partial}{\partial \tilde{\lambda}}\right)^l \exp \hbar^{-1}[S(z, \hat{z}, s, \tilde{\lambda}) - S(z, \hat{z}, s, \lambda) + O(\hbar^1)]|_{\tilde{\lambda}=\lambda}$$

$$= \frac{\hbar^{-l}}{l}\left[\left(\frac{\partial S(z, \hat{z}, s, \lambda)}{\partial \lambda}\right)^l + O(\hbar)\right]$$

$$= \frac{\hbar^{-l}}{l}[(\mathcal{M}(\lambda)\lambda^{-1})^l + O(\hbar)]. \tag{48}$$

We then pick out the most singular term ($\propto \hbar^{-1}$) as $\hbar \to 0$, and consider it as defining a λ-dependent infinitesimal transformation of F, $F \to F + \varepsilon w^{(l)}(\lambda)F$. Its Fourier modes $w_n^{(l)}F$ are given by

$$w_n^{(l)}F = -\operatorname*{Res}_{\lambda = \infty} \frac{1}{l}(\mathscr{M}(\lambda)\lambda^{-1})^l \lambda^{n+l-1}\,\mathrm{d}\lambda$$

$$= -\operatorname*{Res}_{\lambda = \infty} \frac{1}{l} \mathscr{M}^l \mathscr{L}^n\,\mathrm{d}\log\mathscr{L} \tag{49}$$

where $\operatorname{Res}_{\lambda = \infty}$ denotes the residue at $\lambda = \infty$,

$$\operatorname*{Res}_{\lambda = \infty} \lambda^n\,\mathrm{d}\lambda = -\delta_{n,-1}. \tag{50}$$

These $w_n^{(l)}$ coincide with one half of the $w_{1+\infty}$ symmetries that have been constructed by a different method [4]. Another half can be obtained from $\hat{W}_n^{(l)}(\hbar)$ in the same way.

In the fermionic language, the vertex operators correspond to fermion bilinear forms. Let $\psi(\lambda)$ and $\psi^*(\lambda)$ be the Date–Jimbo–Kashiwara–Miwa free fermion fields [8]

$$\psi(\lambda) = \sum_{n=-\infty}^{\infty} \psi_n \lambda^n, \qquad \psi^*(\lambda) = \sum_{n=-\infty}^{\infty} \psi_n^* \lambda^{-n-1} \tag{51}$$

with anti-commutation relations

$$[\psi_i, \psi_j]_+ = [\psi_i^*, \psi_j^*]_+ = 0, \quad [\psi_i, \psi_j^*]_+ = \delta_{ij}, \tag{52}$$

and $\langle n|$ and $|n\rangle$, $n \in \mathbb{Z}$, be the ground states in the charge-n sector of the Fock space,

$$\psi_n|0\rangle = 0 \ (n \leqslant -1), \qquad \psi_n^*|0\rangle = 0 \quad (n \geqslant 0),$$
$$\langle 0|\psi_n = 0 \ (n \geqslant 0), \qquad \langle 0|\psi_n^* = 0 \quad (n \leqslant -1). \tag{53}$$

A generic expression of the tau function is given by [9]

$$\tau(\hbar, z, \hat{z}, s) = \langle \hbar^{-1}s \,|\, \mathrm{e}^{H(z)/\hbar} g(\hbar)\, \mathrm{e}^{-\hat{H}(\hat{z})/\hbar}\, |\hbar^{-1}s\rangle, \tag{54}$$

where $g(\hbar)$ is an appropriate $\hbar$-dependent Clifford operator, and $H(z)$ and $\hat{H}(\hat{z})$ are generators of time evolutions,

$$H(z) = \sum_{n=1}^{\infty} z_n H_n, \qquad \hat{H}(\hat{z}) = \sum_{n=1}^{\infty} \hat{z}_n H_{-n},$$

$$H_n = \sum_{m=-\infty}^{\infty} :\psi_m \psi_{m+n}^*: \tag{55}$$

normal ordered with respect to $\langle 0|$ and $|0\rangle$. Actually, the Clifford operator $g(\hbar)$ takes the same form as in the case of the KP hierarchy [6]:

$$g(\hbar) = \exp \hbar^{-1} \oint :A\left(\lambda, \hbar\frac{\partial}{\partial\lambda}\right)\psi(\lambda)\cdot\psi^*(\lambda): \frac{\mathrm{d}\lambda}{2\pi i}, \tag{56}$$

where A is a linear differential operator, and '$A\psi \cdot \psi^*$' means $A\psi$ times ψ^*. The action of $Z(\hbar, \tilde{\lambda}, \lambda)$ and $\hat{Z}(\hbar, \tilde{\lambda}, \lambda)$ is realized by insertion of a fermion bilinear form:

$$
\begin{aligned}
Z(\hbar, \tilde{\lambda}, \lambda)\tau(\hbar, z, \hat{z}, s) &= \langle \hbar^{-1}s | \, e^{H(z)/\hbar} \psi(\tilde{\lambda}) \psi^*(\lambda) g(\hbar) \, e^{-H(\hat{z})/\hbar} \, |\hbar^{-1}s \rangle, \\
\hat{Z}(\hbar, \tilde{\lambda}, \lambda)\tau(\hbar, z, \hat{z}, s) &= \langle \hbar^{-1}s | \, e^{H(z)/\hbar} g(\hbar) \psi(\tilde{\lambda}) \psi^*(\lambda) \, e^{-\hat{H}(\hat{z})/\hbar} \, |\hbar^{-1}s \rangle.
\end{aligned}
\tag{57}
$$

The bosonic and fermionic representations are thus connected. For the moment, the bosonic language looks more preferable because the fermionic representation is valid only for discrete values $s \in \hbar \mathbb{Z}$.

7. Conclusion

We have thus extended our previous results on the quasi-classical limit of the KP hierarchy to the Toda hierarchy. The Planck constant $\hbar$ now enters into the Lax formalism as the spacing unit of difference operators. The notions of dressing operators, Baker–Akhiezer functions and tau function are redefined so as to fit into the new formulation. We have used two copies of the Date–Jimbo–Kashiwara–Miwa vertex operators to calculate the action of $W_{1+\infty}$ symmetries on the tau function, which exhibits singular behavior as $\hbar \to 0$. The most singular terms therein turn out to give $w_{1+\infty}$ symmetries of the dispersionless Toda hierarchy.

Hopefully, these results will further be extended to multi-component KP and Toda hierarchies. The so-called Whitham hierarchies [10] will then emerge as quasi-classical limit.

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