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ALMOST FREE GROUPS AND THE M. HALL PROPERTY

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In the present article, we continue our study of groups with the M. Hall property started in [1] (see also [2-6] and [7, Chap. 1, Sec. 3]). According to [3], a group G has *the M. Hall property* (G is is an *M. Hall group*) if each of its finitely generated subgroups is a free factor in a subgroup of finite index in G.

Let $\mathfrak{F}$ be the class of fundamental groups of finite graphs of finite groups. It was proved in [3] that every finitely presented M. Hall group is isomorphic to some group in $\mathfrak{F}$. In [1], a criterion for a group in $\mathfrak{F}$ to be M. Hall was given and an algorithm verifying this criterion was described.

In connection with the study of groups without the M. Hall property in $\mathfrak{F}$, the following question was raised: Given an arbitrary group G in $\mathfrak{F}$ and a subgroup H of G defined by a finite set of generators, how can one determine whether H is a free factor in a subgroup of finite index in G? This question is settled in the present article (Theorem 3.1; see also Corollary 3.1).

Theorem 2.2 and its topological analog — Theorem 2.1 — are interesting in their own right. In the proof of Theorem 2.2, we give an algorithm which, for an arbitrary finitely generated subgroup H of the fundamental group $\pi_1(G,\Gamma,v_0)$ of a finite graph of finite groups, produces its representation as the fundamental group of some finite graph of finite groups.

In the present article, we use the same technique of covering spaces as in [1]. The basic definitions from [1] are given in Sec. 1. Theorems 2.1 and 2.2 are proved in Sec. 2, and Theorem 3.1 — in Sec. 3.

1. BASIC DEFINITIONS AND NOTATION

Let Γ be a nonempty connected graph with the vertex set $\Gamma^0=\{v_i|i\in I\}$, the edge set $\Gamma^1=\{e_j\,|\,j\in J\}$, the functions "origin of an edge" $\alpha:\Gamma^1\to\Gamma^0$, "end of an edge" $\omega:\Gamma^1\to\Gamma^0$, and "inverse of an edge" $^-:\Gamma^1\to\Gamma^1$. If $\beta(e_j)=v_i$, where $\beta\in\{\alpha,\omega\}$, then we will also write $\beta(j)=i$. For an arbitrary path f in Γ, as well as in the complexes appearing below, we denote by $\alpha(f)$ the origin of f, by $\omega(f)$ the end of f, and by $[f]$ the homotopy class of f.

Recall that a graph of groups (G,Γ) consists of a graph Γ, a set of vertex groups $\{V_i|\,i\in I\}$, a set of edge groups $\{E_j\,|\,j\in J\}$, and embeddings $\alpha_j\colon E_j\to V_{\alpha(j)}$ and $\omega_j\colon E_j\to V_{\omega(j)}$, for each $j\in J$. In addition, if $\bar e_{j_1}=e_{j_2}$, then it is necessary that $E_{j_1}=E_{j_2}$, $\alpha_{j_1}=\omega_{j_2}$, and $\omega_{j_1}=\alpha_{j_2}$. In what follows, we assume that the graph Γ has no loops, since otherwise we can consider the barycentric subdivision of the graph of groups. The definitions of the group $F(G,\Gamma)$ and the fundamental group $\pi_1(G,\Gamma,v_i)$, where $v_i\in\Gamma^0$, can be found in [1] and [8].

In [1, Sec. 4], an arbitrary group H was associated, in a canonical way, with a 2-complex $K(H)$, containing a single vertex v such that $\pi_1(K(H),v)\cong H$. For $i\in I$, the complex $K(V_i)$ is called a *body*; its

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vertex is denoted by v_i and its loops are labeled by the symbols $\widetilde{a}$, where $a \in V_i$. For $j \in J$, the vertex of the complex $K(E_j)$ is denoted by u_j. The topological space $K(E_j) \times [-1,1]$ can be transformed into a complex, which we will call a *tube*. Two inverse edges of a tube mapped onto the geometrical edge $\{u_j\} \times [-1,1]$ are labeled by $\widetilde{e}_j$ and $\widetilde{\widetilde{e}}_j$. The subcomplex $K(E_j) \times \{0\}$ is called a *face*.

Next, all bodies and tubes are glued (using the structure of the graph Γ) into a 3-complex $K(G,\Gamma)$ such that $\pi_1(K(G,\Gamma), v_i) \cong \pi_1(G,\Gamma,v_i)$. The isomorphism is induced by the labeling. The complex $K(G,\Gamma)$ can be represented as the union of neighborhoods $O(V_i)$, $i \in I$, (see [1, Sec. 5]). Each neighborhood $O(V_i)$ consists of a body $K(V_i)$ and handles of the form $K(E_j) \times S$, where $S = [-1,0]$ for $\alpha(j) = i$ and $S = [0,1]$ for $\omega(j) = i$.

As in [1], subgroups of the group $\pi_1(G,\Gamma,v_i)$ are studied with the help of coverings of the complex $K(G,\Gamma)$. We will need the following properties of coverings.

(1) For an arbitrary covering $p : (X,x) \to (Y,y)$ of connected complexes X and Y, the mapping $p^* : \pi_1(X,x) \to \pi_1(Y,y)$ is an embedding. When z runs over the set $p^{-1}(y)$, the group $p^*(\pi_1(X,z))$ runs over some class of conjugate subgroups of the group $\pi_1(Y,y)$.

(2) For an arbitrary connected complex Y, a vertex y of Y, and a subgroup $H \leq \pi_1(Y,y)$, there exists a covering $p\colon (X,x) \to (Y,y)$ such that $p^*(\pi_1(X,x)) = H$.

Let $p\colon X \to K(G,\Gamma)$ be an arbitrary covering of the complex $K(G,\Gamma)$. Any connected component of the inverse image of a body, a tube, a handle, a face, or a neighborhood in $K(G,\Gamma)$ has the same name in X. The complex X consists of covering neighborhoods $O_s(V_i)$, whose structure was described in Proposition 5.1 of [1].

Let L be an arbitrary handle in X which covers the handle $M = K(E_t) \times [-1,0]$ in $K(G,\Gamma)$. If a covering $p_{|L} : L \to K(E_t) \times [-1,0]$ corresponds to a class $\mathcal{E}$ of conjugate subgroups of the group $E_t \cong \pi_1(K(E_t)\times[-1,0], (u_t,0))$, then we label the handle L by $(e_t^+, \mathcal{E})$. If L covers the handle $M = K(E_t)\times[0,1]$, and the corresponding class is $\mathcal{E}$, then we label L by $(e_t^-, \mathcal{E})$.

An *identity handle* (*neighborhood, face*) is a handle (neighborhood, face) whose fundamental group is trivial.

The definitions of an irreducible tube path, a telescopic (nonascending) tube path, and a tube tree in X were given in [1, Sec. 7]

Definition 1.1. A tube tree T is said to be *telescopic* (*nonascending*) with respect to a body a in T if every irreducible tube path in T starting in a is telescopic (nonascending).

2. DEFINING A GROUP: FROM A SYSTEM OF GENERATORS TO A GRAPH OF GROUPS

LEMMA 2.1. Suppose that $G = \pi_1(G,\Gamma,v_0)$ is the fundamental group of a finite graph of finite groups, H is a finitely generated subgroup of G with generators $h_1,\dots,h_n$. Then the occurrence problem for G in H is decidable.

Proof. It is known (see, for example, [1, Theorem 8.1]) that G has a free subgroup F of index d and rank n if and only if

$$n = d\left(\sum_{e_j\in\Gamma^1_+}\frac{1}{|E_j|} - \sum_{v_i\in\Gamma^0}\frac{1}{|V_i|}\right) + 1 \text{ and } \operatorname*{LSM}_{v_i\in\Gamma^0}|V_i| \;\Big|\; d.$$

In particular, there exists a free subgroup F of index $d = \operatorname*{LSM}_{v_i\in\Gamma^0}|V_i|$. As in [1, Sec. 8], we construct a d-sheeted covering $p : (K_F, \bar{v}_0) \to (K(G,\Gamma), v_0)$ such that $p^*\pi_1(K_F, \bar{v}_0) = F$. For each

$g \in G \cong \pi_1(K(G,\Gamma), v_0)$, we denote by $\widetilde{g}$ some closed path in $K(G,\Gamma)$ originating at v_0 and representing the element g; $\widetilde{\widetilde{g}}$ denotes the lifting of g in K_F originating at $\bar{v}_0$. By the properties of coverings, arbitrary elements g_1 and g_2 in G belong to the same right coset of the subgroup F if and only if $\omega(\widetilde{\widetilde{g}}_1) = \omega(\widetilde{\widetilde{g}}_2)$. Since H is finitely generated and K_F is finite, we can find a system of representatives of right cosets in H with respect to $H \cap F$ and a system of generators for $H \cap F$. Thus, the occurrence problem for elements of G in H can be reduced to the occurrence problem for elements of the free group F in the finitely generated subgroup $H \cap F$. The latter problem can be solved by Nielsen's method (see [8, Chap. 1, Sec. 2]).

Definition 2.1 (see [1, Sec. 9]). Suppose that (G,Γ) is a graph of groups, $H \leq G = \pi_1(G,\Gamma,v_0)$, and $p\colon (K_H, \bar{v}_0) \to (K(G,\Gamma), v_0)$ is a covering such that $p^*\pi_1(K_H,\bar{v}_0) = H$. We define the *support* $\overset{\circ}{K}_H$ of the covering p to be the minimal subcomplex of K_H with the following properties:

(1) $\bar{v}_0 \in \overset{\circ}{K}_H$,

(2) $\overset{\circ}{K}_H$ is connected,

(3) $\overset{\circ}{K}_H$ is the union of covering neighborhoods $O_s(V_i)$,

(4) $\pi_1(\overset{\circ}{K}_H, \bar{v}_0) = \pi_1(K_H, \bar{v}_0)$.

The complex K_H can be represented as $K_H = \overset{\circ}{K}_H \cup (\bigcup_{k=1} X_k)$, where each X_k is a connected union of covering neighborhoods, the intersection $\overset{\circ}{K}_H \cap X_k$ is a face in $\overset{\circ}{K}_H$, and $X_{k_1} \cap X_{k_2} = \emptyset$ for $X_{k_1} \neq X_{k_2}$. Let a_k be the body of that neighborhood in X_k which contains the face $\overset{\circ}{K}_H \cap X_k$. Then X_k is a telescopic (nonascending) tube tree with respect to a_k (see Definition 1.1), and the embedding $\overset{\circ}{K}_H \cap X_k \to X_k$ induces the isomorphism $\pi_1(\overset{\circ}{K}_H \cap X_k, x_k) \to \pi_1(X_k, x_k)$ for some basic vertex $x_k \in \overset{\circ}{K}_H \cap X_k$.

THEOREM 2.1. There exists an algorithm which, given a finite system $\{h_1, \ldots, h_n\}$ of generators for an arbitrary subgroup H of the group $G = \pi_1(G,\Gamma,v_0)$, where (G,Γ) is a finite graph of finite groups, allows us to construct the support of a covering $p\colon (K_H,\bar{v}_0) \to (K(G,\Gamma), v_0)$ such that $p^*\pi_1(K_H,\bar{v}_0) = H$.

Proof. The covering complex K_H can be represented as the union of an ascending chain of subcomplexes $K_H(0) \subset K_H(1) \subset \ldots$, where $K_H(0)$ is the covering neighborhood $O_t(V_0)$ containing the point $\bar{v}_0$, and $K_H(i+1)$ is the union of $K_H(i)$ and all the covering neighborhoods that intersect $K_H(i)$ in K_H. According to Definitions 2.2 and 2.3 given below, $K_H(i)$ are locally covering H-complexes. In order to construct $K_H(i)$ from $K_H(i-1)$, we need not know the whole complex K_H (which may well be infinite). It is sufficient to use the operations of H-adjoining and H-gluing described below. We then construct the support $\overset{\circ}{K}_H$ by deleting "superfluous" neighborhoods from an appropriate complex $K_H(m)$.

Definition 2.2. A complex C is said to be *locally covering* for $K(G,\Gamma)$ if the following conditions are satisfied:

(1) C is a connected union of some subcomplexes which are homeomorphic to covering neighborhoods of the form $O_s(V_i)$. We assume the homeomorphisms to be fixed and make no distinction between the covering neighborhoods and their images in C. The covering mappings $p_{s,i}\colon O_s(V_i) \to O(V_i)$ are also assumed to be fixed.

(2) For any distinct neighborhoods $O_{s_1}(V_{i_1})$ and $O_{s_2}(V_{i_2})$ with nonempty intersection in C, every connected component of the set $O_{s_1}(V_{i_1}) \cap O_{s_2}(V_{i_2})$ is simultaneously the face ∂L of some handle $L \subset O_{s_1}(V_{i_1})$ and the face ∂M of some handle $M \subset O_{s_2}(V_{i_2})$.

(3) $p_{s_1,i_1}|_{O_{s_1}(V_{i_1}) \cap O_{s_2}(V_{i_2})} = p_{s_2,i_2}|_{O_{s_1}(V_{i_1}) \cap O_{s_2}(V_{i_2})}$.

For a locally covering complex C, we can define a mapping $p\colon C \to K(G,\Gamma)$ by gluing the covering mappings $p_{s,i}$. The mapping p is generally not a covering.

A *face of the complex* C is the face of any handle in C (see Sec. 1). If a face lies only in one handle, we call the face and the handle *outer*. If a face lies in two handles, we call the face and the handles *inner*.

Definition 2.3. A locally covering complex C is said to be an H-complex if the following conditions are met:

(1) C has a vertex $\bar{v}_0$ such that $p(\bar{v}_0) = v_0$;

(2) $p^*\pi_1(C, \bar{v}_0) \leq H$;

(3) for any covering neighborhood $O_s(V_i) \subseteq C$, any vertex $w \in O_s(V_i)$, and any path f in C going from $\bar{v}_0$ to w, the covering $p_{s,i}\colon (O_s(V_i), w) \to (O(V_i), p(w))$ corresponds to the subgroup

$$\{[v] \mid v \text{ is a loop in } O(V_i) \text{ originating at } p(w), \text{ and } [p(f)vp(f)^{-1}] \in H\}.$$

Remark 2.1. In the last definition, from (2) it follows that it is enough to require that condition (3) hold for some vertex $w \in O_s(V_i)$ and some path f in C going from $\bar{v}_0$ to w.

For any connected subcomplex R of the complex $K(G, \Gamma)$ and for any path f with origin $\bar{v}_0$ in R, we denote by $\mathcal{H}_f(R)$ the following subgroup of $\pi_1(R, \omega(f))$:

$$\{[v] \mid v \text{ is a loop in } R \text{ originating at } \omega(f) \text{ and } [fvf^{-1}] \in H\}.$$

Now we define the operations of H-adjoining and H-gluing which will be used to construct new H-complexes from H-complexes.

H-adjoining. For an arbitrary H-complex C with a nonempty set of outer faces, let f be an arbitrary path in C going from $\bar{v}_0$ to some vertex w on the outer face $\partial L_1' \subset O_{s_1}(V_{i_1}) \subseteq C$ and let $p_{s_1,i_1} : O_{s_1}(V_{i_1}) \to O(V_{i_1})$ be a standard covering; put $L_1 = p_{s_1,i_1}(L_1')$. In view of Definition 2.3 and the structure of covering neighborhoods, Diagram 1 (coverings and embeddings) corresponds to Diagram 2 (group embeddings).

$$\begin{array}{ccc} O_{s_1}(V_{i_1}) & \xrightarrow{p_{s_1,i_1}} & O(V_{i_1}) \\ \uparrow & & \uparrow \\ \partial L_1' & \xrightarrow[p_{s_1,i_1}|_{\partial L_1'}]{} & \partial L_1 \end{array}$$

Diagram 1

$$\begin{array}{ccc} \mathcal{H}_{p(f)}(O(V_{i_1})) & \longrightarrow & \pi_1(O(V_{i_1}), p(w)) \\ \uparrow & & \uparrow \\ \mathcal{H}_{p(f)}(\partial L_1) & \longrightarrow & \pi_1(\partial L_1, p(w)) \end{array}$$

Diagram 2

Suppose that $O(V_{i_2})$ is a neighborhood in $K(G, \Gamma)$, adjacent to the neighborhood $O(V_{i_1})$, and L_2 is a handle of $O(V_{i_2})$ such that $\partial L_2 = \partial L_1$. By Lemma 2.1, we can enumerate all elements of the finite groups shown in Diagram 3 and, therefore, construct the corresponding coverings (Diagram 4).

$$\begin{array}{ccc} \mathcal{H}_{p(f)}(O(V_{i_2})) & \longrightarrow & \pi_1(O(V_{i_2}), p(w)) \\ \uparrow & & \uparrow \\ \mathcal{H}_{p(f)}(\partial L_2) & \longrightarrow & \pi_1(\partial L_2, p(w)) \end{array}$$

Diagram 3

$$\begin{array}{ccc} O_{s_2}(V_{i_2}) & \xrightarrow{p_{s_2,i_2}} & O(V_{i_2}) \\ \uparrow & & \uparrow \\ \partial L_2' & \xrightarrow[p_{s_2,i_2}|_{\partial L_2'}]{} & \partial L_2 \end{array}$$

Diagram 4

Since $\partial L_1 = \partial L_2$ and $\mathcal{H}_{p(f)}(\partial L_1) = \mathcal{H}_{p(f)}(\partial L_2)$, the covering faces $\partial L_1'$ and $\partial L_2'$ are homeomorphic. If we adjoin the neighborhood $O_{s_2}(V_{i_2})$ to C with the help of this homeomorphism, the resulting complex will also be an H-complex. We say that it is *obtained by H-adjoining to C through the face $\partial L_1'$*.

The result of H-adjoining does not depend on the choice of the path f. Indeed, if we change the path f in C, with the end vertex w fixed, the groups $\mathcal{H}_{p(f)}(O(V_{i_1}))$, $\mathcal{H}_{p(f)}(\partial L_1)$, $\mathcal{H}_{p(f)}(O(V_{i_2}))$, and $\mathcal{H}_{p(f)}(\partial L_2)$ do not change, as follows from condition (2) in Definition 2.3. If we change the vertex $w \in \partial L_1'$, the groups in Diagram 3 will be replaced by groups which are naturally isomorphic to them, and so Diagram 4 will not be affected.

H-gluing. For an arbitrary H-complex C, let $\partial L_1'$ and $\partial L_2'$ be two distinct outer faces and suppose that the handles L_1' and L_2' containing them are mapped by p to some handles L_1 and L_2 which form a tube in $K(G,\Gamma)$. Let us choose arbitrary paths f_1 and f_2 in C going from $\bar{v}_0$ to some vertices $w_1 \in \partial L_1'$ and $w_2 \in \partial L_2'$. Suppose that the subgroups $\mathcal{H}_{p(f_1)}(\partial L_1)$ and $\mathcal{H}_{p(f_2)}(\partial L_2)$ are conjugate in the group $\pi_1(\partial L, w)$, where $\partial L = \partial L_1 = \partial L_2$ and $w = p(w_1) = p(w_2)$. In view of the cover properties, there exists a homeomorphism $\varphi\colon \partial L_1' \to \partial L_2'$ such that $p|_{\partial L_1'} = \varphi \circ p|_{\partial L_2'}$. Using it, we glue $\partial L_1'$ and $\partial L_2'$ in C into a single subcomplex $\partial L'$. Let g be some path in $\partial L'$ going from $\varphi(w_1)$ to w_2. If $[p(f_1 g f_2^{-1})] \in H$, the homeomorphism φ is assumed admissible, and we say that *φ is an H-gluing in C.*

Note that by condition (2) in Definition 2.3, the admissibility of φ does not depend on the choice of the paths f_1, f_2, and g. Denote by $C(\varphi)$ the complex obtained from C as a result of gluing by an admissible homeomorphism φ.

LEMMA 2.2. There exists at most one admissible homeomorphism $\varphi\colon \partial L_1' \to \partial L_2'$ such that $p|_{\partial L_1'} = \varphi \circ p|_{\partial L_2'}$.

Proof. Let φ and ψ be two homeomorphisms satisfying the hypothesis of the lemma and let g and h be some paths in $\partial L_2'$ going from $\varphi(w_1)$ to w_2 and from $\psi(w_1)$ to w_2, respectively. Since φ and ψ are admissible, we have $r_1 = [p(f_1 g f_2^{-1})] \in H$ and $r_2 = [p(f_1 h f_2)] \in H$. If a loop v is homotopic to the closed path $p(g^{-1})p(h)$ in the subcomplex $\partial L_2 = p(\partial L_2')$, then we have the following:

(a) $r = r_1^{-1} r_2 = [p(f_2) \cdot v \cdot p(f_2)^{-1}] \in H$;

(b) the covering $p|_{\partial L_2'}\colon \partial L_2' \to \partial L_2$ is the restriction of the covering mapping of neighborhoods containing $\partial L_2'$ and ∂L_2.

In view of Definition 2.3 and the cover properties, the loop v in ∂L_2 can be lifted to some loop v' in $\partial L_2'$ originating at w_2. From the equality $[p|_{\partial L_2'}(v')] = [v]$ and the definition of v, it follows that $[p|_{\partial L_2'}(v'h^{-1})] = [p|_{\partial L_2'}(g^{-1})]$. Since the paths $v'h^{-1}$ and g^{-1} have common origin and $p|_{\partial L_2'}$ is a covering, they have common end, i.e., $\psi(w_1) = \varphi(w_1)$. Since the vertex w_1 in the definition of H-gluing can be chosen arbitrarily, the homeomorphisms φ and ψ coincide at the vertices of the complex $\partial L_1'$. Since, in addition, φ and ψ make the diagram

$$\begin{array}{ccc} \partial L_1' & \xrightarrow{\ \varphi,\psi\ } & \partial L_2' \\ & {\scriptstyle P|_{\partial L_1'}} \searrow \quad \swarrow {\scriptstyle P|_{\partial L_2'}} & \\ & \partial L & \end{array}$$

commutative and $p|_{\partial L_1'}$ and $p|_{\partial L_2'}$ are coverings, it follows that $\varphi = \psi$.

We proceed by constructing the support $\overset{\circ}{K}_H$ of p. First, we effectively construct an initial segment $K_H(0) \subset K_H(1) \subset \ldots \subset K_H(m)$ of the chain of H-complexes defined at the beginning of the proof; here m is the minimal number with the following property:

(A) the irreducible closed paths $\widetilde{h}_1, \ldots, \widetilde{h}_n$ representing the generators of H in $K(G,\Gamma)$ can be lifted to closed paths beginning at $\bar{v}_0$ in the complex $K_H(m)$. In particular, we then have $\pi_1(K_H(m), \bar{v}_0) \cong \pi_1(K_H, \bar{v}_0) \cong H$.

Next, we obtain $\overset{\circ}{K}_H$ by deleting "superfluous" neighborhoods from $K_H(m)$. The complexes $K_H(i)$ will be constructed in such a way that the operation of H-gluing will not be applicable to them.

Construction of $K_H(0)$. Let $p_{s,0} : (O_s(V_0), \bar{v}_0) \to (O(V_0), v_0)$ be the covering corresponding to the subgroup $\{[v] | v$ is a loop in $O(V_0)$ beginning at v_0, and $[v] \in H\} \leq \pi_1(O(V_0), v_0)$. By Lemma 2.1, we can compute this subgroup effectively and, hence, construct the covering (see [1, Sec. 5]). Put $K_H(0) = O_s(V_0)$.

Suppose that $K_H(i)$ is already constructed for some $i \geq 0$, and that the operation of H-gluing cannot be applied to $K_H(i)$. We number all outer handles of the complex $K_H(i)$ by $1, \ldots, m_i$ [if there are no outer handles, we put $\overset{\circ}{K}_H = K_H(i) = K_H$]. Then we apply to $K_H(i)$ the operation of H-adjoining for the first handle, making all possible H-gluings in the resulting complex. If some of the handles numbered $1, \ldots, m_i$ are still outer, we choose one with the smallest number j $(j > 1)$ and perform H-adjoining for this handle, then making all possible H-gluings.

The process continues until all outer handles with numbers $1, \ldots, m_i$ are exhausted. The resulting complex is denoted by $K_H(i+1)$. The operation of H-gluing is not applicable to $K_H(i+1)$, and its outer handles (if there are any) are distinct from those in $K_H(i)$. If condition (A) is satisfied, the construction finishes at step m. It can be proved that $m \leq [(\max_{i=1,\ldots,n} |\tilde{h}_i| + 1)/2]$, where $|\tilde{h}_i|$ is the number of edges in the path $\tilde{h}_i$.

Let X be an arbitrary connected subcomplex in $K_H(m)$ that is a union of covering neighborhoods. We call a neighborhood $O_s(V_i)$ in X *superfluous* if it satisfies the following conditions:

(1) $O_s(V_i)$ does not contain $\bar{v}_0$;

(2) the complex $O_s(V_i) \cap \overline{X \backslash O_s(V_i)}$ coincides with some face ∂L of the neighborhood $O_s(V_i)$; moreover, the embedding $\partial L \hookrightarrow O_s(V_i)$ induces an isomorphism of the fundamental groups $\pi_1(\partial L, w) \cong \pi_1(O_s(V_i), w)$ for some vertex $w \in \partial L$.

We delete all superfluous neighborhoods from $K_H(m)$ and from the resulting complex. The process continues until all superfluous neighborhoods are exhausted. Eventually, we obtain the required complex $\overset{\circ}{K}_H$.

THEOREM 2.2. There exists an algorithm which, given an arbitrary finite graph of finite groups (G, Γ) (with vertex groups G_v, $v \in \Gamma^0$, and edge groups G_e, $e \in \Gamma^1$) and a finite set of elements $h_1, \ldots, h_n \in \pi_1(G, \Gamma, v_0)$, allows us to construct a finite graph of finite groups (D, Δ) such that $\langle h_1, \ldots, h_n \rangle \cong \pi_1(D, \Delta, \bar{v}_0)$ for some vertex $\bar{v}_0 \in \Delta^0$. Moreover, we can point out some mappings $f : \Delta^0 \to \Gamma^0$ and $\varphi : \Delta^1 \to \{g_1 e g_2 |\ e \in \Gamma^1,\ g_1 \in G_{\alpha(e)},\ g_2 \in G_{\omega(e)}\}$, together with embeddings $\varphi_v : D_v \to G_{f(v)}$, $v \in \Delta^0$, which induce an embedding $\psi : \pi_1(D, \Delta, \bar{v}_0) \to \pi_1(G, \Gamma, _0)$ whose image coincides with $\langle h_1, \ldots, h_n \rangle$.

Proof. Letting $H = \langle h_1, \ldots, h_n \rangle$, we apply Theorem 2.1 to construct the support $\overset{\circ}{K}_H$ of an embedding $p\colon (K_H, \bar{v}_0) \to (K(G, \Gamma), v_0)$ such that $p^* \pi_1(K_H, \bar{v}_0) = H$. The graph of groups (D, Δ) is obtained from $\overset{\circ}{K}_H$ as follows:

(1) with each body in $\overset{\circ}{K}_H$, we associate its vertex and fundamental group;

(2) with each tube in $\overset{\circ}{K}_H$, we associate its edge and fundamental group;

(3) we define the incidence of vertices and edges, as well as the embeddings of edge groups in vertex groups, in a natural way.

Now we give a precise definition of the graph of groups (D, Δ), the mappings f and φ, and the embeddings φ_v, $v \in \Delta^0$.

Definition of the graph of groups (D, Δ). In each body $r \subseteq \overset{\circ}{K}_H$, we choose a vertex w_r, selecting $\bar{v}_0$ if it lies in the body. In each tube $L \subset \overset{\circ}{K}_H$, we choose some lifting of the positively oriented edge $\{u_j\} \times [-1, 1]$ [from $(u_j, -1)$ to $(u_j, 1)$, where $e_j \in \Gamma^1_+$] in the tube $p(L) \subset K(G, \Gamma)$, denoting this lifting by e_{L+} and the inverse edge by e_{L-}. In what follows, we always assume that $\sigma \in \{+, -\}$ and $\bar{\sigma} = \{+, -\} \backslash \{\sigma\}$. We denote the origin, the midpoint, and the end of an edge e_{L^σ} by $e_{L^\sigma}(-1)$, $e_{L^\sigma}(0)$, and $e_{L^\sigma}(1)$, respectively. For each vertex $e_{L^\sigma}(t)$, where $t = \pm 1$, and for a body r containing $e_{L^\sigma}(t)$, we choose a path in r going from w_r to $e_{L^\sigma}(t)$. Denote this path by $g(w_r, e_{L^\sigma}(t))$. Suppose that the origin of the edge e_{L^σ} lies in a body r_1 and the end lies in a body r_2. We denote by $e_{(r_1, L^\sigma, r_2)}$ the path that is equal to the product of the paths $g(w_{r_1}, e_{L^\sigma}(-1))$, e_{L^σ}, and $g(w_{r_2}, e_{L^\sigma}(1))^{-1}$. This definition implies that each of the paths $e_{(r_1, L^\sigma, r_2)}$ and $e_{(r_2, L^{\bar{\sigma}}, r_1)}$ is the inverse of the other. Put

$\Delta^0 = \{w_r |\ r$ is a body in the covering $\overset{\circ}{K}_H\}$,

$\Delta^1 = \{e_{(r_1, L^\sigma, r_2)} |\ L$ is a tube, r_1 and r_2 are bodies in $\overset{\circ}{K}_H$ such that the edge e_{L^σ} originates in r_1 and terminates in $r_2\}$,

$\overline{e_{(r_1, L^\sigma, r_2)}} = e_{(r_2, L^{\bar{\sigma}}, r_1)}$, $\alpha(e_{(r_1, L^\sigma, r_2)}) = w_{r_1}$, $\omega(e_{(r_1, L^\sigma, r_2)}) = w_{r_2}$.

Remark 2.2. We will denote by $e_{(r_1, L^\sigma, r_2)}$ both the path in $\overset{\circ}{K}_H$ and the edge in the graph Δ.

Now we define the vertex and edge groups of the graph of groups (D, Δ). With a vertex $w_r \in \Delta^0$ we associate the group $D_{w_r} = \pi_1(r, w_r)$, and with an edge $e_{(r_1, L^\sigma, r_2)}$, the group $D_{e_{(r_1, L^\sigma, r_2)}} = \pi_1(L, e_{L^\sigma}(0))$. It is obvious that mutually inverse edges are associated with the same group. In order to define the embeddings $\alpha\colon D_{e_{(r_1, L^\sigma, r_2)}} \to D_{w_{r_1}}$ and $\omega : D_{e_{(r_1, L^\sigma, r_2)}} \to D_{w_{r_2}}$ of the edge groups in the vertex groups, we shall need the following fact.

Remark 2.3. Since the body $r = K_s(G_v)$ is a retract of the neighborhood $O_s(G_v)$, there is a natural isomorphism $\theta_r : \pi_1(r, w_r) \to \pi_1(O_s(G_v), w_r)$.

For brevity, we denote by $g(w_{r_1}, e_{L^\sigma}(0))$ the path that is the product of the path $g(w_{r_1}, e_{L^\sigma}(-1))$ and the subpath of e_{L^σ} going from $e_{L^\sigma}(-1)$ to $e_{L^\sigma}(0)$. Then, for an arbitrary $d \in D_{e_{(r_1, L^\sigma, r_2)}}$, we put

$$\alpha(d) = \theta_{r_1}^{-1}([g(w_{r_1}, e_{L^\sigma}(0))] d [g(w_{r_1}, e_{L^\sigma}(0))]^{-1}).$$

The element $\omega(d)$ is defined in an analogous way.

Definition of the mapping $f\colon \Delta^0 \to \Gamma^0$. For $w_r \in \Delta^0$, we put $f(w_r) = p(w_r)$.

Definition of the mapping $\varphi : \Delta^1 \to \{g_1 e g_2 |\ e \in \Gamma^1,\ g_1 \in G_{\alpha(e)},\ g_2 \in G_{\omega(e)}\}$. For $e_{(r_1, L^\sigma, r_2)} \in \Delta^1$, we put $\varphi(e_{(r_1, L^\sigma, r_2)}) = g_1 e g_2$, where g_1, g_2, and e are such that the loops homotopic to the closed paths $p(g(w_{r_1}, e_{L^\sigma}(-1)))$ and $p(g(w_{r_2}, e_{L^\sigma}(-1))^{-1})$ in the bodies $p(r_1)$ and $p(r_2)$ are labeled by $\widetilde{g}_1$ and $\widetilde{g}_2$, respectively, and the edge $p(e_{L^\sigma})$ is labeled by $\widetilde{e}$.

Such a definition of φ does not appear accidental if we look at the image of the path $e_{(r_1, L^\sigma, r_2)}$ in $\overset{\circ}{K}_H$ under the action of p (see Remark 2.2).

Definition of the embeddings $\varphi_{w_r} : D_{w_r} \to G_{f(w_r)}$, where $w_r \in \Delta^0$. For $g \in D_{w_r} = \pi_1(r, w_r)$, we put $\varphi_{w_r}(g) = p^*(g)$.

Definition of the embedding $\psi\colon \pi_1(D, \Delta, \bar{v}_0) \to \pi_1(G, \Gamma, v_0)$. We put $\psi = \gamma^* p^* \delta^*$ (see Diagram 5, with the mappings γ^* and δ^* defined below).

$$\begin{array}{ccc} \pi_1(\overset{\circ}{K}_H, \bar{v}_0) & \xrightarrow{\ p^*\ } & \pi_1(K(G, \Gamma), v_0) \\ \gamma^* \uparrow & & \downarrow \delta^* \\ \pi_1(D, \Delta, \bar{v}_0) & \xrightarrow{\ \psi\ } & \pi_1(G, \Gamma, v_0) \end{array}$$

Diagram 5

For an arbitrary complex K, we denote by $\Pi(K)$ the partial groupoid consisting of the homotopy classes of paths in K. The isomorphism γ^* is induced by the mapping $\gamma\colon \bigcup_{v\in\Delta^0} D_v \cup \Delta^1 \to \Pi(\overset{\circ}{K}_H)$ which is given on the elements $e_{(r_1,L^\sigma,r_2)} \in \Delta^1$ and $d \in D_{w_r}$, where $w_r \in \Delta^0$, by the following rules:

$$\gamma(e_{(r_1,L^\sigma,r_2)}) = [g(w_{r_1}, e_{L^\sigma}(-1))e_{L^\sigma}(g(w_{r_2}, e_{L^\sigma}(1)))^{-1}],$$

$$\gamma(d) = d \quad \text{(see the definition of the group } D_{w_r}\text{)}.$$

The groupoid $\Pi(K(G,\Gamma))$ has the following generators: the classes of edges labeled $\widetilde{e}_j$, where $e_j \in \Gamma^1$, and the classes of loops labeled $\widetilde{a}$, where a runs over the elements of all vertex groups G_v.

The isomorphism δ^* is induced by the mapping $\delta : \Pi(K(G,\Gamma)) \to F(G,\Gamma)$ which is defined on the generators as follows: the class of an edge labeled $\widetilde{e}_j$ is associated with the element e_j, and the class of a loop labeled $\widetilde{a}$, with the element a.

We omit a detailed proof of the fact that the graph of groups (D,Δ) and the mappings f, φ, φ_v $(v \in \Delta^0)$, and ψ, defined above, satisfy the conditions of the theorem.

3. SUBGROUPS AS FREE FACTORS

Our main objective in this section is to state and prove Theorem 3.1.

For any nontrivial subgroup $H \le \pi_1(G,\Gamma,v_0)$, we denote by C_H an H-complex with the following properties:

(1) $\overset{\circ}{K}_H \subseteq C_H \subseteq K_H$;

(2) all outer handles in C_H (if there are any) are identity handles, and all inner handles lying in $C_H \backslash \overset{\circ}{K}_H$ are nonidentity handles. Such a complex C_H exists and is unique since $K_H \backslash \overset{\circ}{K}_H$ is a disconnected union of telescopic nonascending tube trees.

We say that two arbitrary outer faces (handles) of an arbitrary locally covering complex *correspond to each other* if the handles containing them are labeled by $(e_t^+, \mathcal{E})$ and $(e_t^-, \mathcal{E})$ for some t and $\mathcal{E}$. Such faces are homeomorphic and they can be glued.

Every complex obtained from C_H by gluing some corresponding outer identity faces is called a *glued complex* (or, for brevity, *gluing*) and is denoted by $\widetilde{C}_H$. It should be stressed that, although there can be several gluings, all of them are denoted by the same symbol. If, in addition, $\widetilde{C}_H \neq C_H$, then $\widetilde{C}_H$ is not an H-complex. Let $\widetilde{p}_H : (\widetilde{C}_H, \bar{v}_0) \to (K(G,\Gamma), v_0)$ be the mapping induced by the projection $p_H\colon (K_H, \bar{v}_0) \to (K(G,\Gamma), v_0)$.

LEMMA 3.1. Let (G,Γ) be a finite graph of finite groups, let $1 \neq H$, and let $M \le \pi_1(G,\Gamma,v_0)$. Suppose that, for some gluing $\widetilde{C}_H$, there exists an embedding $\widetilde{C}_H \hookrightarrow K_M$ that is natural in the following sense:

(1) the image of the vertex $\bar{v}_0$ in $\widetilde{C}_H$ is the vertex $\bar{v}_0$ in K_M;

(2) images of neighborhoods in $\widetilde{C}_H$ are neighborhoods in K_M;

(3) the following diagram commutes:

$$\begin{array}{ccc} \widetilde{C}_H & \hookrightarrow & K_M \\ & {\scriptstyle \widetilde{P}_H}\searrow \quad \swarrow{\scriptstyle P_M} & \\ & K(G,\Gamma) & \end{array}$$

Then the group H is a free factor in M.

If all inner handles of the complex $\overset{\circ}{K}_H$ are not identity handles, then the converse is also true.

Proof. Suppose that for some gluing $\widetilde{C}_H$, there exists a natural embedding $\widetilde{C}_H \hookrightarrow K_M$. We denote its image also by $\widetilde{C}_H$. Since all outer handles of the gluing $\widetilde{C}_H$ are identity ones, the subgroup $\pi_1(\widetilde{C}_H, \bar{v}_0)$ is a free factor in the group $\pi_1(K_M, \bar{v}_0)$. The gluing mapping induces the canonical isomorphism $\pi_1(\widetilde{C}_H, \bar{v}_0) \cong \pi_1(C_H, \bar{v}_0) * F_t$, where t is the number of pairs of identity outer handles being glued. Consequently, the subgroup $H \cong \pi_1(C_H, \bar{v}_0)$ is a free factor in the subgroup $M \cong \pi_1(K_M, \bar{v}_0)$.

Now we prove the converse statement, assuming that the following condition is met:

$$\text{all inner handles of the complex } \overset{\circ}{K}_H \text{ are nonidentity ones.} \tag{1}$$

Let L be a subgroup of the group M such that

$$M = \langle H, L \rangle = H * L. \tag{2}$$

By assumption, every vertex subgroup P of the group $\pi_1(K(G,\Gamma), v_0) \cong \pi_1(G, \Gamma, v_0)$ is finite. Therefore, $P \cap H \neq 1$ implies $P \cap H = P \cap M$.

In the proof of Theorem 2.1, the complex $\overset{\circ}{K}_H$ was constructed from an H-complex $\overset{\circ}{K}_H(0)$ via the operations of H-adjoining and H-gluing. In a similar manner, the complex K_M is constructed from an M-complex $K_M(0)$ via M-adjoining and M-gluing. The remaining part of the proof will rely on Propositions 3.1 and 3.2, in which it is assumed that the conditions of the lemma, as well as conditions (1) and (2), are satisfied.

Proposition 3.1. $K_H(0) = K_M(0)$.

Proof. As P we choose the vertex group $\{[v] |\ v \text{ is a loop in } O(V_0) \text{ originating at } v_0\}$. The complex $K_M(0)$ is the covering of the neighborhood $O(V_0)$ that corresponds to the subgroup $P \cap M$, and the complex $K_H(0)$ is the covering of the same neighborhood corresponding to the subgroup $P \cap H$. Now we prove that the group $P \cap H \cong \pi_1(K_H(0), \bar{v}_0)$ is nontrivial. Consider the following two cases.

(a) Let $K_H(0) = \overset{\circ}{K}_H$. By the conditions of the lemma, the group $H \cong \pi_1(\overset{\circ}{K}_H, \bar{v}_0)$ is nontrivial and, therefore, $\pi_1(K_H(0), \bar{v}_0)$ is nontrivial.

(b) Let $\overset{\circ}{K}_H$ be strictly larger than $K_H(0)$. Then $K_H(0)$ contains at least one inner handle of the complex $\overset{\circ}{K}_H$. Since such handles are nonidentity ones by assumption, the group $\pi_1(K_H(0), \bar{v}_0)$ is nontrivial in this case as well.

Thus, $P \cap H \cong \pi_1(K_H(0), \bar{v}_0) \neq 1$. But then $P \cap M = P \cap H$, and so $K_M(0) = K_H(0)$.

Proposition 3.2. Suppose that S is an H-complex which is simultaneously an M-complex and let S_1 be the H-complex obtained from S by H-adjoining (H-gluing) which involves only nonidentity handles. Then this operation is also an M-adjoining (M-gluing). In particular, S_1 is an M-complex.

Proof. Let ∂L be the nonidentity outer face of the complex S with respect to which the H-adjoining is carried out, f an arbitrary path in S from $\bar{v}_0$ to some vertex $w \in \partial L$. Then the subgroup

$$\{[v] |\ v \text{ is a loop in } p(\partial L) \text{ originating at } \omega(p(f)), \text{ and } [p(f)vp(f)^{-1}] \in H\} \tag{3}$$

is isomorphic to $\pi_1(\partial L, w)$ and is therefore nontrivial. In view of the above, this subgroup coincides with the subgroup

$$\{[v] |\ v \text{ is a loop in } p(\partial L) \text{ originating at } \omega(p(f)), \text{ and } [p(f)vp(f)^{-1}] \in M\}. \tag{4}$$

Since an H-adjoining of some neighborhood O to S with respect to the face ∂L uses the subgroup (3), which is equal to the subgroup (4), it is also an M-adjoining. Similarly, an H-gluing of some nonidentity outer faces in S is an M-gluing. The proposition is proved.

By Propositions 3.1 and 3.2, the support $\overset{\circ}{K}_H$ is an M-complex. Since K_M is the largest M-complex, it is obtained from $\overset{\circ}{K}_H$ via some M-adjoinings and M-gluings. We will prove that only identity outer faces of the complex $\overset{\circ}{K}_H$ can participate in the possible M-gluings. Then some gluing $\widetilde{C}_H$ will be embedded in K_M.

Suppose, to the contrary, that an M-gluing involves two nonidentity outer faces of the complex $\overset{\circ}{K}_H$. Then in H there are two subgroups P_1 and P_2 that are not conjugate in H but are conjugate in $M = H * L$, where $L \neq 1$. This, however, contradicts the presence of the normal form in a free product. The lemma is proved.

Recall that $\overline{K_H \backslash \overset{\circ}{K}_H}$ is a disconnected union of telescopic nonascending tube trees. We use induction to define the level of an arbitrary neighborhood $O_s(V_i)$ in K_H.

Definition 3.1. Each neighborhood in $\overset{\circ}{K}_H$ is a *neighborhood of level* 0. *The level of a neighborhood* $O_s(V_i)$ in $\overline{K_H \backslash \overset{\circ}{K}_H}$ is t $(t \in \mathrm{N})$ if $O_s(V_i)$ has a common face with some neighborhood of level $t-1$, but it is not a neighborhood of level less than $t-1$.

Proposition 3.3. For any graph of groups (G, Γ) and any covering neighborhood $O_s(V_i)$, the number of handles in $O_s(V_i)$ is not greater than $|V_i| \cdot \deg v_i$, where $\deg v_i$ is the degree of the vertex $v_i \in \Gamma^0$.

Proof. Consider the covering $O_s(V_i) \to O(V_i)$. The number of handles in $O(V_i)$ equals $\deg v_i$, and the number of handles in $O_s(V_i)$ with the same projection is at most $|\pi_1(O(V_i), v_i)| = |V_i|$. Thus, the total number of handles in $O_s(V_i)$ is at most $|V_i| \cdot \deg v_i$.

Proposition 3.4. For any finite graph of finite groups (G, Γ) and for an arbitrary finitely generated subgroup $H \leq \pi_1(G, \Gamma, v_0)$, the number of neighborhoods of each level in K_H is finite.

Proof. Since H is finitely generated, the support $\overset{\circ}{K}_H$ is finite; hence, the number of neighborhoods of level 0 is finite. It remains to observe that for each $t \in \mathrm{N}$, the number of neighborhoods of level t is not greater than the total number of faces of neighborhoods of level $t-1$, and the number of faces of an arbitrary neighborhood is finite by Proposition 3.3.

Proposition 3.5. Given a finite graph of finite groups (G, Γ), we can effectively find a natural number k such that for an arbitrary subgroup $H \leq \pi_1(G, \Gamma, v_0)$, the following statement holds: If $\overline{K_H \backslash \overset{\circ}{K}_H}$ contains a finite irreducible tube path, going from $O_{s_1}(V_{i_1})$ to $O_{s_2}(V_{i_2})$, such that the level of the neighborhood $O_{s_2}(V_{i_2})$ is greater than the level of the neighborhood $O_{s_1}(V_{i_1})$ by k, and $|\pi_1(O_{s_1}(V_{i_1}))| = |\pi_1(O_{s_2}(V_{i_2}))|$, then $\overline{K_H \backslash \overset{\circ}{K}_H}$ contains an infinite irreducible tube path such that $|\pi_1(O)| = |\pi_1(O_{s_1}(V_{i_1}))|$ for every neighborhood O which it passes through.

Proof. With every neighborhood $O_s(V_i)$ in $\overline{K_H \backslash \overset{\circ}{K}_H}$ we associate the characteristic pair $(\mathcal{E}, L)$, where $\mathcal{E}$ is the class of conjugate subgroups in $\pi_1(O(V_i), v_i)$ that corresponds to the covering $O_s(V_i) \to O(V_i)$ and $L \subset O_s(V_i)$ is the handle connecting $O_s(V_i)$ with a neighborhood of the preceding level. Let $m(v_i)$ be the number of classes of conjugate subgroups in $\pi_1(O(V_i), v_i)$. By Proposition 3.3, the number of characteristic pairs is not greater than $m = \sum_{v_i \in \Gamma^0} (m(v_i) \cdot |V_i| \cdot \deg v_i)$. We will prove that we can take k to be $m+1$.

Let $O_{s_1}(V_{i_1})$ be a neighborhood of level t and $O_{s_2}(V_{i_2})$ a neighborhood of level $t+m+1$ that is connected with $O_{s_1}(V_{i_1})$ by an irreducible tube path. Suppose that $|\pi_1(O_{s_1}(V_{i_1}))| = |\pi_1(O_{s_2}(V_{i_2}))|$. Then this tube path passes through two neighborhoods A_1 and A_2 with equal characteristic pairs. Denote by P_1

an irreducible tube path originating in A_1 and terminating in A_2. Since the characteristic pairs for A_1 and A_2 coincide, $\overline{K_H \backslash \overset{0}{K}_H}$ contains an irreducible tube path P_2 that is homeomorphic to the path P_1, originating in A_2 and terminating in some neighborhood A_3. By induction, we can similarly define a tube path P_l for every $l \geq 3$. Consider the infinite irreducible path $P = \bigcup_{i=1}^{\infty} P_i$. Since $|\pi_1(O_{s_1}(V_{i_1}))| = |\pi_1(O_{s_2}(V_{i_2}))|$ and the irreducible tube path connecting $O_{s_1}(V_{i_1})$ and $O_{s_2}(V_{i_2})$ is nonascending, for any neighborhood O of the path P_1 we have $|\pi_1(O)| = |\pi_1(O_{s_1}(V_{i_1}))|$. This is also true for any neighborhood of the path P.

LEMMA 3.2. There exists an algorithm which, given a finite graph of finite groups (G, Γ) and a finite set of elements $h_1, \ldots, h_n \in \pi_1(G, \Gamma, v_0)$, determines, in a finite number of steps, whether the complex C_H, where $H = \langle h_1, \ldots, h_n \rangle$, is finite.

Proof. For every nonnegative integer i, we denote by $\overset{i}{K}_H$ the subcomplex of K_H that consists of all covering neighborhoods of level not greater than i. By Proposition 3.4, the complex $\overset{i}{K}_H$ is finite. Similarly to $\overset{0}{K}_H$, it can be constructed in a finite number of steps by H-adjoining and H-gluing.

Algorithm Verifying the Finiteness of C_H

Consider all neighborhoods of level $m = 1 + k \cdot \max_{v_i \in \Gamma^0} |V_i|$, where k is defined as in the proof of Proposition 3.5. If all of them are identity ones, then the complex C_H obviously lies in $\overset{m}{K}_H$ and, hence, is finite. If, however, any neighborhood O of level m is a nonidentity one, then the complex C_H is infinite. Indeed, suppose that $\pi_1(O) \neq 1$ for some neighborhood O of level m. Then there exist a chain of covering neighborhoods $O_1, O_{1+k}, \ldots, O_m = O$ of levels $1, 1+k, \ldots, 1 + k \cdot \max_{v_i \in \Gamma^0} |V_i| = m$ and an irreducible tube path in $\overline{K_H \backslash \overset{0}{K}_H}$, originating in O_1 and terminating in O_m, which successively passes through them. Since $|\pi_1(O_1)| \geq |\pi_1(O_{1+k})| \geq \cdots \geq |\pi_1(O_m)| = |\pi_1(O)| \neq 1$ and the group $\pi_1(O_1)$ is isomorphic to some subgroup of a vertex group V_i, there exist $i_1, i_2 \in \{1, 1+k, \ldots, 1 + k \cdot \max_{v_i \in \Gamma^0} |V_i|\}$ such that $i_2 - i_1 = k$ and $|\pi_1(O_{i_1})| = |\pi_1(O_{i_2})| \neq 1$. Applying Proposition 3.5, we conclude that the complex C_H is infinite.

LEMMA 3.3 (on Inclusion, [1, Sec. 8]). Let (G, Γ) be a finite graph of finite groups. Then the universal covering of any neighborhood in $K(G, \Gamma)$ can be included in a finite complex K_M for some free subgroup M of finite index in $\pi_1(G, \Gamma, v_0)$.

THEOREM 3.1. There exists an algorithm which, given a finite graph of finite groups (G, Γ) and a finite set of elements $h_1, \ldots, h_n \in \pi_1(G, \Gamma, v_0)$, determines, in a finite number of steps, whether the subgroup $H = \langle h_1, \ldots, h_n \rangle$ is a free factor in some subgroup of finite index in the group $\pi_1(G, \Gamma, v_0)$. Moreover, this algorithm effectively enumerates systems of generators of all subgroups of finite index in $\pi_1(G, \Gamma, v_0)$ in which H is a free factor.

Proof. It suffices to consider the case $H \neq 1$. We construct the complex $\overset{0}{K}_H$ according to Theorem 2.1. First, we consider the special case in which all inner handles of the complex $\overset{0}{K}_H$ are nonidentity ones, while the outer handles can be arbitrary.

Suppose that a subgroup $L \leq \pi_1(G, \Gamma, v_0)$ is such that $\langle H, L \rangle = H * L$. Then the following statements are true:

(1) the complex K_{H*L}, as well as all complexes that cover $K(G, \Gamma)$, has no outer handles;

(2) for some gluing $\widetilde{C}_H$, there exists a natural embedding $\widetilde{C}_H \hookrightarrow K_{H*L}$ (see Lemma 3.1).

Therefore, we can cut the complex K_{H*L} along some inner faces in such a way that we obtain the complex C_H and some set D of covering neighborhoods.

If $H * L$ is a subgroup of finite index in the group $\pi_1(G, \Gamma, v_0) \cong \pi_1(K(G, \Gamma), v_0)$, then the covering complex K_{H*L} is finite, since $K(G, \Gamma)$ is finite. Hence, the complex C_H and the set D are also finite.

The following reasoning is similar to that of [1, Sec. 10].

Let $f_H(e_j^\sigma, \langle e\rangle)$ be the number of outer handles of C_H that are labeled by $(e_j^\sigma, \langle e\rangle)$, where $\sigma \in \{+, -\}$. We recall that all outer handles of the complex C_H are identity ones and, therefore, are labeled either by $(e_j^+, \langle e\rangle)$ or $(e_j^-, \langle e\rangle)$, where $j \in J$. If $O_s(V_i) \to O(V_i)$ is an arbitrary covering and e_j is an edge such that v_i is one of the endpoints of e_j, we denote by $d_{s,i}(e_j^\sigma, \mathcal{E})$ the number of handles labeled by $(e_j^\sigma, \mathcal{E})$ in the neighborhood $O_s(V_i)$ (see Sec. 1). Let $x_{s,i}$ be the number of neighborhoods of the form $O_s(V_i)$ in the set D. Then the balance condition formulated below must be satisfied, reflecting the fact that the complex K_{H*L} (without outer handles) is obtained from C_H and the neighborhoods in D by appropriate gluings of their outer faces corresponding to each other.

The balance condition. For each edge $e_j \in \Gamma^1$ and each nontrivial class $\mathcal{E}$ of conjugate subgroups of the group E_j, the following hold:

$$\sum_{\substack{s,i:\\ \alpha(j)=i}} x_{s,i} d_{s,i}(e_j^-, \mathcal{E}) = \sum_{\substack{s,i:\\ \omega(j)=i}} x_{s,i} d_{s,i}(e_j^+, \mathcal{E}), \tag{5}$$

$$\sum_{\substack{s,i:\\ \alpha(j)=i}} x_{s,i} d_{s,i}(e_j^-, \langle e\rangle) + f_H(e_j^-, \langle e\rangle) = \sum_{\substack{s,i:\\ \omega(j)=i}} x_{s,i} d_{s,i}(e_j^+, \langle e\rangle) + f_H(e_j^+, \langle e\rangle). \tag{6}$$

On the other hand, if the complex C_H is finite and there exists a finite set D of covering neighborhoods satisfying conditions (5) and (6), then the complex C_H and the covering neighborhoods in D can be glued to yield one or more connected complexes without outer handles. We denote by R the one containing some gluing $\widetilde{C}_H$. Let the covering $(R, \bar{v}_0) \to (K(G, \Gamma), v_0)$ correspond to a subgroup M of the group $\pi_1(G, \Gamma, v_0) \cong \pi_1(K(G, \Gamma), v_0)$. Then $R = K_M$ (in our notation) and, by Lemma 3.1, H is a free factor in M. In addition, M is a subgroup of finite index in $\pi_1(G, \Gamma, v_0)$ since R is finite.

Thus, if all inner handles of the complex $\overset{\circ}{K}_H$ are nonidentity ones, then the subgroup H is a free factor in some subgroup of finite index in the group $\pi_1(G, \Gamma, v_0)$ if and only if the following conditions are satisfied:

(a) the complex C_H is finite,

(b) the system (5), (6), in which e_j runs over the set Γ^1 and the class $\mathcal{E}$ runs over the set of nontrivial classes of conjugate subgroups of the group E_j, is solvable in nonnegative integers $x_{s,i}$.

In view of Lemma 3.2, condition (a) can be verified effectively. Condition (b) can also be effectively verified by means of integral linear programming.

Now we consider the general case, in which some of the inner handles of the complex $\overset{\circ}{K}_H$ can be identity handles, reducing it to the case given above. In order to reduce the length of the proof, we will not indicate basic vertices in some of the complexes considered below. After we cut the complex $\overset{\circ}{K}_H$ along all identity faces, it decomposes into the union of complexes $K_1, \dots, K_r$, all of whose inner handles are nonidentity ones, and identity neighborhoods $K_{r+1}, \dots, K_m$. It is obvious that $H \cong \pi_1(\overset{\circ}{K}_H, \bar{v}_0) \cong \pi_1(K_1) \dots * \pi_1(K_r) * F_t$, where F_t is a free group of some finite rank t. Therefore, if H is a free factor in some subgroup of finite index in the group $\pi_1(G, \Gamma, v_0)$, then the nontrivial subgroups $\pi_1(K_1), \dots, \pi_1(K_r)$ also have this property.

We now show that the converse is also true. Suppose that the subgroups $\pi_1(K_1), \dots, \pi_1(K_r)$ are free factors in some subgroups $M_1, \dots, M_r$ of finite index in the group $\pi_1(G, \Gamma, v_0)$. Then, by Lemma 3.1, some gluings $\widetilde{C}_{\pi_1(K_1)}, \dots, \widetilde{C}_{\pi_1(K_r)}$ are parts of finite complexes $K_{M_1}, \dots, K_{M_r}$. By Lemma 3.3, identity neighborhoods $K_{r+1}, \dots, K_m$ can be included in some finite complexes $K_{M_{r+1}}, \dots, K_{M_m}$. We cut the complexes $K_{M_1}, \dots, K_{M_r}, K_{M_{r+1}}, \dots, K_{M_m}$ (all of which have no outer handles) along all of the former identity faces of the complex $\overset{\circ}{K}_H$ and reattach handles as in [1, Sec. 9]. As a result, we obtain several connected components, one of which, denoted by R, contains some gluing $\widetilde{C}_H$. Since R is a finite complex covering $K(G, \Gamma)$, we have $R = K_M$ for some subgroup M of finite index in $\pi_1(K(G, \Gamma), v_0) \cong \pi_1(G, \Gamma, v_0)$. By Lemma 3.1, the subgroup H is a free factor in M.

Thus, the converse statement is proved, and the general case is reduced to the above special case.

The systems of generators of all subgroups M of finite index in $\pi_1(G, \Gamma, v_0)$, in which H is a free factor, can be effectively enumerated, since this reduces to enumerating all solutions of the systems of the form (5), (6) in nonnegative integers and subsequent gluing of the corresponding complexes.

COROLLARY 3.1. Suppose that (G, Γ) is a finite graph of finite groups and F is an arbitrary finitely generated free subgroup of the fundamental group $\pi_1(G, \Gamma, v_0)$ of (G, Γ). Then F is a free factor in some subgroup of finite index in the group $\pi_1(G, \Gamma, v_0)$.

REFERENCES

1. O. V. Bogopol'skii, "Finitely generated groups with the M. Hall property," *Algebra Logika*, **31**, No. 3, 227-275 (1992).
2. O. V. Bogopol'skii, *Finitely Presented M. Hall Groups* [in Russian], Preprint No. 20, Institute of Mathematics, Novosibirsk (1992).
3. A. M. Brunner and R. G. Burns, "Groups in which every finitely generated subgroup is almost a free factor," *Can. J. Math.*, **31**, No. 6, 1329-1338 (1979).
4. R. G. Burns, "On finitely generated subgroups of free products," *J. Austr. Math. Soc.*, **12**, 358-364 (1971).
5. M. Hall, "Coset representations in free groups," *Trans. Am. Math. Soc.*, **67**, 421-432 (1949).
6. C. L. Tretkoff and M. D. Tretkoff, "A topological proof of a theorem of Brunner and Burns about M. Hall groups," *Cont. Math.*, **44**, 85-89 (1985).
7. J.-P. Serre, *Trees*, Springer, Berlin (1980).
8. R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer, Berlin (1977).

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