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THE SWAP CONJECTURE OF TENNANT AND TURNER

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We argue against the conjecture which says that any two finite generating sets for G of the same cardinality are swap equivalent. The latter means that one is changed to another by a finite sequence of generating sets such that all the neighboring sets differ only in a single entry. Namely, it is proved that a free metabelian group of rank 3 has non swap equivalent bases.

INTRODUCTION

To the presentation of an arbitrary group as the quotient of a free group F, $G = F/R$, related are the notions of a relation module $\bar{R} = R/R'$ and a relation space group $\bar{F} = F/R'$. Relation modules and relation space groups are generally not uniquely defined by a group G. For this reason, the question of how they are related becomes especially significant.

In the present article we study the relationship in question for finite generating sets of groups in the light of the conjecture stated by Tennant and Turner in [1]. Our basic goal is to refute the conjecture for the general case.

In Sec. 1, some necessary notions are developed and the main results are presented. Sec. 2 is auxiliary; here we discuss an analog of the Tennant–Turner conjecture for modules. The final Sec. 3 contains proofs of both the main and auxiliary statements, as well as some implications which can be inferred from them.

1. MAIN RESULTS

Let $\Gamma_m(G)$ be the family of all generating sets $\gamma = (g_1, \ldots, g_m)$ for a group G. Assume that $m \geq r(G)$, where $r(G)$ is a minimal number of generators for the group, and so $\Gamma_m(G)$ is nonempty in all the cases treated below. The generating sets $\gamma_1, \gamma_2 \in \Gamma_m(G)$ are said to be *Nielsen equivalent*, written $\gamma_1 \sim_N \gamma_2$, if there is a sequence of elementary Nielsen transformations leading from γ_1 to γ_2. By an elementary Nielsen transformation here we mean a replacement of one of the components g_i of γ by $g_i g_j$, $g_j g_i$ $(i \neq j)$ or by g_i^{-1}. We know [2] that in the free group F_n of rank n, every set $\gamma = (g_1, \ldots, g_m) \in \Gamma_m(F_n)$ is Nielsen equivalent to the standard set $\gamma_0 = (f_1, \ldots, f_n, 1, \ldots, 1)$, where $f = (f_1, \ldots, f_n)$ is a fixed free basis for F_n. Thus, the set $\Gamma_m(F_n)$ forms a single Nielsen equivalence class.

In [1], Tennant and Turner formulated and discussed the swap conjecture cited below; the notion of swap equivalence introduced in [1] is wider than the notion of Nielsen equivalence. The sets $\gamma_1, \gamma_2 \in \Gamma_m(G)$ are said to be *swap equivalent*, written $\gamma_1 \sim_S \gamma_2$, if there is a sequence of elementary swaps leading from γ_1 to γ_2. An elementary swap is thought of as a transformation changing one element of the set to an arbitrary element of G. Of course, such a transformation is confined to the set $\Gamma_m(G)$.

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We note that generating sets can be swap equivalent but not Nielsen equivalent. Indeed, for any rank $n \geq 2$ free nilpotent group $N_{nk} = F_n/\gamma_{k+1}F_n$ of class $k \geq 3$, there exists an automorphism φ not induced by the automorphism of the group F_n relative to $\mathrm{Aut}F_n \to \mathrm{Aut}F_n/\gamma_{k+1}F_n$, a natural homomorphism [3, 4]. This implies that the basis $x = (x_1, \ldots, x_n)$ for a group N_{nk}, which corresponds to a fixed basis $f = (f_1, \ldots, f_n)$ for F_n, is not Nielsen equivalent to the basis $x^\varphi = (x_1^\varphi, \ldots, x_n^\varphi)$. Since every automorphism of the rank n free Abelian group $A_n \simeq N_{nk}/N'_{nk} \simeq F_n/F'_n$ is induced by some automorphism of F_n (see [2]), we can assume that $x^\varphi = (u_1x_1, \ldots, u_nx_n)$, where $u_i \in N'_{nk} (i = 1, \ldots, n)$. Any generating set of the form $v = (v_1x_1, \ldots, v_nx_n)$, $v_i \in N'_{nk}$, belongs to $\Gamma_m(N_{nk})$, and so such are all swap equivalent; in particular, $x^\varphi \sim_S x$.

In [1], the following conjecture is stated.

THE SWAP CONJECTURE. Any two finite generating sets $\gamma_1, \gamma_2 \in \Gamma_m(G)$ for G of the same cardinality are swap equivalent.

The swap conjecture was verified for various group classes. We call G a *swap group* if any two generating sets for G of the same cardinality m are swap equivalent, i.e., if the swap conjecture is true for G. In [1], it is noted that finitely generated Abelian groups and Fuchsian groups are swap groups. A natural explanation for this is the possibility of lifting up generating sets of such groups to those of a corresponding free group, which can be done in all the essential cases. We have already mentioned that the reason for free nilpotent groups to be swap groups is different. And it is not hard to establish that this will be similarly true for arbitrary finitely generated nilpotent groups.

In [1], the swap equivalence was treated in connection with relation modules and relation space groups, and the set $\gamma = (g_1, \ldots, g_m)$ was understood to be related to an epimorphism $\varepsilon(\gamma)\colon F_m \to G$, $f_i \mapsto g_i$ $(i = 1, \ldots, n)$. Putting $R(\gamma) = \ker\varepsilon(\gamma)$, we obtain the presentation $G = F_m/R(\gamma)$. Then $\overline{R(\gamma)} = R(\gamma)/R(\gamma)'$ is the relation module of G and $\bar{F_m} = F_m/R(\gamma)'$ is the relation space group, which agree with γ.

It is known [5, 6] that for finite G with $m \geq r(G) + 1$, all elements of $\Gamma_m(G)$ yield isomorphic relation modules and relation space groups. We are also aware (see [1]) that Nielsen equivalence $\gamma_1 \sim_N \gamma_2$ gives similar isomorphisms for arbitrary G. In [1], it was noted that swap equivalence $\gamma_1 \sim_S \gamma_2$ implies Nielsen equivalence $(\gamma_1, 1) \sim_N (\gamma_2, 1)$ [$(\gamma, 1)$ means that 1 is added to γ]. In the general case, $\gamma_1 \sim_S \gamma_2$ yields isomorphisms $\mathbb{Z}G \oplus \overline{R(\gamma_1)} \simeq \mathbb{Z}G \oplus \overline{R(\gamma_2)}$ and $\mathbb{Z}G \leftthreetimes \overline{F_m(\gamma_1)} \simeq \mathbb{Z} \leftthreetimes \overline{F_m(\gamma_2)}$.

In [1], it was also mentioned that a motivation for studying swap equivalence is the standard proof of the Tietze theorem (see [2]). In that proof, two generating sets $\gamma_1, \gamma_2 \in \Gamma_m(G)$ are related by a sequence of Tietze transformations in such a way that all the intermediate sets have at most $2m$ generators; the swap conjecture, if true, can improve this value to $m + 1$.

The swap conjecture is a general problem concerning the various generating sets for the class of all finitely generated groups. In our opinion, however, most interesting is the question of whether or not bases for the groups free in varieties are swap equivalent. Let $G_n = F_n/V(F_n)$ be a free group of rank n in some group variety defined by the set V of words and let $x = (x_1, \ldots, x_n)$ be the basis for G_n which agrees with a fixed basis $f = (f_1, \ldots, f_n)$ for the group F_n. A basis $y = (y_1, \ldots, y_n)$ for G_n is called *induced* (or *tame*) if it is induced by the basis for F_n. This is the case if and only if the automorphism φ of G_n given by $\varphi\colon x_i \mapsto y_i$ $(i = 1, \ldots, n)$ is induced by some automorphism of F_n, i.e., if φ is tame. A part $g = (g_1, \ldots, g_m)$ of the basis for G_n is a *primitive system*, which is likewise called *induced*, if it is induced by the primitive system for F_n. For the inducibility problem of primitive systems for some varieties of groups, see [7, 8].

Inducibility of all automorphisms of a group implies inducibility of all of its bases and, hence, primitive systems. It is known [9-11] that every automorphism of a free metabelian group $M_n = F_n/F''_n$ for $n = 2$ or $n \geq 4$ is tame. Consequently, such are swap groups.

We are especially interested in the exceptional group M_3 of which not each automorphism is tame [12]; moreover, its automorphism group $\mathrm{Aut}M_3$ and its IA-automorphism group $I\mathrm{Aut}M_3$, consisting of all automorphisms inducing an identity modulo M_3', are finitely generated [13, 14]. That a noninduced primitive element of M_3 exists has been established in [15], whose methods and results are used to derive the following:

BASIC THEOREM. The group M_3 has non swap equivalent bases.

This refutes the swap conjecture of Tennant and Turner.

The automorphism of a free group G_n in some variety of groups with fixed basis is called one-row if the image of only one element of the basis is distinct from the element proper. An interesting example of a one-row automorphism of a free metabelian group M_N is one given by the map:

$$\varphi_{ijk}(\alpha)\colon x_k \mapsto (x_i, x_j)^\alpha x_k, \;\; x_l \mapsto x_l \;\; (l \neq k), \tag{1}$$

where $i, j \neq k$ and $\alpha \in \mathbb{Z}A_n = \mathbb{Z}M_n/M_n'$. It is of interest to note that every map (1) gives an automorphism of M_n. In [7], $\varphi_{ijk}(\alpha)$ was called a *Chein* automorphism, since such were first treated by Chein in [12]. Chein established that $\varphi_{231}((a_1-1)^2)$, where a_1 is an element of the basis $a = (a_1, \dots, a_n)$ for a free Abelian group $A_n \simeq M_n/M_n' \simeq F_n/F_n'$ which agrees with the bases x and f for the groups M_n and F_n, respectively, is not a tame automorphism. We proved [7] that a Chein automorphism transforms elements of the basis x for M_3 to induced elements (though the basis resulting from x is not itself always induced!).

In the present article we bring up the notion of an elementary primitive element of the group M_3, implying that induced primitive elements are all elementary. At the same time, there do exist nonelementary primitive elements. The notion introduced is necessary for proving the key result of our article, namely the following:

Proposition. An arbitrary Chein automorphism of a free metabelian group M_3 transforms any elementary primitive element to an element of the same form.

COROLLARY. The group $\mathrm{Aut}M_3$ is not generated by one-row automorphisms.

As has been noted above, Sec. 2 is auxiliary. We will argue against an analog of the swap conjecture for free modules over Laurent polynomial rings. It also seems interesting to answer the question as to whether the swap conjecture is valid for rings. Naturally, we will confine ourselves to the basic classes of rings or pose the question (as was done for the class of groups) of which rings are swap rings.

2. THE SWAP CONJECTURE FOR MODULES

Let Λ be a commutative ring. Consider the matrix $A \in GL_n(\Lambda)$ as a tuple of row vectors. Matrices A_1 and A_2 are called *swap equivalent* if there is a sequence of elementary swaps leading from one to another. For instance, it is obvious that $\Lambda = \mathbb{Z}$ has at most one swap equivalence class of matrices for any n. Let $\Lambda_n = \mathbb{Z}[a_1^{\pm 1}, \dots, a_n^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables. The following theorems are widely known.

THEOREM C1 (Suslin [16]). Every group $GL_m(\Lambda_n)$ acts transitively on the set of all unimodular vectors $v = (v_1, \dots, v_m) \in \Lambda_n^m$.

Recall that $v = (v_1, \dots, v_m)$ is *unimodular* if $\mathrm{ideal}(v_1, \dots, v_m) = \Lambda_n$.

THEOREM C2 (Suslin [17]). For every $m \geq 3$ and for an arbitrary n we have $GE_m(\Lambda_n) = GL_m(\Lambda_n)$, where, as usual, GE_m denotes a subgroup of elementary matrices of the group GL_m which, by definition, is generated by all transvections and diagonal matrices.

Using these results, we obtain the following:

THEOREM 1. For every $m \geq 3$ and for an arbitrary n, there is only one swap equivalence class of matrices in $GL_m(\Lambda_n)$.

COROLLARY 1. All bases of the rank $m \geq 3$ free module over the ring Λ_n (n is any) are swap equivalent.

The theorem and the corollary clash with the following results.

THEOREM 2. For every $n \geq 2$, there are infinitely many swap equivalence classes of matrices in $GL_2(\Lambda_n)$.

Proof. It is known [18] that $GE_2(\Lambda_n) \neq GL_2(\Lambda_n)$ for every $n \geq 2$. Moreover, $GE_2(\Lambda_n)$ has infinite index in $GL_2(\Lambda_n)$. Every matrix in $GL_2(\Lambda_n)$ not belonging to $GE_2(\Lambda_n)$ is thought of as nonelementary.

In [15], it was proved that a unimodular row $a = (a_{11}, a_{12})$ belonging to some elementary matrix cannot belong to a nonelementary one. Moreover, the matrices

$$A = \begin{pmatrix} a \\ b \end{pmatrix}, \ B = \begin{pmatrix} a \\ c \end{pmatrix} \in GL_2(\Lambda_n)$$

belong to a single coset relative to $GE_2(\Lambda_n)$. In order to show this, write

$$AB^{-1} = \begin{pmatrix} 1 & 0 \\ d & g \end{pmatrix} \in GE_2(\Lambda_n).$$

The theorem follows from the facts given above.

COROLLARY 2. In the rank 2 free module over the ring Λ_n ($n \geq 2$), there are infinitely many swap equivalence classes of bases.

Certainly, similar results will be true not only for free modules over Laurent polynomial rings but also in many of the cases for which analogs of Theorems C1 and C2 hold. Here, however, our goal is unpretentious: we aim at giving a transparent idea of how to prove Theorem 2, whose technical use in the next section will meet some difficulties.

3. THE SWAP CONJECTURE FOR M_3

Denote by $b_i = (a_i - 1)(i = 1, \dots, n)$ standard generators of the fundamental ideal $\Delta_n = \sum_{i=1}^{n} b_i \Lambda_n$ for the ring $\Lambda_n = \mathbb{Z}[a_1^{\pm 1}, \dots, a_n^{\pm 1}]$. Hereinafter, $n = 2$ or $n = 3$, and so the use of certain notions and formulas is confined to just these cases.

For fixed k, every element $c \in \Lambda_3$ is presented uniquely as

$$c = c_0 + b_3 c_1 + b_3^2 c_2 + \dots + b_3^{k-1} c_{k-1} + b_3^k c_k, \tag{2}$$

where $c_0, c_1, \dots, c_{k-1} \in \Lambda_2$, $c_k \in \Lambda_3$ (see [19]).

An arbitrary automorphism $\varphi \in I\mathrm{Aut} M_3$ is uniquely defined by the map

$$\varphi \colon x_i \mapsto u_i x_i, \ u_i \in M_3' \ (i = 1, 2, 3). \tag{3}$$

Write the Jacobian matrix

$$J(\varphi) = (\partial u_i x_i / \partial x_j) = E + (\partial u_i / \partial x_j) \ (i, j = 1, 2, 3), \tag{4}$$

where $\partial/\partial x_j$ is a free Fox derivation with values in the ring Λ_3 (for the definitions and properties of such derivations, see [20]). The matrix belongs to a group $GL_3(\Lambda_3)$ if and only if the map φ from (3) yields an automorphism of M_3. The monomorphism

$$\beta\colon I\mathrm{Aut}M_3 \to GL_3(\Lambda_3),\ \ \varphi \mapsto J(\varphi), \tag{5}$$

is a free *Bachmuth embedding* (see [9]). The image $\beta(I\mathrm{Aut}M_3)$ coincides with the stabilizer of a vector $f = (b_1, b_2, b_3)^t$ (t is a transposition, and the matrices act on the vectors by multiplication from the left). The ring Λ_3 has a field of fractions.

We turn to the conjugation

$$\begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 \end{pmatrix}^{-1} J(\varphi) \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 \end{pmatrix} = \begin{pmatrix} A(\varphi) & & 0 \\ & & 0 \\ * & * & 1 \end{pmatrix}, \tag{6}$$

where

$$A(\varphi) = E + \begin{pmatrix} u_1^1 - b_1 b_3^{-1} u_3^1 & u_1^2 - b_1 b_3^{-1} u_3^2 \\ u_2^1 - b_2 b_3^{-1} u_3^1 & u_2^2 - b_2 b_3^{-1} u_3^2 \end{pmatrix}, \tag{7}$$

where u_i^j stands for $\partial u_i/\partial x_j$ $(i, j = 1, 2, 3)$.

It is known [19] that $\ker(\varphi \mapsto A(\varphi))$ consists of all the inner automorphisms of M_3 corresponding to elements in M_3'. Using (2) and (7), we obtain the unique expression

$$A(\varphi) = E + b_3^{-1} A_{-1} + A_0 + b_3 A_1 + b_3^2 A_2, \tag{8}$$

where $A_{-1}, A_0, A_1 \in M_2(\Lambda_2)$ and $A_2 \in M_2(\Lambda_3)$. In [19], it is also established that there exist elements (residues) $\alpha, \beta, \gamma, \delta \in \Lambda_2$ such that the matrix

$$B = \begin{pmatrix} b_1 b_2 & -b_1^2 \\ b_2^2 & -b_1 b_2 \end{pmatrix}$$

satisfies

$$A_{-1} = \alpha B,\, A_0 B = \beta B,\, B A_0 = \gamma B,\ \text{and}\ B A_1 B = \delta B. \tag{9}$$

Furthermore, $BCB = \delta B (\delta \in \Lambda_2)$ holds for any matrix $C \in M_2(\Lambda_2)$. Thus, presentation (8) signifies that the automorphism $\varphi \in I\mathrm{Aut}M_3$ can be related to the elements $\alpha, \beta, \gamma, \delta \in \Lambda_2$.

In [15, 19], explicit formulas are given for computing α, β, γ, and δ via the matrices A_{-1}, A_0, and A_1 from (8). Suppose

$$u_l^i = u_{l0}^i + b_3 u_{l1}^i + b_3^2 u_{l2}^i + b_3^3 u_{l3}^i \ \ (l, i = 1, 2, 3) \tag{10}$$

is a decomposition resulting from (2). Then

$$\begin{gathered} \alpha = -u_{30}^i b_2^{-1} = u_{30}^2 b_1^{-1}, \\ \beta = -b_1 u_{31}^1 - b_2 u_{31}^2 = u_{30}^3, \\ \gamma = u_{10}^1 - b_1 b_2^{-1} u_{20}^1 = -b_1^{-1} b_2 u_{10}^2 + u_{20}^2, \\ \delta = b_1 b_2 u_{11}^1 - b_1^2 u_{21}^1 + b_2^2 u_{11}^2 - b_1 b_2 u_{21}^2 = -b_2 u_{10}^3 + b_1 u_{20}^3. \end{gathered} \tag{11}$$

Three equalities complementary to [15] [ranking second in the last three Eqs. of (11)] follow from the basic identity for free derivations:

$$\partial c/\partial x_1 \cdot b_1 + \ldots + \partial c/\partial x_n \cdot b_n = \bar{c} - 1, \tag{12}$$

where $\bar{c}$ is the image of $c \in M_n$ in the group A_n. From (12), for instance, we have the equalities

$$u_{10}^1 b_1 + u_{10}^2 b_2 = 0,$$

$$u_{i1}^1 b_1 + u_{i1}^2 b_2 + u_{i0}^3 = 0 \ (i = 1, 2, 3).$$

In [11, 19] (see also [15]), the following statements have been proved:

(1) The map

$$\rho\colon I\mathrm{Aut} M_3 \to GL_2(\Lambda_2), \ \varphi \mapsto \rho(\varphi) = \begin{pmatrix} 1+\beta & \alpha \\ \delta & 1+\gamma \end{pmatrix}, \tag{13}$$

is a homomorphism.

(2) The matrix $C \in GL_2(\Lambda_2)$ is an image of some automorphism $\varphi \in I\mathrm{Aut} M_3$ if and only if its elements respect the following inclusion pattern:

$$\begin{pmatrix} 1+\Delta_2 & \Lambda_2 \\ \Delta_2^2 & 1+\Delta_2 \end{pmatrix}. \tag{14}$$

Denote by $GL_2(\Lambda_2, \Delta_2, \Delta_2^2)$ the subgroup of $GL_2(\Lambda_2)$ consisting of all the matrices which agree with (14), i.e., assume $GL_2(\Lambda_2, \Delta_2, \Delta_2^2) = Im\rho$.

We know from [2] that a subgroup of all tame automorphisms of $I\mathrm{Aut} M_3$ is generated by automorphisms φ_{123}, φ_{132}, φ_{231}, ψ_{21}, ψ_{31}, ψ_{12}, ψ_{32}, ψ_{13}, and ψ_{23}, where

$$\begin{aligned} &\varphi_{ijk}\colon x_k \mapsto (x_i, x_j)x_k, \ x_l \mapsto x_l \ (l \neq k), \\ &\psi_{ik}\colon x_k \mapsto (x_i, x_k)x_k = x_i x_k x_i^{-1}, \ x_l \mapsto x_l \ (l \neq k). \end{aligned} \tag{15}$$

Images of these automorphisms under ρ, computed in [11, 19] (see also [15]), are presented as

$$\begin{aligned} &\varphi_{123} \mapsto t_{12}(1), \ \varphi_{132} \mapsto t_{21}(b_1^2), \\ &\varphi_{231} \mapsto t_{21}(-b_2^2), \ \psi_{21} \mapsto d_2(a_2), \ \psi_{31} \mapsto t_{21}(b_1 b_2), \\ &\psi_{12} \mapsto d_2(a_1), \ \psi_{13} \mapsto d_1(a_1), \ \psi_{23} \mapsto d_1(a_2), \end{aligned} \tag{16}$$

where $d_i(g)$ are the diagonal matrices which differ from E only by g in the ith entry, and $t_{ij}(z)$ is the usual transvection. From (16), we see that the image of a tame automorphism subgroup in $I\mathrm{Aut} M_3$ lies in $GE_2(\Lambda_2)$, a group of elementary matrices.

The matrix $A \in GL_2(\Lambda_2, \Delta_2, \Delta_2^2)$ is called *tame* if A belongs to the image of a tame automorphism group of M_3. Note that every tame matrix is elementary.

Let A be a nontame matrix in $GL_2(\Lambda_2, \Delta_2, \Delta_2^2)$. Suppose $A = \rho(\varphi)$, where φ is an automorphism given by the map in (3), in which case, as is shown in [15], $u_3 x_3$ is a noninduced element. The converse is also true.

LEMMA 1. Let A be a tame matrix from $GL_2(\Lambda_2, \Delta_2, \Delta_2^2)$ and let $A = \rho(\varphi)$, where φ is given by the map (3). Then $u_3 x_3$ is an induced element.

Proof. We can assume A to be an identity matrix since right multiplication by a tame matrix does not change the property of being induced for elements obtained under the action of the initial matrix on the standard basis. From the expressions for α and β in (11) and the basic identity (12), we see that $u_3 = v^{b_3}$ for some element $v \in M_3'$. Then $u_3 x_3 = x_3^v$, and so $u_3 x_3$ is induced.

We proceed by treating Chein automorphisms.

It is easily verified that the image $\rho(\chi)$ of any Chein automorphism χ is a tame matrix in $GL_2(\Lambda_2, \Delta_2, \Delta_2^2)$. It will be interesting to know how Chein automorphisms are conjugated by tame automorphisms of M_3. With this in mind, we change the definition of a homomorphism ρ to fit the case where $\varphi \in I\text{Aut}M_3$ stabilizes x_2. Consider that φ is given by (3), and $u_2 = 1$. Write the Jacobi matrix

$$J(\varphi) = \begin{pmatrix} 1+u_1^1 & u_1^2 & u_1^3 \\ 0 & 1 & 0 \\ u_3^1 & u_3^2 & 1+u_3^3 \end{pmatrix}. \tag{17}$$

Invertibility of $J(\varphi)$ implies one for

$$B(\varphi) = \begin{pmatrix} 1+u_3^3 & u_3^1 \\ u_1^3 & 1+u_1^1 \end{pmatrix}, \tag{18}$$

which specializes to

$$\rho_1(\varphi) = \begin{pmatrix} 1+u_{30}^3 & u_{30}^1 \\ u_{10}^3 & 1+u_{10}^1 \end{pmatrix} \tag{19}$$

by $b_3 \mapsto 0$. The matrix $\rho_1(\varphi)$ is conjugate to $\rho(\varphi)$ by $\mathcal{D} = \text{diag}(1, -b_2^{-1})$. The $\mathcal{D}$ transforms transvections to transvections and diagonal matrices to diagonal matrices. Hence we can speak of elementary and tame or, respectively, of nonelementary and nontame matrices in $Im\rho_1$. As above, u_3x_3 in (3) is an induced element if and only if $\rho_1(\varphi)$, where φ is given by (3), is a tame matrix, and φ is, of course, not defined uniquely.

At this point we introduce a new notion which is wider than one defining an "induced element." A primitive I-element $u_3x_3 \in M_3$ is *elementary* if so is the matrix $\rho(\varphi)$, where φ is determined by (3). The definition is obviously correct: the elements u_1x_2, u_2x_2 are chosen in such a way as not to affect the first row of $\rho(\varphi)$ computed only from u_3x_3; the row (as has been shown above) in turn specifies whether or not the matrix is elementary. An arbitrary primitive element $g \in M_3$ is called *elementary* if there exists a tame automorphism τ such that g^τ is an elementary I-element. Clearly, this definition is also correct.

Proof of the proposition. Let g be an elementary primitive element of M_3 and χ a Chein automorphism. We need to show that g^χ is elementary as well. There are three cases to consider.

Case 1. An element g is an I-element w.r.t. a fixed basis $x = (x_1, x_2, x_3)$ for M_3. Permutations of x do not violate the property of being a tame (or elementary) element, and so we can assume that $g = u_3x_3$, $u_3 \in M_3'$. Add g to the basis $x^\psi = (u_1x_1, u_2x_2, u_3x_3)$, $u_i \in M_3'$ $(i = 1, 2, 3)$, for M_3. The elementary matrix $\rho_1(\psi)$ agrees with the corresponding automorphism ψ. The tame matrix $\rho_1(\chi)$, as was already mentioned, is in correspondence with χ; hence, the product $\psi\chi$ is related to the elementary matrix $\rho_1(\psi\chi) = \rho_1(\psi)\rho_1(\chi)$. The image of x_3 under $\psi\chi$ is $x_3^{\psi\chi} = g^\chi$, an elementary element, as desired.

To be specific, assume $\chi = \varphi_{231}(\alpha)$.

Case 2. An element g has the form $wx_1^kx_3^l$, $w \in M_3'$. Since g has the primitive property, it follows that $(k, l) = 1$, and the elements $u_1x_1^rx_3^s$, u_2x_2, $u_i \in M_3'$ $(i = 1, 2)$ complement g to the basis for M_3.

Define the Nielsen automorphism λ_{ij} by the map

$$\lambda_{ij} : x_i \mapsto x_jx_i, \;\; x_k \mapsto x_k, \;\; \{i, j, k\} = \{1, 2, 3\}.$$

We then use some automorphism $\tau \in \langle\lambda_{13}, \lambda_{31}\rangle$ and replace, wherever possible, x_1 by x_3 and g by g^{-1} to present g as ux_1, $u \in M_3'$. The image of g under χ coincides with the image of ux_1 under $\tau\chi$, and the latter automorphism is elementary if so is $\tau\chi\tau^{-1}(ux_1)$. Therefore, it suffices to verify that the matrix $\rho_1(\tau\chi\tau^{-1})$ is elementary.

Let φ be an automorphism given by (3), subject to the condition that $x_2^{\varphi} = x_2$. We need to clarify how automorphisms $\lambda_{13}^{\varepsilon}$, $\lambda_{31}^{\varepsilon}$ ($\varepsilon = \pm 1$) act on φ. Computations will be done for λ_{31}:

$$\begin{aligned} \lambda_{31}\varphi\lambda_{31}^{-1}: \quad x_1 &\mapsto u_1^{\lambda_{31}^{-1}} x_1, \\ x_2 &\mapsto x_2, \\ x_3 &\mapsto u_1^{\lambda_{31}^{-1}} (u_3^{\lambda_{31}^{-1}})^{a_1} x_3, \end{aligned} \tag{20}$$

$$B(\lambda_{31}\varphi\lambda_{31}^{-1}) = \begin{pmatrix} 1 + u_3^3 + a_1^{-1}u_1^3 & u_1^1 - a_1^{-1}u_1^3 + a_1 u_3^1 - u_3^3 \\ a_1^{-1}u_1^3 & 1 + u_1^1 - a_1^{-1}u_1^3 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} 1 + u_3^3 & u_3^1 \\ u_1^3 & 1 + u_1^1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{21}$$

Consequently, $B(\lambda_{31}\varphi\lambda_{31}^{-1})$ and $B(\varphi)$ are conjugated by an elementary matrix and, hence, are simultaneously elementary or nonelementary. This is likewise true for the matrices $\rho_1(\lambda_{31}\varphi\lambda_{31}^{-1})$, $\rho_1(\varphi)$. Other conjugations are to be treated similarly.

Case 3. An element g has the form $g = w x_1^k x_2^l x_3^m$, $w \in M_3'$. Here it is impossible to directly refer to the homomorphism ρ_1. Note, however, that there exists an automorphism $\pi \in \langle \lambda_{23}, \lambda_{32} \rangle$ such that $g^{\pi^{-1}} = v x_1^k x_2^t$ or $g^{\pi^{-1}} = v x_1^k x_3^t$. Replacing x_2 by x_3 if necessary, assume $g^{\pi^{-1}} = v x_1^k x_3^t$, $v \in M_3'$.

Consider an element $\pi\chi\pi^{-1}$. Since $\chi = \varphi_{231}(\alpha)$, it is easily verified that $\pi\chi\pi^{-1} = \varphi_{231}(\beta)$ for some β. Taking an automorphism $\chi_1 = \pi\chi\pi^{-1}$ in place of χ and an element $g^{\pi^{-1}}$ in place of g, we find ourselves in the conditions of Case 2. Hence, χ_1 transforms $g^{\pi^{-1}}$ to an elementary element, but $\chi_1(g^{\pi^{-1}}) = (g^{\pi})^{\chi}$, and so g^{χ} is elementary by definition.

By substitutions in the basis x, other cases can be reduced to the three considered above. The proposition is proved.

Proof of the theorem. Suppose that any two bases for M_3 are swap equivalent. It is easy to see, then, that one is changed to another by a sequence of transformations each of which is either an elementary Nielsen or an elementary I-swap transformation, not changing a set modulo M_3'. Standard considerations (as are those used in proving that the group $\mathrm{Aut}F_n$ is generated by Nielsen automorphisms; see [2]) apply to conclude that $\mathrm{Aut}M_3$ is generated by Nielsen automorphisms and by one-row IA-automorphisms. It is clear that the latter factor into the products of tame IA-automorphisms and Chein automorphisms. Hence, $\mathrm{Aut}M_3$ is generated by tame and Chein automorphisms of which each transforms an elementary primitive element to an element of the same form. Therefore, a basis that has a nonelementary element (such exist by [15]) is not swap equivalent to the standard basis. Contradiction. The theorem is proved.

A proof of the corollary is contained in the proof of the theorem.

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