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A Tomographic Approach to Wigner's function

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We propose a new derivation of Wigner's function based on the property of positivity of its integrals along straight lines in phase space. Identifying the values of these marginalizations with densities pertaining to invariant observables, we are able to reconstruct Wigner's pseudo-distribution from its slices.

1. INTRODUCTION

The possibility of expressing quantum mechanical expectation values as averages over phase space distribution functions is well known. It allows the transformation of the standard quantum formula

$$\langle A \rangle = \int \psi^*(x)\, A_{\mathrm{op}}(x, -i\hbar\partial/\partial x)\, \psi(x)\, dx \tag{1}$$

into the statistical expression

$$\langle A \rangle = \int a(x, p)\, f(x, p)\, dx\, dp \tag{2}$$

where $a(x, p)$ is a classical function corresponding to the operator A_{op} and f is the "distribution function." The first introduction of such a representation was made by Wigner,[1] who used for f the expression

$$f_{\mathrm{W}}(x, p) = \int du\, e^{-2i\pi up}\, \psi(x + \pi\hbar u)\, \psi^*(x - \pi\hbar u) \tag{3}$$

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Later, alternative expressions were proposed but (3) remained the most attractive, especially when the goal was to display the departure from classical mechanics.

In an attempt to select the best ways to the phase space representation, axiomatic approaches have been proposed.[2-5] Basically they all make use of the following three main constraints plus some extra conditions:

(i) f must be a real Hermitian form of the wave function;

(ii) integration with respect to each variable can be interpreted as a marginalization, i.e.,

$$\int f(x, p)\, dx = |\psi(p)|^2, \qquad \int f(x, p)\, dp = |\psi(x)|^2$$

(iii) the correspondence $\psi \to f$ must be invariant under Galilean transformations at fixed t and by space inversions.

A typical example is given by the determination of (3) from (i)–(iii) and the added requirement that for free particles the time dependence of f becomes the nonrelativistic classical one.[2] A general feature of all auxiliary conditions is that they do not directly contribute to the phase space interpretation of quantum probabilities. The aim of this paper is to give an alternative construction which avoids this criticism and leads un equivocally to Wigner's function.

To present our work, we have to explain condition (iii). The group G of Galilean transformations (a, v) at a given instant t of time performs translations in phase space according to

$$(a, v)\colon (x, p) \to (x - vt - a, p - mv) \tag{4}$$

It is supposed to act on wave functions as usual (cf. Appendix) and to transform f like a classical distribution. Galilean covariance is then expressed by the commutativity of the following diagram

$$\begin{array}{ccc} \psi(x, t) & \longrightarrow & e^{-i(m/\hbar)vx + i\varphi(v,t)}\, \psi(x + vt + a, t) \qquad (5) \\ \big\downarrow & & \big\downarrow \\ f(x, p) & \longrightarrow & f(x + vt + a, p + mv) \qquad (6) \end{array}$$

where $\varphi(v, t)$ is a phase we need not know explicitly since we always write Hermitian forms of ψ at time t. The principle of the method rests upon the fact that true densities exist for certain classes of observables, namely those invariant under a subgroup of G. Indeed, such observables can all be simultaneously diagonalized, thus leading to an expression of the expectation values in the form

$$\langle A \rangle = \int d\beta \, A(\beta) \, \rho(\beta) \tag{7}$$

where $\rho(\beta)$ appears as a probability law associated with the diagonal part of the density matrix. On the other hand, as a consequence of the transformation law (6) of f, these invariant observables are represented in (2) by functions in phase space which are constant on the subgroup orbits. This implies that (2) becomes a one-dimensional integral involving a marginalization of the phase space distribution. At this stage, a natural requirement is that the marginalization of f be identified with the quantum mechanical density $\rho(\beta)$. This is nothing but the generalization of constraint (ii) to all G subgroups. Then, determining f appears as a tomographic reconstruction problem which can be solved using a Radon transform inverse.

In Sec. 2, the basis diagonalizing observables invariant under an arbitrary subgroup of G is obtained together with its associated density. In Sec. 3, the general marginal condition is shown to determine f by its integrals along any direction in phase space. Finally, in Sec. 4, the inversion of the Radon transform yields Wigner's function.

2. PROBABILITY DENSITY FOR CLASSES OF INVARIANT OBSERVABLES

In the following, we are exclusively using the coordinate representation to express quantum states and quantum observables. In particular, expectation values are written

$$\langle A \rangle = \int dx_1 \, dx_2 A(x_1, x_2) \, \psi^*(x_1) \, \psi(x_2) \tag{8}$$

where the kernel $A(x_1, x_2)$ is a distribution satisfying

$$A(x_1, x_2) = A^*(x_2, x_1)$$

In order to specify some particular classes of observables, we need to consider subgroups of the Galilean group G. The subgroups will be denoted G_α, where the label α fixes the ratio of the parameters a and v appearing in (4):

$$a = \alpha v \tag{9}$$

The action of such a subgroup on any quantum state results from the action (5) of the whole group and reads

$$(\alpha v, v) \in G_\alpha : \psi(x, t) \to e^{-i(m/\hbar)vx + i\varphi(v,t)}\, \psi(x + v(t+\alpha), t) \tag{10}$$

Observables whose expectation values are invariant under this action will be called G_α-invariant. They are characterized by kernels satisfying

$$A(x_1, x_2) = e^{-i(m/\hbar)v(x_2 - x_1)}\, A(x_1 - v(t+\alpha), x_2 - v(t+\alpha)) \tag{11}$$

To find the basis Z^α of $L^2(R)$ diagonalizing A, we notice that, due to the spectral decomposition

$$A(x_1, x_2) = \int Z^{\alpha *}(x_2, \beta)\, Z^\alpha(x_1, \beta)\, A(\beta)\, d\beta \tag{12}$$

condition (11) leads to the equation

$$Z^\alpha(x, \beta) = e^{i\theta(\alpha,\beta,v) + i(m/\hbar)vx}\, Z^\alpha(x - v(t+\alpha), \beta) \tag{13}$$

where θ is a real function.

To solve this equation, we perform v differentiations on both sides and, setting $v = 0$, we obtain

$$[i\lambda(\beta, \alpha) - i(m/\hbar)\, x]\, Z^\alpha(x, \beta) + (t+\alpha)\frac{\partial Z^\alpha}{\partial x}(x, \beta) = 0 \tag{14}$$

where

$$\lambda(\beta, \alpha) \equiv -(\partial\theta/\partial v)(\alpha, \beta, v)|_{v=0}$$

The solution is

$$Z^\alpha(x, \beta) = K(\beta, \alpha)\, e^{i((m/2\hbar)x^2 - \lambda(\beta,\alpha)\, x)/(t+\alpha)} \tag{15}$$

The modulus of the integration constant K is determined by the normalization condition

$$\int Z^\alpha(x, \beta)\, Z^{\alpha *}(x, \beta')\, dx = \delta(\beta - \beta')$$

The result is

$$|K(\alpha, \beta)| = \left| \frac{1}{2\pi(t+\alpha)} \frac{\partial\lambda(\beta, \alpha)}{\partial\beta} \right|^{1/2} \tag{16}$$

This expression guarantees the completeness relation

$$\int Z^{\alpha}(x, \beta)\, Z^{\alpha *}(x', \beta)\, d\beta = \delta(x - x')$$

What we have obtained is an improper basis of $L^2(R)$ allowing us to write the expectation value of any G_α-invariant observable as

$$\langle A \rangle = \int d\beta A(\beta)\, \rho_\alpha(\beta) \tag{17}$$

where

$$\rho_\alpha(\beta) = \int dx_1\, dx_2\, Z^{\alpha}(x_2, \beta)\, Z^{\alpha *}(x_1, \beta)\, \psi(x_1)\, \psi^*(x_2) \tag{18}$$

The function $\rho_\alpha(\beta)$ is everywhere $\geqslant 0$ and thus can be viewed as a probability density associated with the class of G_α-invariant observables. The arbitrary function λ corresponds to the freedom in the choice of the variable β. It will be fixed later on.

3. TOMOGRAPHIC DEFINITION OF THE PSEUDO-DISTRIBUTION FUNCTION

We will now express the density associated with G_α-invariant observables in terms of the pseudo-distribution f. Using relation (2) and performing a G_α transformation on f as defined by (6) and (9), we deduce that the function $a(x, p)$ characterizing such observables must satisfy the equation

$$a(x, p) = a(x - v(t + \alpha), p - mv) \tag{19}$$

Hence a must be constant on the G_α orbits in phase space given by

$$x - (t + \alpha)\, p/m = \gamma \tag{20}$$

where γ is a real number. As a consequence, the phase space representation of G_α-invariant observables can be written as

$$a(x, p) = \int d\gamma\, \tilde{a}(\gamma)\, \delta(\gamma + (t + \alpha)\, p/m - x) \tag{21}$$

and their expectation values reduce to

$$\langle A \rangle = \int d\gamma\, \tilde{a}(\gamma)\, I(\gamma, \alpha) \tag{22}$$

where

$$I(\gamma, \alpha) \equiv \int dx\, dp\, f(x, p)\, \delta(\gamma + (t+\alpha)(p/m) - x) \tag{23}$$

Thus, for each value of α, the integral of f along parallel lines in phase space plays the role of a one-dimensional density associated with the class of G_α-invariant observables. The identification of the latter with the density ρ_α obtained in the preceding section will give us what can be seen as a general marginal condition. In practice, we have to match the parameters appearing in Eqs. (18) and (23). In a first step, we do it through the arbitrary function λ introduced in solving (14) and obtain

$$I(\beta, \alpha) = \rho_\alpha(\lambda(\beta, \alpha)) \tag{24}$$

where

$$\rho_\alpha(\lambda(\beta, \alpha)) = \int dx_1\, dx_2\, \frac{|\partial\lambda(\beta, \alpha)/\partial\beta|}{2\pi\, |t+\alpha|}\, \psi(x_1)\, \psi^*(x_2) \cdot e^{i[(m/2\hbar)(x_2^2 - x_1^2) - \lambda(\beta, \alpha)(x_2 - x_1)]/(t+\alpha)}$$

The second step will consist in a determination of the function λ by requiring the stability of (24) under Galilean transformations and space reflections. Writing (24) for the transformed (5)–(6) of f and ψ, we get

$$I(\beta + a - \alpha v, \alpha) = \rho'_\alpha(\lambda(\beta, \alpha))$$

where

$$\rho'_\alpha(\lambda(\beta, \alpha)) = \int dx_1\, dx_2\, \frac{|\partial\lambda(\beta, \alpha)/\partial\beta|}{2\pi\, |t+\alpha|}\, \psi(x_1)\, \psi^*(x_2)\, e^{-i(m/\hbar)v(x_1 - x_2)} \cdot e^{i[(m/2\hbar)((x_2 - vt - a)^2 - (x_1 - vt - a)^2) - \lambda(\beta, \alpha)(x_2 - x_1)]/(t+\alpha)}$$

In order to obtain the same functional relation as (24), we must have

$$\rho'_\alpha(\lambda(\beta, \alpha)) = \rho_\alpha(\lambda(\beta + a - \alpha v, \alpha))$$

This relation will hold for any ψ if and only if

$$\lambda(\beta + a - \alpha v, \alpha) = \lambda(\beta, \alpha) + (m/\hbar)(a - \alpha v)$$

The general solution of this equation is

$$\lambda(\beta, \alpha) = (m/\hbar)\, \beta + h(\alpha) \tag{25}$$

with h an arbitrary function.

In the same way, requiring the commutativity of the diagram related to space inversions

$$\begin{array}{ccc} \psi(x) & \longrightarrow & \psi(-x) \\ \downarrow & & \downarrow \\ f(x, p) & \longrightarrow & f(-x, -p) \end{array}$$

we obtain

$$h(\alpha) \equiv 0 \tag{26}$$

Combining (23), (24), and (25)–(26), we deduce the generalized marginal condition in the form

$$\int dx\, dp\, f(x, p)\, \delta(\beta + (t+\alpha)\, p/m - x)$$

$$= \int dx_1\, dx_2\, \frac{m}{2\pi\hbar(t+\alpha)}\, e^{im[\frac{1}{2}(x_2^2 - x_1^2) - \beta(x_2 - x_1)]/(t+\alpha)\hbar}\, \psi(x_1)\, \psi^*(x_2)$$

This is nothing but a tomographic description of f.

4. INVERSION OF THE RADON TRANSFORM. WIGNER'S FUNCTION

According to (27), f is determined by its Radon transform. The easiest inversion technique makes use of a Fourier integral.[6]

Indeed, multiplying both sides of (27) by $e^{2i\pi\beta\hat{\beta}}$ and integrating in β, we get

$$\int dx\, dp\, f(x, p)\, e^{2i\pi(x - (t+\alpha)p/m)\hat{\beta}}$$

$$= \int dx_1\, dx_2\, \delta\Big(x_1 - x_2 + 2\frac{\pi\hbar}{m}(t+\alpha)\, \hat{\beta}\Big)\, e^{i(m/2\hbar)(t+\alpha)(x_2^2 - x_1^2)}\, \psi(x_1)\, \psi^*(x_2)$$

The left-hand side is just the double Fourier transform of f, which can be readily inverted, yielding

$$f(x, p) = \int du\, e^{-2i\pi up}\, \psi(x + \pi\hbar u)\, \psi^*(x - \pi\hbar u)$$

This is just the expression (3) of the original Wigner function.

5. CONCLUDING REMARKS

We have proposed a construction of Wigner's function resting upon one probabilistic constraint beside Galilei invariance. Its main interest is to emphasize the prominent position of the Wigner function among pseudo-probability distributions. Though nonpositive, it is the only one to have the correct marginalizations whatever the direction of integration in the $x - p$ plane. This permits us to consider it as the closest to a true distribution function.

For simplicity, the above developments have been restricted to one spatial dimension. Generalization to higher dimensions involves only technical difficulties.

APPENDIX. TRANSFORMATION OF THE WAVE FUNCTION BY GALILEI GROUP

Extensive discussions of the Galilei invariance may be found in the literature.[7,8] However, we are only concerned with the changes of Galilean reference frames at a given instant, and a straightforward derivation is possible.

In quantum mechanics, we work with a density and a current density defined, respectively, by

$$\rho(x, t) = |\psi(x, t)|^2 \tag{A1}$$

$$J(x, t) = -(\hbar/2im)[\psi(x, t)\, \nabla\psi^*(x, t) - \psi^*(x, t)\, \nabla\psi(x, t)] \tag{A2}$$

These quantities are related by the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} J = 0 \tag{A3}$$

Such an equation must be form-invariant under the Galilean transformation given by

$$\begin{aligned} x' &= x - vt - a \\ t' &= t \end{aligned} \tag{A4}$$

This implies that ρ and J must transform according to

$$\rho \to \rho'(x, t) = \rho(x + vt, t) \tag{A5}$$

$$J \to J'(x, t) = J(x + vt, t) - v\rho(x + vt, t) \tag{A6}$$

The first relation shows that ψ must transform as

$$\psi \to \psi'(x, t) = e^{i\phi(x,t,a,v)}\, \psi(x + vt, t) \tag{A7}$$

where ϕ is an arbitrary real function.

Combining Eqs. (6) and (7), we obtain the x dependence of this phase:

$$\phi(x, t, v) = -(m/\hbar)\, vx + \varphi(v, t) \tag{A8}$$

where φ remains arbitrary. This result is sufficient for studying the behavior of Hermitian forms of ψ under Galilei transformations at t fixed.

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