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An Analytical Solution of the Stochastic Navier–Stokes System

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This paper, using the author's decomposition method and recent generalizations, presents algorithms for an analytic solution of the stochastic Navier–Stokes system without linearization, perturbation, discretization, or restrictive assumptions on the nature of stochasticity. The pressure, forces, velocities, and initial/boundary conditions can be stochastic processes and are not limited to white noise. Solutions obtained are physically realistic because of the avoidance of assumptions made purely for mathematical tractability by usual methods. Certain extensions and further generalizations of the decomposition method have provided the basis for the solution.

1. INTRODUCTION

Turbulence is encountered everywhere in the flow of fluids, and the methods of dealing with it realistically are still inadequate due to commonly used restrictive assumptions and formulations tailored to convenient mathematics. A correct, convenient, and physically realistic theory requires the analytic solution of nonlinear stochastic partial differential equations—a stochastic Navier–Stokes system under general conditions, i.e., without linearizations or use of physically unrealistic processes such as white noise, perturbation, closure approximations, or even discretized numerical approximation methods leading to massive printouts. Such a solution of Navier–Stokes has been considered impossible and, despite thousands of papers on turbulence and some excellent books,[1-3] has not yet been realized. (Formal solutions in terms of function space integrals and generalized measures which do not lead to actual solutions are not considered to fit our criteria.)

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Fortunately, recent developments[4-11] on the mathematics make a totally new approach possible. We outline it here assuming as known all the work in the referenced papers, particularly Ref. 5.

2. DISCUSSION

As usually stated, the model is an incompressible fluid of kinematic viscosity ν, and constant density ρ. These conditions will, of course, have to be modified according to the prevailing circumstances, and the methodology is not limited to these cases. The basic equations are given as

$$\partial \mathbf{u}/\partial t) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\nabla^2\mathbf{u} + (1/\rho)\,\nabla p = \mathbf{f} \qquad \Omega\times(0, T)$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0, T), \qquad \mathbf{u} = 0 \quad \text{in } d\Omega\times(0, T)$$

where $\mathbf{u}$ is a vector with components u, v, w.[2]

We assume the velocity $\mathbf{u}(x, y, z, t, \omega)$, the pressure $p(x, y, z, t, \omega)$, and the external force $\mathbf{f}$ are stochastic processes. In terms of velocity components u, v, w, we write

$$\begin{aligned}
&(\partial u/\partial t) + u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z)\\
&\qquad - \nu((\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) + (\partial^2 u/\partial z^2)) + (1/\rho)(\partial p/\partial x) = F_x\\
&(\partial v/\partial t) + u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z)\\
&\qquad - \nu((\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) + (\partial^2 v/\partial z^2)) + (1/\rho)(\partial p/\partial y) = F_y\\
&(\partial w/\partial t) + u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z)\\
&\qquad - \nu((\partial^2 w/\partial x^2) + (\partial^2 w/\partial y^2) + (\partial^2 w/\partial z^2)) + (1/\rho)(\partial p/\partial z) = F_z
\end{aligned} \tag{1}$$

We define an initial-boundary problem by specifying initial conditions for u, v, w and, for $t \geqslant 0$, specifying u, v, w on the boundary $(x, y, z) \in \Gamma$.

3. THE ERGODICITY QUESTION

The problem as generally stated[1-3] utilizes a white noise input, i.e., the derivative of a Wiener process, and an Itô approach. These restrictions will not apply in the present treatment.

[2] We have considered $y' = (y-1)^2 + a$ with $a > 0$. If $a < 0$, the solution varies between two horizontal asymptotes with inflection point at $(0, 1)$. The asymptotes coincide if $a = 0$. The solution $y = 1$ is a singular solution not derivable from the general solution.

To describe general stochastic processes completely, we would need nth order probability distributions as $n \to \infty$; however, it is physically more reasonable to determine only first- and second-order statistics. Although we can go further, it makes no sense to ask for more knowledge for the solution process of our system than we can possibly know for the input processes.

The assumption of ergodic behavior is common and basic. In practice, we do not see an entire ensemble; we see results of an observation over a period T. Thus, we observe moments with respect to time and assume under the ergodic hypothesis that the means with respect to time over a period T converge as $T \to \infty$ to the corresponding ensemble means.

Let us represent the point x, y, z, t by ξ; then a correlation R would be $R(\xi_1, \xi_2)$, which is dependent on two points of space-time. The correlation depends on two points of space-time and is the mean of the product of values at two different instants but the same point in space. We call this the correlation in time. If we consider a single instant and two space points, we have a correlation in space. If the random field is stationary, this implies $R(x_1, y_1, z_1, t_1; x_2, y_2, z_2, t_2) = R(x_1, y_1, z_1; x_2, y_2, z_2, t_2 - t_1)$ or $R(t_2 - t_1) = R(\tau)$. It implies an invariance of probability distributions to second order with a shift in the time origin, or a simultaneous shift of two-time points. Thus, the first and second moments are averages over time. Similarly, we can have moments obtained by space-averaging. Thus, a random field satisfying an invariance of probability distributions with respect to position, or a simultaneous shift of two space points to obtain a correlation, is stationary with respect to space, which is more often called statistical homogeneity rather than stationarity. For space-averaging to yield the same result as ensemble-averaging, a random field would have to be statistically homogeneous (or stationary with respect to position). For time-averaging or space-averaging to equate to ensemble averaging, we must have the respective stationarity over some finite region. However, this is a necessary but not sufficient condition. If we have stationarity, and $R(\tau) \to 0$ as $\tau \to \infty$, we can expect ergodicity—this is reasonable since for physical processes as $t_2 - t$ becomes large, the correlation should vanish. The time average u approaches the ensemble average $\langle u \rangle$ as $T \to \infty$ and $\lim_{T \to \infty} \langle (u - \langle u \rangle)^2 \rangle = 0$. Ergodicity requires stationarity. For turbulent flows, ergodicity is an unjustified hypothesis under nonsteady flow conditions. The onset and decay of turbulence on a region cannot be stationary, hence, we cannot have ergodicity. Conditions governing the process would need to be time-independent, as would the external conditions.

Emphasizing that by a "stochastic differential equation" we mean not a deterministic system with a stochastic input (usually white noise) but one which has stochastic processes in the system parameters or coefficients of the equation, and perhaps also in the input or given conditions, the solution process cannot be stationary even if all parameter and input processes and conditions are stationary (an exception can occur under physically unrealistic circumstances that require correlations between input and parameters). We are solving Navier–Stokes as a stochastic system, hence ergodicity cannot exist in the solution except under special conditions. Nonstationarity is the general case.

Under special conditions where the mechanisms involved have become steady or over sufficiently short time intervals, we may have a quasi-stationary condition and use the property of ergodicity.

4. APPROACH USING DECOMPOSITION METHOD

Let us rewrite the system (1) in the standard (Adomian) decomposition form[5]

$$Lu + N_1(u, v, w) = g_1$$
$$Lv + N_2(u, v, w) = g_2$$
$$Lw + N_3(u, v, w) = g_3$$

We do have some choice on the definition of the nonlinear terms. First, we consider

$$L = (\partial/\partial t) - \nu(\partial^2/\partial x^2) - \nu(\partial^2/\partial y^2) - \nu(\partial^2/\partial z^2) = L_t + L_x + L_y + L_z$$
$$N_1 = u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z)$$
$$N_2 = v(\partial v/\partial y) + u(\partial v/\partial x) + w(\partial v/\partial z)$$
$$N_3 = w(\partial w/\partial z) + u(\partial w/\partial x) + v(\partial w/\partial y)$$
$$g_1 = F_x - (1/\rho)(\partial p/\partial x)$$
$$g_2 = F_y - (1/\rho)(\partial p/\partial y)$$
$$g_3 = F_z - (1/\rho)(\partial p/\partial z)$$

To complete the specification of g_1, g_2, g_3 we must know the pressure function. Let us assume first that the pressure depends upon depth only. It will, of course, become a function of x, y, z, t as any disturbance occurs. However, we must determine the functional dependence of pressure on the velocities u, v, w so that the g_1, g_2, g_3 are calculable.

We can rewrite the Navier–Stokes equations (for incompressible flow) also in the form

$$\frac{\partial}{\partial t} u_i(\mathbf{x}, t) + u_j(\mathbf{x}, t) \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t) - \nu \nabla^2 u_i(\mathbf{x}, t) = F_i - \frac{\partial}{\partial x_i} p(\mathbf{x}, t)$$

$$\frac{\partial}{\partial x_j} u_j(\mathbf{x}, t)$$

recognizing that in a more general situation p becomes a function of x, y, z, t. (We will also consider two points $\mathbf{x}_1, t_1$ and $\mathbf{x}_2, t_2$ when we seek the two-point correlations). (We can absorb the $1/\rho$ into p by writing $(1/\rho)(\partial/\partial x_i)\, p(\mathbf{x}, t)$ for the last term.)

In this form $i = 1$ yields the equation for u, $i = 2$ yields the equation for v, and $i = 3$ yields the equation for w. The j runs from 1 to 3 and the repeated index implies summation. The x_j corresponds to x, y, z respectively as j goes from 1 to 3. The second term is the same as $u(\partial/\partial x)\, u + v\, (\partial/\partial y)\, u + w\, (\partial/\partial z)\, u$. Our intention is to solve for $u_i(\mathbf{x}, t)$, or $u_i(x, y, z, t)$, by the decomposition method. This will result in a stochastic series from which statistics will be obtained. Usual methods are either to multiply the Navier–Stokes equation by various terms and ensemble-average to get moments, or to write a characteristic functional for the probability distribution $P(\mathbf{u}, p, t)$, expand as a series of joint moments, and ensemble-average.

This is unsatisfactory since we have limited statistical knowledge in the real world. We ask only for $\langle u(\mathbf{x}, t)\rangle$ and two-point correlations found from $u(\mathbf{x}_1, t_1)$ and $u(\mathbf{x}_2, t_2)$. Decomposition avoids the closure problem which, in essence, is a perturbative treatment which eliminates the possibility of understanding turbulence which is due to strong nonlinear stochastic effects.

If we find the divergence of each term in the Navier–Stokes equation, the ∇p becomes $\nabla^2 p$ [or $(1/\rho)\, \nabla^2 p$ depending on the definition used for p]. The first and third terms vanish from the divergence condition.[3] The second term gives us $\nabla(\bar{u} \cdot \nabla) \cdot \mathbf{u}$. Thus

$$\nabla^2 p = \nabla \cdot \mathbf{F} - \nabla(\mathbf{u} \cdot \nabla) \cdot \mathbf{u}$$

The $\mathbf{i}$ component of $\nabla(\mathbf{u} \cdot \nabla)$ is $(\partial/\partial x)[u\, \partial/\partial x + v\, \partial/\partial y + w\, \partial/\partial z]$. The $\mathbf{j}$ component is $(\partial/\partial y)[u\, \partial/\partial x + v\, \partial/\partial y + w\, \partial/\partial z]$. The $\mathbf{k}$ component is $(\partial/\partial z)[u\, \partial/\partial x + v\, \partial/\partial y + w\, \partial/\partial z]$. Taking the scalar product with $\mathbf{u}$, we have

[3] To see that the third term is zero, note that $\nabla \times (\nabla \times \bar{u}) = \nabla(\nabla \cdot \bar{u}) - \nabla^2 \bar{u}$. Since the first term on the right vanishes, the $\mathrm{div}(\nabla^2 \bar{u})$ is the div curl of a vector and therefore vanishes.

$$\frac{\partial}{\partial x}\left[u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right]u+\frac{\partial}{\partial y}\left[u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right]v$$
$$+\frac{\partial}{\partial z}\left[u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right]w$$

After differentiations, we have 18 terms. Nine of these terms or

$$u\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}\right]+v\left[\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\,\partial z}\right]$$
$$+w\left[\frac{\partial^2 u}{\partial z\,\partial x}+\frac{\partial^2 v}{\partial z\,\partial y}+\frac{\partial^2 w}{\partial z^2}\right]$$

are identically $(\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u})$ and vanish since $\nabla\cdot\mathbf{u}=0$. The final result for the second term of the right side of the equation for $\nabla^2 p$ is

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$
$$+2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}+2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}$$

Thus,

$$[L_x+L_y+L_z]\,p=\nabla\cdot\mathbf{F}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$
$$+2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}+2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}$$

Symbolizing the right side by f, solving for $L_x p$, and inverting the operator L_x, we have

$$p=A+Bx+L_x^{-1}f-L_x^{-1}L_y p-L_x^{-1}L_z p$$

Writing $p=\sum_{n=0}^{\infty}p_n$ and identifying

$$p_0=A+Bx+L_x^{-1}f$$

we have for $n\geqslant 0$

$$p_{n+1}=-L_x^{-1}L_y p_n-L_x^{-1}L_z p_n$$

and we can write an n-term approximation for p by

$$\varphi_n=\sum_{i=0}^{n-1}p_n$$

which converges to $\sum_{n=0}^{\infty} p_n$ or p. (Similar equations can be written for $L_y p$ and $L_z p$). To get p_0 we need $L_x^{-1} f$ or

$$L_x^{-1} \frac{\partial}{\partial x}\left[u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right]u$$

We originally assumed a pressure varying only with depth, i.e., as an initial pressure or A in our equation for p_0. The coefficient B is zero since the disturbance vanishes as $x \to \infty$. We use this $p(z)$ to find u, v, w as outlined in this paper. The resulting velocities are used in our equation for p, as a function of velocities, to yield an improved $p = p_0 + p_1$ (which recalculates p_0 because of the change in f). This is used to improve results for velocities u, v, w. These calculations can proceed until we have sufficiently accurate results for u, v, w, p.

The effect of the force represented by the ∇p becomes more and more important as velocities increase. Dependent on the force $\mathbf{F}$, the boundary conditions and the velocities involved, we get $p(x, y, z, t)$ and $\bar{u}(x, y, z, t)$ and a turbulent situation.

Assume a coordinate system with z downward and centered at the surface. Let the pressure p for small z be $p(z) = \rho g z + p_0$, where p_0 is the atmospheric pressure and ρ is assumed constant. Hence $dp/dz = \rho g$ and $dp/dx = dp/dy = 0$. (Effects of temperature and salinity on ρ are ignored here). Thus

$$L_t u + L_x u + L_y u + L_z u = g_1 - N_1$$
$$L_t v + L_x v + L_y v + L_z v = g_2 - N_2$$
$$L_t w + L_x w + L_y w + L_z w = g_3 - N_3$$

from which

$$L_t u = g_1 - N_1 - L_x u - L_y u - L_z u$$
$$L_x u = g_1 - N_1 - L_t u - L_y u - L_z u$$
$$L_y u = g_1 - N_1 - L_x u - L_t u - L_z u$$
$$L_z u = g_1 - N_1 - L_x u - L_y u - L_t u$$

$$L_t v = g_2 - N_2 - L_x v - L_y v - L_z v$$
$$L_x v = g_2 - N_2 - L_t v - L_y v - L_z v$$
$$L_y v = g_2 - N_2 - L_x v - L_t v - L_z v$$
$$L_z v = g_2 - N_2 - L_x v - L_y v - L_t v$$

$$L_t w = g_3 - N_3 - L_x w - L_y w - L_z w$$
$$L_x w = g_3 - N_3 - L_t w - L_y w - L_z w$$
$$L_y w = g_3 - N_3 - L_x w - L_t w - L_z w$$
$$L_z w = g_3 - N_3 - L_x w - L_y w - L_t w$$

It has been shown by Adomian and Rach[8] that, when the boundary conditions are general (when conditions on any one variable depend upon all the others), to solve for u, v, w we can use any of the four operator equations depending on the given conditions and integrations required. If we know initial conditions, the equations involving the operator L_t on the left side will be simplest since only a single integration will be required.[6] We can also solve the system as a boundary value problem using any of the equations involving L_x, L_y, or L_z on the left side using the recent extensions of decomposition by Adomian and Rach.[9,12] Hence, using the first equation of each set above and operating with L_t^{-1}, we have

$$u = u(0) + L_t^{-1} g_1 - L_t^{-1} N_1 - L_t^{-1}(L_x + L_y + L_z)\, u$$
$$v = v(0) + L_t^{-1} g_2 - L_t^{-1} N_2 - L_t^{-1}(L_x + L_y + L_z)\, v$$
$$w = w(0) + L_t^{-1} g_3 - L_t^{-1} N_3 - L_t^{-1}(L_x + L_y + L_z)\, w$$

Now write, according to the Adomian decomposition,[6,7]

$$u = \sum_{n=0}^{\infty} u_n, \qquad v = \sum_{n=0}^{\infty} v_n, \qquad w = \sum_{n=0}^{\infty} w_n$$

also, write N_1, N_2, N_3 in terms of the Adomian (A_n) polynomials,[5-7] and finally identify

$$u_0 = u(0) + L_t^{-1} g_1$$
$$v_0 = v(0) + L_t^{-1} g_2$$
$$w_0 = w(0) + L_t^{-1} g_3$$

The remaining components of u, v, w for $n \geqslant 0$ can now be determined:

$$u_{n+1} = -L_t^{-1} A_n\{N_1\} - L_t^{-1}(L_x + L_y + L_z)\, u_n$$
$$v_{n+1} = -L_t^{-1} A_n\{N_2\} - L_t^{-1}(L_x + L_y + L_z)\, v_n$$
$$w_{n+1} = -L_t^{-1} A_n\{N_3\} - L_t^{-1}(L_x + L_y + L_z)\, w_n$$

where the notation $A_n\{\cdot\}$ refers to the A_n for the quantity in brackets. We now have a completely calculable system which, if we ignore stochasticity for the moment, and write $u=\sum_{n=0}^{\infty} u_n$, $v=\sum_{n=0}^{\infty} v_n$, $w=\sum_{n=0}^{\infty} w_n$ and approximate by n term approximations $\varphi_n^{(u)}=\sum_{i=0}^{n-1} u_i$, $\varphi_n^{(v)}=\sum_{i=0}^{n-1} v_i$, $\varphi_n^{(w)}=\sum_{i=0}^{n-1} w_i$, we have found u, v, w to n-term approximations.

In the stochastic case, the expressions for u, v, w are stochastic series, i.e., series containing stochastic processes which we must solve for first- and second-order statistics where the velocity components are replaced by a sum of a deterministic component velocity and a stochastic component. As pointed out previously,[13] the equation obtained by replacing velocities with stochastic processes may not be correct although we will do this in this paper. An example of this is the problem of wave propagation in a random medium where it is incorrect to simply replace the velocity in the d'Alembertian operator with a stochastic quantity, i.e., a stochastic model must be derived which has the deterministic model as a limit rather than using the deterministic model to obtain a stochastic model. Thus, we must obtain

$$\langle u\rangle = \langle u_0\rangle + \langle u_1\rangle + \langle u_2\rangle + \cdots$$
$$\langle v\rangle = \langle v_0\rangle + \langle v_1\rangle + \langle v_2\rangle + \cdots$$
$$\langle w\rangle = \langle w_0\rangle + \langle w_1\rangle + \langle w_2\rangle + \cdots$$

remembering that g_1, g_2, g_3 are stochastic since F and p are stochastic and the A_n are stochastic.

The two-point correlation for each velocity component u, v, w is obtained by averaging the product of series for velocity components at two space-time points. If we consider, for example, fixed space position and that the time scales are such that stationarity can be assumed, the ergodic hypothesis may hold so that ensemble averages can be replaced with time averages of observations.

Since the nonlinear terms contain both functions of a single variable, such as $f(u)$, and also functions of two variables, such as $f(u, v)$, our previously given algorithm for A_n for $f(u)$ needs generalization here to A_n for $f(u, v)$. For convenience, we repeat

$$A_n\{f(u)\} = \sum_{\nu=1}^{n} c(\nu, n)\, f^{(\nu)}(u_0)$$

where the second index in the coefficient is the same as the order of the polynomial being calculated, and the first index progresses from 1 to n along with the order of the derivative. We see that the sum of the subscripts in each term of the A_n is equal to n. The $c(\nu, n)$ are products, or sums of products, of ν components of u whose subscripts sum to n and divided by the factorial of the number of repeated subscripts. Thus $c(1, 3)$ can only be u_3. $c(2, 3)$ is $u_1 u_2$. $c(3, 3) = (1/3!)\, u_1^3$. Now an analytic $f(u)$ can be expressed by $f(u) = \sum_{n=0}^{\infty} A_n$.

The result is

$$
\begin{aligned}
A_0 &= f(u_0) \\
A_1 &= u_1 (d/du_0)\, f(u_0) \\
A_2 &= u_2 (d/du_0)\, f(u_0) + (u_1^2/2!)(d^2/du_0^2)\, f(u_0) \\
A_3 &= u_3 (d/du_0)\, f(u_0) + u_1 u_2 (d^2/du_0^2\, f(u_0) \\
&\quad + (u_1^3/3!)(d^3/du_0^3)\, f(u_0) \\
&\vdots
\end{aligned}
$$

Calculation of $A_n\{f(u, v)\}$ and $A_n\{f(u, v, w)\}$ is discussed completely in Ref. 12. We can use the $A_n\{f(u, v)\}$ since the N_1, N_2, N_3 can be considered term by term. Now the coefficients will involve three quantities, the derivative is $f^{(u,v)}$, and the summation is over u, v. The result is

$$
\begin{aligned}
A_0 &= f^{(0,0)} \\
A_1 &= u_1 f^{(1,0)} + v_1 f^{(0,1)} \\
A_2 &= (u_1^2/2!)\, f^{(2,0)} + u_1 v_1 f^{(1,1)} + (v_1^2/2!)\, f^{(0,2)} \\
&\quad + u_2 f^{(1,0)} + v_2 f^{(0,1)} \\
A_3 &= (u_1^3/3!)\, f^{(3,0)} + (u_1^2/2!)\, v_1 f^{(2,1)} + v_1^2/2!)\, u_1 f^{(1,2)} + (v_1^3/3!)\, f^{(0,3)} \\
&\quad + u_1 u_2 f^{(2,0)} + v_1 u_2 f^{(1,1)} + v_1 v_2 f^{(0,2)} + u_1 v_2 f^{(1,1)} + u_3 f^{(1,0)} + v_3 f^{(0,1)} \\
&\vdots
\end{aligned}
$$

Reference 12 is a complete reference for functions such as $f(u, v)$ or $f(u, v, w)$, so, either way, the system is completely solvable. This represents a general solution. It is known that smooth solutions (to the incompressible problem under consideration) do exist for short times and are continuously dependent on the initial data. The next step (to be investigated) is the use of specific conditions and calculation of the results for comparison with known results for simpler cases solvable by current techniques or, more to the point, to show experimentally verifiable phenomena such as the onset of turbulence.

5. SOME THOUGHTS ON THE ONSET OF TURBULENCE

Consider first a very simple equation whose solution is clearly trivial. Thus consider $dy/dx = (y-1)^2$ with $y(0) = 1$ which obviously is satisfied by $y = 1$.

Now consider the effect of a 1% change in a parameter by writing $dy/dx = (y-1)^2 + .01$.[4] This now yields a periodic solution $y = 1 + 0.1 \tan(x/10)$ which has vertical asymptotes at $(2k+1)\,5\pi$, $k = 0, \pm 1, \pm 2, \ldots$

Now, let's make a 1% change in the initial condition, or $y(0) = 1.01$. We now have a hyperbola $y = 1 - 1/(x - 100)$ and only one vertical asymptote at $x = 100$.[5] Thus the effect in a *nonlinear* equation of even very small changes in inputs or parameters can result in large effects on the solution.

Suppose now that very small fluctuations are present in the input and parameter because of small inherent randomness. Then the solution could change randomly between the possibilities above and appear very complex indeed.

Now considering the Navier–Stokes system with its nonlinear terms where there could be small fluctuations in density, pressure, viscosity, and velocities, it is clear we can expect similar effects and a "chaotic-looking" or turbulent case.

The nonlinear terms cause small fluctuations to become large fluctuations while friction terms tend to remove differences in velocities. The Reynolds number is a measure of the ratio of nonlinear terms to frictional terms, so it is reasonable that if the number becomes large, the tendency to turbulence increases. However, factors such as smoothness of boundaries and the magnitude of initial fluctuations also influence the resulting flow.

In the simple deterministic case, consider one nonlinear term $u\,\partial u/\partial x$ divided by a molecular friction term $\nu\,\partial^2 u/\partial x^2$. If u and $\partial u/\partial x$ are assumed to be of the order U and L is a typical distance over which the velocity varies by U, the ratio is of the order $(U^2/L)/(\nu \cdot U/L^2) = U \cdot L/\nu$ or the Reynolds number.

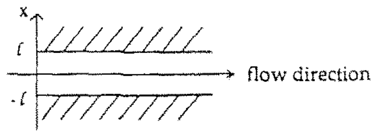
In the general case, if we have a fluctuating v or ν, we can see that large changes can occur in the tendency to turbulent behavior.

The best way apparently to determine when turbulence starts is to solve the stochastic Navier–Stokes system as we have outlined and study the behavior as a function of the parameters of the flow. A comparison of a deterministic solution and a stochastic solution with varying conditions should illuminate the problem of onset of turbulence.

[4] See footnote 2.

[5] If $y(0) = 0.99$, the asymptote moves to $x = -100$.

Suppose we consider flow in a flat channel as an idealization of a pipe in a plane. We have



Assume $\partial p/\partial x = 0$. Replace x by x/ℓ to make the half-width unity. Using decomposition, write

$$L_t u = \nu(\partial^2/\partial x^2)\, u - u(\partial/\partial x)\, u$$

$$u = L_t^{-1}\nu(\partial^2/\partial x^2) \sum_{n=0}^{\infty} u_n - L_t^{-1} \sum_{n=0}^{\infty} A_n$$

where the A_n polynomials are generated for the nonlinear term. Then

$$u_0 = u(0) + tu'(0)$$

$$u_{n+1} = L_t^{-1}\nu(\partial^2/\partial x^2)\, u_n - L_t^{-1} A_n$$

for $n \geqslant 0$. If ν is constant and u_0 is deterministic, u is deterministic. If u_0 has a random component, this component will cause new terms to keep appearing because of the expressions on the right side of the equation for u_{n+1} for any $n \geqslant 0$, especially from the term involving A_n. This is obvious by inspection of the A_n for increasing n. (The effect of physically unrealistic change in the solution by a linearization is also clear[5,6].)

Consequently, as a result of any randomness and the nonlinearity, the flow is radically altered—the effect increasing as the fluctuation becomes larger. Random boundary conditions resulting from roughness in the walls will have the same effect.

A basic question is whether the Navier–Stokes equations are an adequate model for real turbulent fluids.[4,13] The linear constitutive law used in the derivation means that derivatives of the velocity components u, v, w are necessarily small. Secondly, stochasticity cannot be considered as an afterthought; it must be considered in the initial modeling. A more general model due to Ladyzhenskaya has partially addressed this issue by allowing nonlinearity in the constitutive law and leads to a global uniqueness for nonstationary three-dimensional flow. A truly nonlinear stochastic model coupled with the decomposition method of solution may resolve remaining difficulties. The general problem may have random initial/boundary conditions. The quantity ρ is generally taken as a constant and set equal to unity; however, compressibility becomes a factor with increasing depth.

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