Foundations of Physics, Vol. 19, No. 6, 1989

The Determination of the Past and the Future of a Physical System in Quantum Mechanics[1]

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Received August 1, 1988

The determination of the past and the future of a physical system are complementary aims of measurements. An optimal determination of the past of a system can be achieved by an informationally complete set of physical quantities. Such a set is always strongly noncommutative. An optimal determination of the future of a physical system can be obtained by a Boolean complete set of quantities. The two aims can be reconciled to a reasonable degree with using unsharp measurements.

1. INTRODUCTION

Assume that a measurement is performed on a physical system and a certain result is obtained. This result may be used to infer which properties the system had before the measurement, or it may be used to deduce which properties the system possesses after the measurement. We refer to these two aspects of a measurement as the determination of the past and the determination of the future of a physical system. Here they will be investigated within the Hilbert space formulation of quantum mechanics.

In quantum mechanics pure states represent maximal collections of properties which a physical system may possess at a time. Thus an optimal determination of the past or of the future of the system is obtained whenever the measurement result, or results, lead to the specification of a pure state of the system, either in its past or in its future. It turns out that in

[1] This work was partly supported by the Bundesministerium für Forschung und Technologie, Bonn, the Research Institute for Theoretical Physics, Helsinki, and the University of Turku Foundation, Turku.

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0015-9018/89/0600-0633$06.00/0

825/19/6-1

quantum mechanics these two aspects of a measurement are mutually exclusive in the sense that an optimal determination of the past of the system requires measurements of strongly noncommutative quantities whereas an optimal determination of the future of the system can be obtained only via the preparatory measurements of maximal quantities, or complete sets of commuting quantities. On the other hand, the constitution of physical systems requires their persistence from the past to the future, even in the course of measurements, showing thus that the two aspects of measurements are equally important for a full description of a physical system. The determinations of the past and the future of a physical system then form a new mode of complementarity in quantum mechanics.

In Sec. 2 of this paper the problem of determining the past of a physical system is investigated by means of the notion of informational completeness. As it is reviewed in Sec. 2.2, informationally complete sets of physical quantities allow optimal determinations of the past of the system. After giving the basic formal definitions in Sec. 2.1 we show that no single physical quantity nor any set of commuting quantities is informationally complete. A necessary condition for a set of physical quantities to be informationally complete is that the quantities are strongly noncommutative (Thm. 2.1.8). However, even total noncommutativity of a pair of physical quantities is not sufficient for the informational completeness of this pair. This observation then leads to a search for informational completions of pairs of totally noncommutative quantities. In Sec. 2.3 this is done in the important case of complementary position and momentum observables, whereas in Sec. 2.4 the same is done for the spin quantities of a spin-1/2 system. In both cases, the informational completion of the totally noncommutative pairs is obtained only via replacing the pairs with informationally equivalent unsharp but coexistent pairs. The existence of an informationally complete joint observable of a coexistent unsharp pair of quantities then depends on the degree of unsharpness involved.

The optimal determination of the future of a physical system via the preparatory measurements of maximal quantities (or complete sets of commuting quantities) is already well-understood in the quantum theory of measurement. But in order to compare the two aspects of measurements—the determinations of the past and the future—it becomes relevant to study the maximal degrees of certainty of the values of totally noncommutative quantities and to consider the introduction of unsharp values. Theorem 3.2.1 of Sec. 3 gives necessary and sufficient conditions for the existence of the so-called maximal information states for pairs of physical quantities. It also gives a complete characterization of such states. It is then demonstrated that the introduction of unsharpness, which is necessary for an informational completion of a noncommutative pair, cannot increase the maximal information on the values of such quantities (Secs. 3.3 and 3.4).

Section 4 finally summarizes the results of the paper in discussing the new mode of complementarity which manifests itself in the competing aims of determining the past and the future of a physical system in quantum mechanics.

We close this introduction with recalling the basic notations and terminology of the Hilbert space formulation of quantum mechanics used in this paper.

In the Hilbert space formulation of quantum mechanics the description of a physical system $\mathscr{S}$ is based on a (complex, separable, generally infinite dimensional) Hilbert space $\mathscr{H}$, with the inner product $\langle \cdot | \cdot \rangle$. Any physical quantity of the system is represented as (and identified with) a self-adjoint operator A in $\mathscr{H}$. The spectral measure of A is denoted by $E^A: \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$, where $\mathscr{B}(\mathbb{R})$ is the Boral σ-algebra of the real line $\mathbb{R}$ ($\mathscr{B}(\mathbb{R}^2)$ that of $\mathbb{R}^2$), and $\mathscr{L}(\mathscr{H})$ is the set of bounded linear operators on $\mathscr{H}$. Any state of the system is represented as (and identified with) an element of T of $\mathscr{T}_s(\mathscr{H})_1^+$ of positive normalized trace class operators on $\mathscr{H}$. In this representation the pure states of the system appear as the one-dimensional projection operators $P[\varphi]$ ($P[\varphi]\psi = \langle \varphi | \psi \rangle \varphi$) on $\mathscr{H}$, $\varphi \in \mathscr{H}$, so that they may be identified, modulo a phase factor, with the unit vectors of $\mathscr{H}$. $\mathscr{H}_1$ denotes the set of unit vectors of $\mathscr{H}$.

Any pair (A, T) of a physical quantity A and a state T defines a real normalized measure on the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ of $\mathbb{R}$, i.e., E_T^A: $\mathscr{B}(\mathbb{R}) \to [0, 1]$, $X \mapsto E_T^A(X) := \mathrm{tr}(TE^A(X))$. According to the Born interpretation, E_T^A is the probability measure of the possible values of the quantity A in the state T, i.e. $E_T^A(X)$ is the probability that the value of A is (in) the set X when the system is in the state T.

The concept of a physical quantity as a (real) spectral measure E^A (or self-adjoint operator A) is unnecessarily restricted. In fact, it is insufficient for a quantitative analysis of the topic of this paper. As a natural generalization one defines a physical quantity as a semispectral measure $E: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ on a measurable space $(\Omega, \mathscr{A})$, like $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, i.e., a positive operator valued measure for which $(0 \leqslant E(X) \leqslant)$ $E(\Omega) = I$. Spectral measures are then exactly those semispectral measures E for which $E(X) = E(X)^2$ for any $X \in \mathscr{A}$. The unsharp quantities are examples of generalized quantities. Clearly, any pair (E, T) of a generalized physical quantity E and a state T again defines a probability measure E_T for which the Born interpretation may be adopted.

In the course of this work we shall also refer to the usual formulation of the quantum theory of measurement. In that theory a measurement of a physical quantity A, say, on the system $\mathscr{S}$ is represented as a quadruple

$\langle \mathscr{H}_M, A_M, T_M, V \rangle$, where $\mathscr{H}_M$ is the Hilbert space of the measuring apparatus $\mathscr{M}$, A_M is the pointer observable of $\mathscr{M}$ (i.e. the quantity of $\mathscr{M}$ which corresponds to the measured quantity A of $\mathscr{S}$), T_M is the initial (or premeasurement) state of $\mathscr{M}$, and V is the state-transformation representing the measurement coupling between $\mathscr{S}$ and $\mathscr{M}$. For further details, see, e.g., Beltrametti *et al.*,[1] or Beltrametti and Cassinelli.[2] In the latter reference the reader will also find a more systematic exposition of the Hilbert space formulation of quantum mechanics as it is used in this paper.

2. INFORMATIONAL COMPLETENESS

2.1. General: Mathematical Aspects

Let $\mathscr{H}$ be a complex separable Hilbert space, $\mathscr{L}(\mathscr{H})$ the set of bounded operators on $\mathscr{H}$, and $\mathscr{L}_s(\mathscr{H})$, $\mathscr{L}(\mathscr{H})^+$, and $\mathscr{T}_s(\mathscr{H})_1^+$ its subsets of self-adjoint, positive, and positive normalized trace class operators, respectively. A positive operator valued measure $E: \mathscr{A} \to \mathscr{L}(\mathscr{H})^+$ on a measurable space $(\Omega, \mathscr{A})$, the value space, is a semispectral measure if $E(\Omega) = I$. A semispectral measure E is a spectral measure if $E(X) = E(X)^2$ for any $X \in \mathscr{A}$. This is the case exactly when $E(X \cap Y) = E(X)\, E(Y)$ for any $X, Y \in \mathscr{A}$. If the value space of a spectral measure E is the real Borel space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, then E defines a unique self-adjoint operator A in $\mathscr{H}$. According to the spectral theorem each self-adjoint operator A in $\mathscr{H}$ is so determined. Thus, when a spectral measure is defined on the real value space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ we denote it as E^A to indicate that it uniquely determines and is uniquely determined by the self-adjoint operator A in $\mathscr{H}$.

Let $\mathscr{L} \subset \mathscr{L}(\mathscr{H})$. In the physical applications considered here $\mathscr{L}$ usually appears as the union of the ranges $\mathscr{R}(E) = \{E(X) : X \in \mathscr{A}\}$ of some semispectral measures $E: \mathscr{A} \to \mathscr{L}(\mathscr{H})^+$. Let $T_1, T_2 \in \mathscr{T}_s(\mathscr{H})_1^+$. States T_1 and T_2 are *L-equivalent* (cf. Jauch[3]) if

$$\operatorname{tr}(T_1 L) = \operatorname{tr}(T_2 L) \qquad \text{for any} \quad L \in \mathscr{L} \tag{1}$$

If T_1 and T_2 are $\mathscr{L}$-equivalent states, we denote it as $T_1 \sim_{\mathscr{L}} T_2$. The relation $\sim_{\mathscr{L}}$ is an equivalence relation in $\mathscr{T}_s(\mathscr{H})_1^+$. [In fact, it is an equivalence relation in any subset of states $\mathscr{T} \subset \mathscr{T}_s(\mathscr{H})_1^+$. Cf. Remark 2.1.1.] For a given $\mathscr{L} \subset \mathscr{L}(\mathscr{H})$ we denote

$$[T]^{\mathscr{L}} = \{T' \in \mathscr{T}_s(\mathscr{H})_1^+ : T' \sim_{\mathscr{L}} T\} \tag{2}$$

for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$. If $\mathscr{L} = \mathscr{R}(E^1) \cup \cdots \cup \mathscr{R}(E^n)$, or $\mathscr{L} = \mathscr{R}(E^A)$ we simply write $[T]^{E^1, \ldots, E^n}$, or $[T]^A$ instead of $[T]^{\mathscr{R}(E^1) \cup \cdots \cup \mathscr{R}(E^n)}$, or $[T]^{\mathscr{R}(E^A)}$, respectively.

Let $(T_i)_{i\in\mathscr{I}} \subset \mathscr{T}_s(\mathscr{H})_1^+$ be a countable subset of states, with a countable index set $\mathscr{I}$, and assume that $T_i \sim_{\mathscr{L}} T_j$ for any $i, j \in \mathscr{I}$ for some $\mathscr{L} \subset \mathscr{L}(\mathscr{H})$. Then for any $i \in \mathscr{I}$, $T_i \sim_{\mathscr{L}} \sum_{j\in\mathscr{I}} t_j T_j$ for all $(t_j)_{j\in\mathscr{I}} \subset \mathbb{R}^+$ such that $\sum_{j\in\mathscr{I}} t_j = 1$. Thus $[T]^{\mathscr{L}}$ is a σ-convex set for any $\mathscr{L} \subset \mathscr{L}(\mathscr{H})$, and for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$. Let $\mathrm{Ex}([T]^{\mathscr{L}})$ be the set of extreme points of $[T]^{\mathscr{L}}$. Clearly, $\mathrm{Ex}(\mathscr{T}_s(\mathscr{H})_1^+) \cap [T]^{\mathscr{L}} \subset \mathrm{Ex}([T]^{\mathscr{L}})$ for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$. But the converse inclusion need not hold true as $\mathrm{Ex}([T]^{\mathscr{L}})$ may be nonempty even though $[T]^{\mathscr{L}}$ contains no pure states. In fact, this is the case whenever $\mathscr{L}$ is informationally complete (see the following) and T is a nonpure state, for then $[T]^{\mathscr{L}} = \{T\} \neq \varnothing = [T]^{\mathscr{L}} \cap \mathrm{Ex}(\mathscr{T}_s(\mathscr{H})_1^+)$, and $\mathrm{Ex}[T]^{\mathscr{L}} = \{T\}$. However, we note that for any $\mathscr{L} \subset \mathscr{L}(\mathscr{H})$

$$[T]^{\mathscr{L}} = \bigcap_{L\in\mathscr{L}} [T]^L = \bigcap_{L\in\mathscr{L}} \mathrm{cl\,conv}\{P[\varphi] : \varphi \in \mathscr{H}_1,\ P[\varphi] \sim_{\mathscr{L}} T\} \tag{3}$$

(Hadjisavvas,[4] Theorem 3). This result is based on the following fact (ibd., Corollary 2): For any $T \in \mathscr{T}_s(\mathscr{H})_1^+$, $T = \lambda P[\varphi] + (1-\lambda) T_1$ for some $\lambda > 0$, $\varphi \in \mathscr{H}_1$, if and only if $\varphi \in \mathscr{R}(T^{1/2})$.

Let $\mathscr{L}, \mathscr{L}' \subset \mathscr{L}(\mathscr{H})$. If $\mathscr{L} \subset \mathscr{L}'$, then $[T]^{\mathscr{L}} \supset [T]^{\mathscr{L}'}$ for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$. In other words, we have the following set-theoretical inclusions:

$$\mathscr{T}_s(\mathscr{H})_1^+ = [T]^{\{0,I\}} \supset [T]^{\mathscr{L}} \supset [T]^{\mathscr{L}'} \supset [T]^{\mathscr{L}(\mathscr{H})} = \{T\} \tag{4}$$

for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$.

2.1.1. Remark

Instead of $[T]^{\mathscr{L}}$, $T \in \mathscr{T}_s(\mathscr{H})_1^+$, $\mathscr{L} \subset \mathscr{L}(\mathscr{H})$ we might also consider the following equivalence classes $[T]^{\mathscr{L}}_{\mathscr{T}} := [T]^{\mathscr{L}} \cap \mathscr{T}$ or $[T]^{\mathscr{L}}$, $T \in \mathscr{T}$, for a given subset of states $\mathscr{T} \subset \mathscr{T}_s(\mathscr{H})_1^+$. Instead to developing the general theory along these options we shall return to them only in some special cases.

Let $\mathscr{L}, \mathscr{L}' \subset \mathscr{L}(\mathscr{H})$. $\mathscr{L}$ and $\mathscr{L}'$ are *informationally equivalent with respect to a set of states* $\mathscr{T} \subset \mathscr{T}_s(\mathscr{H})_1^+$ if

$$[T]^{\mathscr{L}} = [T]^{\mathscr{L}'} \qquad \text{for any} \quad T \in \mathscr{T} \tag{5}$$

If $\mathscr{L}$ and $\mathscr{L}'$ are informationally equivalent w.r.t. all states $T \in \mathscr{T}_s(\mathscr{H})_1^+$, then $\mathscr{L}$ and $\mathscr{L}'$ are *informationally equivalent* (cf. Ali and Doebner[5]). The subsets $\mathscr{P}(\mathscr{H}) = \{P \in \mathscr{L}(H)\colon\ P = P^+ = P^2\}$, $\mathscr{E}(\mathscr{H}) = \{A \in \mathscr{L}(\mathscr{H})\colon 0 \leqslant A \leqslant I\}$, and $\mathscr{L}_s(\mathscr{H})$ of $\mathscr{L}(\mathscr{H})$ are informationally equivalent. Moreover,

$$[T]^{\mathscr{P}(\mathscr{H})} = [T]^{\mathscr{E}(\mathscr{H})} = [T]^{\mathscr{L}_s(\mathscr{H})} = [T]^{\mathscr{L}(\mathscr{H})} = \{T\} \tag{6}$$

for any $T \in \mathcal{T}_s(\mathcal{H})_1^+$. Indeed, if $T_1 \in [T]^{\mathcal{P}(\mathcal{H})}$, then $\mathrm{tr}(T_1 P) = \mathrm{tr}(TP)$ for any $P \in \mathcal{P}(H)$, so that, in particular, $\langle \varphi | (T_1 - T)\varphi \rangle = 0$ for any $\varphi \in \mathcal{H}$. But this implies that $T_1 - T = 0$, i.e., $[T]^{\mathcal{P}(\mathcal{H})} = \{T\}$. As $\mathcal{P}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}) \subset \mathcal{L}_s(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ we have Eq. (6) by Eq. (4).

Let $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$. If $\mathcal{L}$ is informationally equivalent with $\mathcal{L}(\mathcal{H})$ (or with any $\mathcal{L}' \supset \mathcal{P}(\mathcal{H})$) w.r.t. a set of states $\mathcal{T} \subset \mathcal{T}_s(\mathcal{H})_1^+$, then

$$[T]^{\mathcal{L}} = \{T\} \qquad \text{for any} \quad T \in \mathcal{T} \tag{7}$$

In that case we say that $\mathcal{L}$ is *informationally complete with respect to* $\mathcal{T} \subset \mathcal{T}_s(\mathcal{H})_1^+$. If $\mathcal{L}$ is informationally complete w.r.t. all states $T \in \mathcal{T}_s(\mathcal{H})_1^+$, then $\mathcal{L}$ is *informationally complete* (cf. Prugovecki[6]).

There is an immediate characterization of informational completeness of a given $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ in terms of the following functional

$$\mathcal{T}_s(\mathcal{H}) \to \mathbb{R}, \quad T \mapsto \|T\|_{\mathcal{L}} := \sup \left\{ \frac{|\mathrm{tr}(TL)|}{\|L\|} : L \in \mathcal{L}, L \neq 0 \right\} \tag{8}$$

Namely, assuming that the identity operator I is in $\mathcal{L}$, then $\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{T}_s(\mathcal{H})$ (the real vector space of the trace class operators on $\mathcal{H}$) if and only if $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ is informationally complete. Indeed, if $\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{T}_s(\mathcal{H})$, then its positive definiteness implies that $[T]^{\mathcal{L}} = \{T\}$ for any $T \in \mathcal{T}_s(\mathcal{H})_1^+$. On the other hand, the functional defined in Eq. (8) is always absolutely homogeneous and subadditive. If $\mathcal{L}$ is informationally complete, i.e. $[T]^{\mathcal{L}} = \{T\}$ for any $T \in \mathcal{T}_s(\mathcal{H})_1^+$, then also $\|T_1 - T_2\|_{\mathcal{L}} = 0$ for any $T_1, T_2 \in \mathcal{T}_s(\mathcal{H})$, only if $T_1 = T_2$ (for $I \in \mathcal{L}$), i.e. $\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{T}_s(\mathcal{H})$. The physical relevance of this characterization of informational completeness of an $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ has been discussed by Busch.[7]

In the light of the physical applications considered here it is important to ask whether

$$[T]^{\mathcal{L}} = \{T\}, \qquad T \in \mathcal{T}_s(\mathcal{H})_1^+ \tag{9}$$

for some physically interesting $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$, and if not, whether, at least,

$$[T]^{\mathcal{L}} = \{T\}, \qquad T \in \mathcal{T} \tag{10}$$

for some physically interesting $\mathcal{T} \subset \mathcal{T}_s(\mathcal{H})_1^+$, and/or whether

$$[T]^{\mathcal{L}'} = \{T\}, \qquad T \in \mathcal{T}_s(\mathcal{H})_1^+ \tag{11}$$

for some physically interesting extension $\mathcal{L}'$ of $\mathcal{L}$. Moreover, as the pure states $\mathrm{Ex}(\mathcal{T}_s(\mathcal{H})_1^+)$ $(= \{T \in \mathcal{T}_s(\mathcal{H})_1^+ : T = T^2\} = \{T \in \mathcal{T}_s(\mathcal{H})_1^+ : T = P[\varphi]$ for some $\varphi \in \mathcal{H}_1\})$ determine all the states $\mathcal{T}_s(\mathcal{H})_1^+$, i.e. $\mathcal{T}_s(\mathcal{H})_1^+ = \mathrm{cl\ conv}$

$\mathrm{Ex}(\mathcal{T}_s(\mathcal{H})_1^+)$ (Davies[8]), it should also be important to know whether for a given $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ its informational completeness w.r.t. $\mathrm{Ex}(\mathcal{T}_s(\mathcal{H})_1^+)$ already implies its informational completeness, i.e., does the condition

$$[T]^{\mathcal{L}} = \{T\} \qquad \text{for any} \quad T \in \mathrm{Ex}(\mathcal{T}_s(\mathcal{H})_1^+)$$

imply that (12)

$$[T]^{\mathcal{L}} = \{T\} \qquad \text{for any} \quad T \in \mathcal{T}_s(\mathcal{H})_1^+$$

Clearly, Eq. (12) holds true for any $\mathcal{L} \supset \mathcal{P}(\mathcal{H})$ as well as in the case of two dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ (cf. Sec. 2.4). Whether Eq. (12) holds true also in some nontrivial cases is an open question. We shall return to that subsequently.

In Secs. 2.3 and 2.4 we shall give important physical examples of the cases for Eqs. (9)–(12). In order to fully appreciate those examples we shall first discuss some basic results on the problem of informational completeness.

A semispectral measure $E\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})^+$ is commutative if $E(X)\,E(Y) = E(Y)\,E(X)$ for all $X, Y \in \mathcal{A}$. Clearly, any spectral measure is commutative. But there are also important commutative semispectral measures, like unsharp position and unsharp momentum to be discussed in Sec. 2.3. Our first result refers to such measures.

2.1.2. Theorem

No commutative semispectral measure $E\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})^+$ is informationally complete, when $\dim(\mathcal{H}) \geqslant 2$.

Proof. For any $0 \neq E(X) \neq I$ there is a $\varphi \in \mathcal{H}_1$ such that $\psi \equiv e^{iE(X)}\varphi \neq e^{ia}\varphi$ for any $a \in \mathbb{R}$. Then $P[\psi] \neq P[\varphi]$, but $\mathrm{tr}(P[\psi]\,E(Y)) = \langle e^{iE(X)}\varphi \mid E(Y)e^{iE(X)}\varphi\rangle = \langle \varphi \mid E(Y)\varphi\rangle = \mathrm{tr}(P[\varphi]\,E(Y))$ for any $Y \in \mathcal{A}$, as $E(X)\,E(Y) = E(Y)\,E(X)$ for any $X, Y \in \mathcal{A}$, and thus also $e^{iE(X)}E(Y) = E(Y)e^{iE(X)}$ for all $X, Y \in \mathcal{A}$. If $E(X) \in \{0, I\}$ for any $X \in \mathcal{A}$, then E is constant and $[T]^E = \mathcal{T}_s(\mathcal{H})_1^+$ for any $T \in \mathcal{T}_s(\mathcal{H})_1^+$ so that E is informationally incomplete whenever $\dim(\mathcal{H}) \geqslant 2$. ∎

Any Boolean sub σ-algebra $\mathcal{L}$ of $\mathcal{P}(\mathcal{H})$, the projection lattice of $\mathcal{H}$, can be represented as the range of some spectral measure $\mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})^+$ (Varadarajan,[9] Lemma 3.16). Thus Theorem 2.1.2 has the corollary:

2.1.3. Corollary

If $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ is an informationally complete set of projection operators, then $\mathcal{L}$ cannot be a Boolean sub σ-algebra of $\mathcal{P}(\mathcal{H})$. Especially, $\mathcal{L}$ cannot be a commuting set of projection operators.

As no Boolean sub σ-algebra of $\mathscr{P}(\mathscr{H})$ is informationally complete we also have the following corollary.

2.1.4. Corollary

No family of mutually commuting spectral measures is informationally complete. Especially, no commuting set $\mathscr{L} \subset \mathscr{L}_s(\mathscr{H})$ is informationally complete.

There are two physically important special cases of these results which deserve to be mentioned separately: No maximal quantity nor any complete set of commuting quantities is informationally complete.

Spectral measures E: $\mathscr{A} \to \mathscr{L}(\mathscr{H})^+$, or more generally Boolean sub σ-algebra of $\mathscr{P}(\mathscr{H})$, are, of course, physically very important quantities. As they cannot be informationally complete it is important to ask along with Eq. (10) what, if any, is the largest set of states $\mathscr{T} \subset \mathscr{T}_s(\mathscr{H})_1^+$ w.r.t. which E, say, is informationally complete. The following theorem answers this question. Let $\mathscr{T}(E)$ denote the set of one-dimensional spectral projections of E. As there are at most countably many different one-dimensional spectral projections of E we may write $\mathscr{T}(E)$ as $\mathscr{T}(E) = \{P[\varphi_i]: \varphi_i \in \mathscr{H}_1, P[\varphi_i] = E_i, i \in \mathscr{I} \subset \mathbb{N}\}$. Occasionally, we shall refer to the elements of $\mathscr{T}(E)$ as the nondegenerate eigenstates of E. Clearly, $\mathscr{T}(E)$ may also be empty.

2.1.5. Theorem

Let E: $\mathscr{A} \to \mathscr{L}(\mathscr{H})^+$ be a spectral measure. Then $[T]^E = \{T\}$ if and only if $T \in \mathscr{T}(E)$.

Proof. We shall show first that the only pure states $P[\varphi]$, $\varphi \in \mathscr{H}_1$, w.r.t. which E is informationally complete are the nondegenerate eigenstates $P[\varphi_i] = E_i$, $i \in \mathscr{I}$, of E. If $T \in [P[\varphi_i]]^E$, then $\operatorname{tr}(TE(X)) = \operatorname{tr}(P[\varphi_i]\, E(X)) = \operatorname{tr}(E_i E(X))$ for any $X \in \mathscr{A}$. Hence $\langle \varphi_i | T\varphi_i \rangle = 1$, which shows that $T = P[\varphi_i]$. Indeed, if $T = \sum t_k P[\psi_k]$ is the canonical decomposition of T, then $1 = \langle \varphi_i | T\varphi_i \rangle = \sum t_k |\langle \psi_k | \varphi_i \rangle|^2$ so that $|\langle \psi_k | \varphi_i \rangle|^2 = 1$ for any k, i.e. $T = P[\varphi_i]$. Assume next that $[P[\varphi]]^E = \{P[\varphi]\}$ for some $\varphi \in \mathscr{H}_1$. We claim that $P[\varphi] = P[\varphi_i]$ for some $i \in \mathscr{I}$. Assume that $P[\varphi] \neq P[\varphi_i]$ for any $i \in \mathscr{I}$. If $P[\varphi] \leqslant E(X)$ for some $X \in \mathscr{A}$, let $P = \bigwedge (E(X)$: $X \in \mathscr{A}$, $P[\varphi] \leqslant E(X))$ so that $\dim(P(\mathscr{H})) \geqslant 2$. Then $\{P[\psi]: P\psi = \psi, \psi \in \mathscr{H}_1\} \subset [P[\varphi]]^E$. On the other hand, if $P[\varphi] \neq E(X)$ for any $X \in \mathscr{A}$, $E(X) \neq I$, then for some $0 \neq E(X) \neq I$, $e^{iE(X)}\varphi \neq e^{ia}\varphi$ for all $a \in \mathbb{R}$, which again implies that $[P[\varphi]]^E \neq \{P[\varphi]\}$. Thus the nondegenerate eigenstates of E, if any, are the only pure states w.r.t. which E is informationally complete.

Let $\tilde{\mathscr{T}}(E) = \operatorname{cl\,conv} \mathscr{T}(E)$. We show next that $[T]^E_{\mathscr{T}(E)} = \{T\}$ for any $T \in \tilde{\mathscr{T}}(E)$. Indeed, if $T, T' \in \tilde{\mathscr{T}}(E)$, and $T' \sim_E T$, then clearly $T = T'$.

Assume further that $T \in \tilde{\mathcal{T}}(E)$. Then $[T]^E = \{T\}$ if and only if $T = P[\varphi]$ for some $\varphi \in \mathcal{H}_1$. If $T = P[\varphi]$, $\varphi \in \mathcal{H}_1$, then $[T]^E = \{T\}$ if and only if $P[\varphi] = P[\varphi_i]$ for some $i \in \mathcal{I}$, as shown earlier. On the other hand, if $T \in \tilde{\mathcal{T}}(E)$, and $T = \sum t_i P[\varphi_i]$, $t_i > 0$, at least for some $P[\varphi_i] \neq P[\varphi_j]$, $i \neq j$, then any $\varphi = \sum \langle \varphi_i | \varphi \rangle \varphi_i$, with $|\langle \varphi_i | \varphi \rangle|^2 = t_i$ determines a state $P[\varphi] \in [T]^E$. Finally, if $T \notin \tilde{\mathcal{T}}(E)$, then $T = \lambda P[\varphi] + (1 - \lambda) T_1$ for some $\lambda > 0$, $\varphi \in \mathcal{R}(T^{1/2})$, $P[\varphi] \notin \mathcal{T}(E)$. As $P[\varphi] \sim_E P[e^{iE(X)}\varphi]$ for some $0 \neq E(X) \neq I$ for which $e^{iE(X)}\varphi \neq e^{ia}\varphi$ for any $a \in \mathbb{R}$, we have $T' = \lambda P[e^{iE(X)}\varphi] + (1 - \lambda) T_1 \sim_E T$ though $T \neq T'$. ∎

This proof contains a partial answer to the question of Eq. (12). The set $\mathcal{T}(E)$ of the nondegenerate eigenstates of a spectral measure E is the largest set w.r.t. which E is informationally complete. E is not informationally complete w.r.t. $\tilde{\mathcal{T}}(E) = \mathrm{cl\,conv}\, \mathcal{T}(E)$, though $[T]^E_{\mathcal{T}(E)} = \{T\}$ for any $T \in \tilde{\mathcal{T}}(E)$. Thus it still remains open whether the informational completeness of an $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ w.r.t. $\mathrm{Ex}(\mathcal{T}_s(\mathcal{H})_1^+)$ implies its informational completeness.

There are two immediate corollaries to Theorem 2.1.5.

2.1.6. Corollary

If $\mathcal{L} \subset \mathcal{L}(\mathcal{H})$ is a Boolean sub σ-algebra of $\mathcal{P}(\mathcal{H})$, then for any $T \in \mathcal{T}_s(\mathcal{H})_1^+$ $[T]^{\mathcal{L}} = \{T\}$ if and only if T is a pure state corresponding to an atom of $\mathcal{L}$. If $\mathcal{L}$ has no (one-dimensional) atoms, then it is informationally complete w.r.t. no state.

2.1.7. Corollary

If a spectral measure has no nondegenerate eigenstates, then it is informationally complete w.r.t. no state.

It is now clear that a commutative $\mathcal{L} \subset \mathcal{L}_s(\mathcal{H})$ cannot be informationally complete. Our next result characterizes the degree of noncommutativity which is necessary for the informational completeness of a set $\mathcal{L} \subset \mathcal{L}_s(\mathcal{H})$. According to the usual terminology, the set $\mathcal{L} \subset \mathcal{L}_s(\mathcal{H})$ is commutative if $L_1 L_2 = L_2 L_1$ for and $L_1, L_2 \in \mathcal{L}$. A natural generalization of this notion is the *commutativity domain* of $\mathcal{L}$, $\mathrm{com}(\mathcal{L}) := \{\varphi \in \mathcal{H}: L_1 \cdots L_n \varphi = L_{p(1)} \cdots L_{p(n)} \varphi$ for any $L_1, ..., L_n \in \mathcal{L}$, and for all permutations p of $\{1, ..., n\}$, $n \in \mathbb{N}\}$, $n \in \mathbb{N}\}$ (cf. eg. Pulmannova and Dvurecenskij.[10] Clearly, $\mathrm{com}(\mathcal{L})$ is a closed subspace of $\mathcal{H}$, and $\mathcal{L}$ is commutative (in the usual sense) if and only if $\mathrm{com}(\mathcal{L}) = \mathcal{H}$. In the other extreme case, $\mathrm{com}(\mathcal{L}) = \{0\}$, $\mathcal{L}$ is said to be totally noncommutative. One immediately observes that $\mathrm{com}(\mathcal{L})$ is invariant under each $L \in \mathcal{L}$, i.e. $L(\mathrm{com}(\mathcal{L})) \subset \mathrm{com}(\mathcal{L})$ for any $L \in \mathcal{L}$. Moreover $\mathcal{L}|_{\mathrm{com}(\mathcal{L})} = \{L|_{\mathrm{com}(\mathcal{L})}: L \in \mathcal{L}\}$ is a commutative set. Thus denoting $\mathcal{K} = \mathrm{com}(\mathcal{L})$ and $P_{\mathcal{K}}$ the projection on $\mathcal{K}$, then $\{P_{\mathscr{K}} L P_{\mathscr{K}}\colon L \in \mathscr{L}\}$ is a commutative set of bounded self-adjoint operators on the Hilbert space $\mathscr{K}$. Clearly, for any $\varphi \in \mathscr{K}$, $\langle \varphi | L\varphi \rangle = \langle \varphi | P_K L P_K \varphi \rangle$ for all $L \in \mathscr{L}$. By Corollary 2.1.4 we then have the following result:

2.1.8. Theorem

Let $\mathscr{L} \subset \mathscr{L}_s(\mathscr{H})$. If $\mathscr{L}$ is informationally complete, then dim $[\mathrm{com}(\mathscr{L})] \leqslant 1$.

There are two special cases which deserve to be mentioned separately. If $E^i\colon \mathscr{A}_i \to \mathscr{L}(\mathscr{H})^+$, $i = 1, 2$, are any two spectral measures, then their commutativity domain $\mathrm{com}(E^1, E^2) \equiv \mathrm{com}(\mathscr{R}(E^1) \cup \mathscr{R}(E^2))$ reduces to the set $\mathrm{com}(E^1, E^2) = \{\varphi \in \mathscr{H}\colon E^1(X) E^2(Y)\varphi = E^2(Y) E^1(X)\varphi$ for all $X \in \mathscr{A}_1$, $Y \in \mathscr{A}_2\} =: \mathrm{com}_{(2)}(E^1, E^2)$ (as the relation $E(X) E(Y) = E(X \cap Y)$, X, $Y \in \mathscr{A}$, holds true for spectral measures $E\colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$). If, in addition, $E^1 = E^A$ and $E^2 = E^B$ for some self-adjoint operators A and B, and if A and B are bounded, then $\mathrm{com}(E^1, E^2) = \{\varphi \in \mathscr{H}\colon AB\varphi = BA\varphi\} = \mathrm{com}(A, B)$. If $E^i\colon \mathscr{A}_i \to \mathscr{L}(\mathscr{H})^+$, $i = 1, 2$, are semispectral measures, then $\mathrm{com}(E^1, E^2)$ does not reduce to such a simple form. Even in the case of commutative semispectral measures E^i, $i = 1, 2$, $\mathrm{com}(E^1, E^2)$ is usually a proper subspace of $\mathrm{com}_{(2)}(E^1, E^2)$ $(= \{\varphi \in \mathscr{H}\colon E^1(X) E^2(Y)\varphi = E^2(Y) E^1(X)\varphi$ for all $X \in \mathscr{A}_1$, $Y \in \mathscr{A}_2\})$. In any case we have:

2.1.9. Corollary

Let $E^i\colon \mathscr{A}_i \to \mathscr{L}(H)^+$, $i = 1, 2$, be any two spectral or semispectral measures. If $\mathscr{L} = \mathscr{R}(E^1) \cup \mathscr{R}(E^2)$ is informationally complete, then dim $[\mathrm{com}(\mathscr{L})] \leqslant 1$.

The examples of Secs. 2.3 and 2.4 will demonstrate that the condition $\dim[\mathrm{com}(E^1, E^2)] \leqslant 1$ is not sufficient for the informational completeness of the pair (E^1, E^2). The final example of this Section demonstrates the existence of countable sets of projection operators which are informationally complete. Moreover, they can be used to define informationally complete discrete semispectral measures or informationally complete countable families of simple spectral measures (Remark 2.1.11).

2.1.10. Example

Let $\{\varphi_n\colon n \in \mathbb{N}\}$ be an orthonormal base of $\mathscr{H}$. The collection of one-dimensional projection operators $\mathscr{P} = \{P[\varphi_n], P[\sqrt{1/2}(\varphi_k + i^r \varphi_l)]\colon r = 0, 1, 2, 3;\ n, k, l \in \mathbb{N},\ k > l\}$ is informationally complete. Moreover, the set $\mathscr{P}$ is totally noncommutative.

Proof. Let $\{\varphi_n\colon n \in \mathbb{N}\}$ be an orthonormal base of $\mathscr{H}$. Then for any $A \in \mathscr{L}(\mathscr{H})$, $A = 0$ if and only if $\langle \varphi_m | A\varphi_n \rangle = 0$ for all $m, n \in \mathbb{N}$. The polarization identity $\langle \varphi_m | A\varphi_n \rangle = 1/4 \sum_{r=0}^{3} i^r \langle \varphi_m + i^r \varphi_n | A(\varphi_m + i^r \varphi_n) \rangle$, $m, n \in \mathbb{N}$, then shows that the projection operators $P[\sqrt{1/2}(\varphi_m + i^r \varphi_n)]$, $r = 0, 1, 2, 3$, $m, n \in \mathbb{N}$ form an informationally complete set. In fact the set $\mathscr{P} = \{P[\varphi_l], P[\sqrt{1/2}(\varphi_m + i^r \varphi_n)]: r = 0, 1, 2, 3; l, m, n \in \mathbb{N}, m > n\}$ is already informationally complete, as $T^* = T$ for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$ so that $\langle \varphi_n | T\varphi_m \rangle = \overline{\langle \varphi_m | T\varphi_n \rangle}$ for all $m, n \in \mathbb{N}$. The set $\mathscr{P}$ contains mutually noncommutative one-dimensional projection operators so that $\mathrm{com}(\mathscr{P}) \subseteq \mathrm{com}_{(2)}(\mathscr{P}) = \{0\}$. ∎

2.1.11. Remark

This Example (2.1.10) can be used to construct a *discrete* semispectral measure which is informationally complete. We omit the somewhat lengthy but straightforward proof; it amounts to showing that the set $\mathscr{P}$ is the union of at most countably many subsets $\mathscr{P}_i$, each of which contains a collection of mutually orthogonal projections adding up to unity. Introducing a sequence of positive numbers $\lambda_i > 0$ such that $\sum \lambda_i = 1$, then defining the sets $\lambda_i \mathscr{P}_i = \{\lambda_i P | P \in \mathscr{P}_i\}$ of effects, one shows that $\sum_i \sum_{P_{ij} \in \mathscr{P}_i} \lambda_i P_{ij} = \sum \lambda_i I = I$. Thus $\mathscr{E} = \bigcup \lambda_i \mathscr{P}_i$ is an informationally complete set of effects satisfying the normalization condition. Therefore it can be used to construct a discrete (generalized) observable which then is informationally complete. Clearly, this observable is totally noncommutative.

2.2. General: Physical Motivations

Assume that a measurement of a physical quantity is performed on a physical system $\mathscr{S}$ and a certain result, a real number a, say, is obtained. What is the relevance of such a single measurement result on $\mathscr{S}$? This question splits up in a natural way into two parts according to the modes of the past and the future. The result of the presently performed measurement may be used to infer some properties of the system prior to the measurement, or it may allow one to deduce some properties of $\mathscr{S}$ immediately after the measurement.

According to quantum mechanics the measurement result $a \in \mathbb{R}$ means that the system $\mathscr{S}$ was initially, i.e., prior to the measurement in a state $T_i \in \mathscr{T}_s(\mathscr{H})_1^+$, *i*-initial, in which it was possible to obtain this result, i.e., $E_{T_i}^A(\{a\}) \neq 0$. The measurement could also have left the system in a state $T_f \in \mathscr{T}_s(\mathscr{H})_1^+$, *f*-final, in which the quantity A has the measured value, i.e., $E_{T_f}^A(\{a\}) = 1$. As the pure states $P[\varphi]$, $\varphi \in \mathscr{H}_1$, characterize the maximal sets of properties that the system may possess at a time, it would be most favorable if the measurement would lead to a determination of a pure state of the system either before or after the measurement. As was said earlier, these two options turn out to be complementary aspects of measurements.

It is a well-established fact of quantum mechanics that a discrete maximal quantity, or a complete set of commuting discrete quantities admit measurements which lead to an optimal determination of the future of the system. In the case of noncommuting quantities such measurements do not exist but still some reasonable determinations of the future of the system are feasible. In Sec. 3 such options shall be investigated. Here we shall concentrate on the question of the determination of the past, i.e., the premeasurement state of the system on the basis of certain measurement results. We now drop the subindex i in referring to the initial state of the system.

In general, a single measurement result does not suffice to infer the initial state of the system. According to quantum mechanics this is simply because there is no projection operator $P \in \mathscr{P}(\mathscr{H})$ (nor any effect $E \in \mathscr{E}(\mathscr{H})$) for which the set of 'possible initial states' $\{T \in \mathscr{T}_s(\mathscr{H})_1 : \operatorname{tr}(TP) \neq 0\}$ would be a singleton set $\{T\}$. But we may repeat the same A-measurement under the same conditions many times, say N times. This then leads to a (factual) finite measurement result sequence $\Gamma^A_{c,N} = \{a_1, \ldots, a_N\}$, where c refers to "the same conditions to which the system $\mathscr{S}$ is repeatedly subjected." But the situation with $\Gamma^A_{c,N}$ is not essentially better than that with $\Gamma^A_{c,1}$. We have to take the step of inductive generalization to replace the (factual) finite sequence $\Gamma^A_{c,N}$ with a (conceptual) infinite sequence $\Gamma^A_c = \{a_1, a_2, \ldots\}$. The problems of induction and statistical inference are, of course, foremost here. As we do not intend to propose any solutions to those questions here we simply omit them now. (There is a vast literature on the subject matter, the references to van Fraassen[11] and Salmon[21] are relevant illustrations of that.) Rather, we refer to the quantum theory of measurement (Ozawa[13] and to the relative frequency interpretation of probability (van Fraassen[11]) to give the measurement statistics interpretation of quantum mechanics (Cassinelli and Lahti[14]). In that interpretation the quantum mechanical probabilities E^A_T are obtained, in a systematic way, as relative frequencies in the measurement result sequences $\Gamma^A_T = \{a_1, \ldots, a_k, \ldots\}$ obtained in repeating the same A-measurement under the same conditions, given now by the state T, infinitely many times; $E^A_T(X) = \operatorname{relf}(X, \Gamma^A_T) := \lim_{k \to \infty} \sum_{i=1}^k \chi_X(\Gamma^A_T(i))$, with $\Gamma^A_T(i) = a_i$.

The determination of the past of the system $\mathscr{S}$ on the basis of measurements on $\mathscr{S}$ now appears as the determination of the state T of the system from a given measurement result sequence Γ^A_T, say. Formally, the question could be restricted to determining the state T of the system from a given QM-probability measure E^A_T, say. However, the physical motivation of the problem is most apparent with respect to the measurement result sequences Γ^A_T.

Consider a set of physical quantities $E^1, \ldots, E^n$, represented as spectral or semispectral measure $\mathscr{A}_i \to \mathscr{L}(\mathscr{H})^+$, $i = 1,..., n$. The informational completeness of the set $\{E^1,..., E^n\}$ now means that the measurement result sequences $\Gamma_T^{E^1},..., \Gamma_T^{E^n}$ always determine the (initial) state of the system as in that case $[T]^{E^1,...,E^n} = \{T\}$ for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$. The state distinction power of informationally complete sets of physical quantities is optimal. Their measurement result sequences can distinguish between all the states of the system. According to Corollary 2.1.4, no single physical quantity A, represented as a spectral measure E^A, nor any complete set of commuting quantities $A_1,..., A_n$ is informationally complete. The measurement result sequences Γ_T^A, or $\Gamma_T^{A_1},..., \Gamma_T^{A_n}$ of such quantities do not suffice to infer, in general, the state T in which the sequences were obtained. The sequence Γ_T^A determines the state T if and only if T is a nondegenerate eigenstate of A (Thm. 2.1.5). Results of Thms. 2.1.2, 2.1.8, and 2.1.9 show that a certain amount of noncommutativity is always needed in order the measurement result sequences would lead to an optimal state determination. This already shows that optimal determinations of the past and of the future of the system are mutually exclusive. In the next two subsections we shall investigate the state distinction power of the important pairs of complementary position and momentum observables and the components of the spin quantities.

2.3. Example. The Pauli Problem

In a footnote on p. 17 to his 1933 *Wellenmechanik*, Pauli[(15)] remarked that the question under which conditions the position and the momentum distributions $|\varphi|^2$ and $|\hat{\varphi}|^2$ define the state function φ uniquely (modulo a phase factor) "has still not been investigated in all its generality." This problem is now known as the Pauli problem, and it simply refers to the question of informational completeness of the canonically conjugate position and momentum observables.

To some extent the Pauli problem is still open, though there is now a good number of important results on that. Here we shall attempt to give a systematic presentation of the main questions and results related to this problem.

It is to be emphasized that there is a rich literature on investigations, both theoretical and experimental, of, e.g., electron spatial and momentum distributions. Usually such investigations do not explicitly refer to the Pauli problem but they rather develop theoretical tools for determining in some important concrete cases the position and momentum distributions of the physical system. The experimental investigations are then considered as testing the validity of those, usually approximative, methods. However, the Pauli problem appears implicitly in some important cases; e.g. in the investigations of the atomic electron densities, the set of admissible states is usually restricted to the energy bound states. In such cases the Pauli problem appears in a restrictive form (cf. Remark 2.1.1), and the experimentally obtained spatial and momentum distributions do then, in principle, suffice to determine the state of the system. (Cf. Secs. 2.3.3 and 2.3.4. See e.g. Williams[16] for a review of the mentioned investigations.)

Let us introduce Q and P as the usual position and momentum observables in the sense of a Schrödinger couple. Hence without any loss in generality we may identify the Hilbert space $\mathscr{H}$ as the Lebesque function space $\mathscr{H}=\mathscr{L}^2(\mathbb{R}, dx)$. Position Q is then defined as the multiplicative operator $(Q\varphi)(x)=x\varphi(x)$ with the domain $\mathrm{dom}(Q)=\{\varphi\in\mathscr{H}: \int_{\mathbb{R}} id^2\, dE^Q_{P[\varphi]}<\infty\}$, and with the spectral projections $E^Q(X)\varphi:=\chi_X\cdot\varphi$, $X\in\mathscr{B}(\mathbb{R})$, $\varphi\in\mathscr{H}$. (Here χ_X denotes the characteristic function of the set X.) The conjugate momentum P may then be defined through the spectral projections $E^P(X)=F^{-1}E^Q(X)F$, $X\in\mathscr{B}(\mathbb{R})$, where F is the Fourier–Plancherel operator on $\mathscr{H}$. (Here we have used the units $h/2\pi=1$.) Then P is nothing else than the usual differential operator $(P\varphi)(x)=-i(d/dx)\,\varphi(x)$ with the domain $\mathrm{dom}(P)=F^{-1}(\mathrm{dom}(Q))$, and with the same spectrum as Q. There are two well known properties of the pair (Q, P) which follow from their Fourier connection $P=F^{-1}QF$. Firstly, Q and P satisfy the uncertainty relation as $\mathrm{Var}(Q, T)\cdot\mathrm{Var}(P, T)\geqslant 1/4$ for any $T\in\mathscr{T}_s(\mathscr{H})^+_1$. (Here, e.g., $\mathrm{Var}(Q, T)$ denotes the variance of the probability measure E^Q_T.) Secondly, Q and P are complementary as $E^Q(X)\wedge E^P(Y)=0$ for any bounded $X, Y\in\mathscr{B}(\mathbb{R})$. (Here $\wedge$ denotes the meet operation in the projection lattice $\mathscr{P}(\mathscr{H})$.) Either one of the previous two 'coupling properties' of Q and P implies that the pair (Q, P) is also totally noncommutative, i.e., their commutativity domain $\mathrm{com}(Q, P)$ is the null space $\{0\}$. (For that see Lahti and Ylinen,[17] which contains also references to other relevant original papers.)

2.3.1. The informational incompleteness of both the position Q and the momentum P is already given by Thm. 2.1.2. But as the point spectra of Q and P are empty we also have the stronger result, based on Thm. 2.1.5, that neither Q nor P is informationally complete w.r.t. any state, i.e., for any $T\in\mathscr{T}_s(\mathscr{H})^+_1$

$$[T]^Q\neq\{T\},\qquad [T]^P\neq\{T\} \tag{13}$$

Consequently, neither the position measurement statistics Γ^Q_T nor the momentum measurement statistics Γ^P_T ever determine the state T of the system. A similar result is well known also in the classical phase space mechanics.

2.3.2. Position Q and momentum P are complementary and thus also totally noncommutative. Hence the pair (Q, P) fulfills a necessary condition, Cor. 2.1.9, for being informationally complete. But as already remarked earlier, this pair is not informationally complete. Indeed, if $\varphi \in L^2(\mathbb{R}, dx)$ is a unit vector for which $\varphi(x) = |\varphi(x)|\, e^{i\theta(x)}$, $0 \leqslant \theta(x) < 2\pi$, $|\varphi(x)| = |\varphi(-x)|$ a.e. $x \in \mathbb{R}$, $\theta(x) + \theta(-x) \neq \text{constant} \pmod{2\pi}$ then $\varphi_1(x) := |\varphi(x)|\, e^{-i\theta(-x)}$ represents another state than $\varphi(x)$, i.e. $P[\varphi_1] \neq P[\varphi]$, but $|\varphi_1(x)|^2 = |\varphi(x)|^2$ and $|\hat{\varphi}_1(y)|^2 = |\hat{\varphi}(y)|^2$, i.e., Q and P do not distinguish between these states. (Here $\hat{\varphi}$ denotes the Fourier transform of φ.) (This example can be found in Prugovecki[6] or in Reichenbach,[18] but Corbett and Hurst[19] contains a wider class of similar examples.) The informational incompleteness of the pair (Q, P) means, in particular, that the combined measurement statistics (Γ_T^Q, Γ_T^P) do not, in general, suffice to infer the intial state T of the system. This is contra to the situation in classical physics, and it may be taken as an illustration of the "surplus information" (von Weizsäcker[20]) coded in a quantum (pure) state when compared with its classical counterpart.

We shall now turn to study questions posed in Eqs. (10) and (11) of Subsection 2.1 for the canonical pair (Q, P). Though these questions are related to each others we shall study them here separately starting with the first one.

2.3.3. Let φ be a unit vector in $\mathrm{dom}(|Q|^{1/2}) \cap \mathrm{dom}(|P|^{1/2})$. This condition quarantees that both $\mathrm{Exp}(Q, P[\varphi])$ $(:= \int_{\mathbb{R}} i\, dE^Q_{P[\varphi]})$ and $\mathrm{Exp}(P, P[\varphi])$ exist and are finite. (Recall also that e.g., the domain of Q is contained in the domain of $|Q|^{1/2}$.) Applying the two-parameter family of unitary operators $\mathrm{U}_{qp} = e^{i(pQ - qP)}$, $(q, p) \in \mathbb{R}^2$, on φ we obtain the following subset of pure states $\mathcal{T}(\varphi) := \{P[\mathrm{U}_{qp}\varphi]\colon (q, p) \in \mathbb{R}^2\}$. We claim that

$$[T]^{Q,P} \cap \mathcal{T}(\varphi) = \{T\} \qquad \text{for any} \quad T \in \mathcal{T}(\varphi) \tag{14}$$

Indeed, if $E^Q_{P[\mathrm{U}_{q'p'}\varphi]} = E^Q_{P[\mathrm{U}_{qp}\varphi]}$, then the expectations $\mathrm{Exp}(Q, P[\mathrm{U}_{qp}\varphi]) = q$ and $\mathrm{Exp}(Q, P[\mathrm{U}_{q'p'}\varphi]) = q'$ are the same, i.e., $q = q'$. Similarly, the condition $E^P_{P[\mathrm{U}_{qp}\varphi]} = E^P_{P[\mathrm{U}_{q'p'}\varphi]}$ now implies that $p = p'$. We recall that the set $\{\mathrm{U}_{qp}\varphi\colon (q, p) \in \mathbb{R}^2\} \subset \mathcal{H}_1$ is overcomplete (i.e., $\int_{\mathbb{R}^2} |\mathrm{U}_{qp}\varphi\rangle\langle \mathrm{U}_{qp}\varphi|\, dq\, dp = 2\pi I$) and its linear span is dense in $\mathcal{H}$ (see e.g., Klauder and Skagerstam[21]).

There is an important application of the result of Sec. 2.3.3 referring to the so-called coherent or minimal uncertainty states. Consider the uncertainty product $\mathrm{Var}(Q, T) \cdot \mathrm{Var}(P, T)$ of Q and P. This product has the well known lower bound 1/4 (in the units $h/2\pi = 1$) which is reached exactly by the pure states (cf. Thirring,[22] Remark 3.1.14.2) of the form $P[\mathrm{U}_{qp}\varphi^G_\sigma]$, where φ^G_σ is the Gaussian $\varphi^G_\sigma(x) = \pi^{-1/4}\sigma^{-1/2}e^{-x^2/4\sigma}$ with the variance

$\mathrm{Var}(|\varphi_\sigma^G|^2)=\sigma^2$. The set $\mathscr{T}(\varphi_\sigma^G)$ is a special case of the earlier sets $\mathscr{T}(\varphi)$, $\varphi\in\mathrm{dom}(|Q|^{1/2})\cap\mathrm{dom}(|P|^{1/2})$, so that

$$[T]^{Q,P}\cap\mathscr{T}(\varphi_\sigma^G)=\{T\}\qquad \text{for any}\quad T\in\mathscr{T}(\varphi_\sigma^G) \tag{15}$$

The restriction of $[T]^{Q,P}$, $T\in\mathscr{T}_s(\mathscr{H})_1^+$, to $[T]^{Q,P}\cap\mathscr{T}(\varphi_\sigma^G)$, $T\in\mathscr{T}(\varphi_\sigma^G)$, has now a special relevance: If the combined measurement statistics (Γ_T^Q,Γ_T^P) is such that $\mathrm{Var}(\Gamma_T^Q)\cdot\mathrm{Var}(\Gamma_T^P)=1/4$, then $T\in\mathscr{T}(\varphi_\sigma^G)$ with $\sigma^2=\mathrm{Var}(\Gamma_T^Q)$, and it is uniquely determined by the expectations $\mathrm{Exp}(\Gamma_T^Q)$ and $\mathrm{Exp}(\Gamma_T^P)$. We recall that the set $\mathscr{T}(\varphi_\sigma^G)$ of minimal uncertainty states can also be characterized as the set of maximal information states w.r.t. a global Shannon type information functional for the pair (Q, P). (For that see e.g., Grabowski,[23] cf. also Remark 4.1.1.)

2.3.4. We next consider the question of Eq. (11) in Subsection 2.1 asking first whether there is some physical quantity A, with the spectral measure E^A, such that the triple (Q, P, A) would be informationally complete. A physically important class of such quantities A are of the form

$$A=f(Q)+g(P) \tag{16}$$

where f and g are real valued Borel functions in $\mathbb{R}$ such that $f(Q)+g(P)$ is self-adjoint. We note first that if either $f=0$ or $g=0$, then $\mathscr{R}(E^Q)\cup\mathscr{R}(E^P)=\mathscr{R}(E^Q)\cup\mathscr{R}(E^P)\cup\mathscr{R}(E^A)$, as e.g., $\mathscr{R}(E^{f(Q)})\subseteq\mathscr{R}(E^Q)$ for any f. Thus in order A would increase the state distinction power of the pair (Q, P), both f and g must be nonzero. Furthermore, as

$$[T]^{Q,P,A}=[T]^{Q,P}\cap[T]^A=[T]^Q\cap[T]^P\cap[T]^A\supseteq\{T\} \tag{17}$$

for any $T\in\mathscr{T}_s(\mathscr{H})_1^+$, the triple (Q, P, A) is informationally complete w.r.t. any set $\mathscr{T}\subset\mathscr{T}_s(\mathscr{H})_1^+$ w.r.t. which either (Q, P) or A is informationally complete. Due to the fundamental role of the pair (Q, P) it seems also interesting to ask whether the informational completeness of A w.r.t. a set $\mathscr{T}\subset\mathscr{T}_s(\mathscr{H})_1^+$ implies the informational completeness of the pair (Q, P) w.r.t. that set. According to Thm. 2.1.5, $[T]^A=\{T\}$ if and only if $T\in\mathscr{T}(A)$. Assuming that $\mathscr{T}(A)$ is nonempty, i.e., $A=f(Q)+g(P)$ has nondegenerate eigenstates, then also

$$[T]^{Q,P}\cap\mathscr{T}(A)=\{T\} \tag{18}$$

for any $T\in\mathscr{T}(A)$. (This should not be confused with the trivial result $[T]^{Q,P}\cap[T]^A=\{T\}$ for any $T\in\mathscr{T}(A)$.) Indeed, if $P[\varphi_i]$, $P[\varphi_j]\in\mathscr{T}(A)$, and $E^Q_{P[\varphi_i]}=E^Q_{P[\varphi_j]}$ and $E^P_{P[\varphi_i]}=E^P_{P[\varphi_j]}$, then also $E^{f(Q)}_{P[\varphi_i]}=E^{f(Q)}_{P[\varphi_j]}$ and $E^{g(P)}_{P[\varphi_i]}=E^{g(P)}_{P[\varphi_j]}$, so that $\langle\varphi_i|f(Q)\varphi_i\rangle=\langle\varphi_j|f(Q)\varphi_j\rangle$ and $\langle\varphi_i|\,g(P)\varphi_i\rangle=$

$\langle\varphi_j | g(P)\varphi_j\rangle$. But then also $\langle\varphi_i | (f(Q)+g(P))\varphi_i\rangle = \langle\varphi_j | (f(Q)+g(P))\varphi_j\rangle$, i.e., the nondegenerate eigenvalues of A associated with $P[\varphi_i]$ and $P[\varphi_j]$ are the same, so that $P[\varphi_i]=P[\varphi_j]$. (Recall, that for any $P[\varphi_i]\in\mathcal{T}(A)$, $\varphi_i\in\text{dom}(A)=\text{dom}(f(Q))\cap\text{dom}(g(P))$ so that the expectations like e.g., $\text{Exp}(f(Q), P[\varphi_i])=\langle\varphi_i | f(Q)\varphi_i\rangle$ are now finite.)

The relevance of the result of Eq. (18) is similar to that of Eq. (15). There may be some physical arguments to restrict the study of $[T]^{Q,P}$, $T\in\mathcal{T}_s(\mathcal{H})_1^+$, to those of the form $[T]^{Q,P}\cap\mathcal{T}$, $\mathcal{T}\subset\mathcal{T}_s(\mathcal{H})_1^+$. If for example, $A=f(Q)+g(P)$ represents the Hamilton operator of the system $\mathcal{S}$ then $\mathcal{T}(A)$ consists of the nondegenerate energy eigenstates of $\mathcal{S}$ and it may be justified to study the position and momentum distributions of $\mathcal{S}$ just for those states in $\mathcal{T}(A)$ (cf. the discussion in the beginning of this Section).

This result of Eq. (18) is an illustration of the attempts to find an informational completion of the pair (Q, P). Some interesting special cases of the operators A of the form in Eq. (16) have been investigated in detail in the literature (see e.g., Corbett and Hurst[19]). However, no informationally complete triplets $(Q, P, f(Q)+g(P))$ are known. In the case $g(P)=P^2$ and $f(Q)\in\mathcal{L}(\mathcal{H})^+$ the informational incompleteness of such pairs have even been demonstrated (Pavicic[24]). In lack of a general result we close this subsection with challenging the reader to (dis-)prove the following statement: No (Q, P, A), with a self-adjoint A, is informationally complete.

2.3.5. We continue to study the question of Eq. (11) of Subsection 2.1 for the canonical pair (Q, P). A physically relevant informational completion of $\mathcal{R}(E^Q)\cup\mathcal{R}(E^P)$ in $\mathcal{P}(\mathcal{H})$ were of the form $\mathcal{R}(E^Q)\cup\mathcal{R}(E^P)\cup\mathcal{R}(E^A)$, where A is a physical quantity (a self-adjoint operator) of the form $A=f(Q)+g(P)$. Such an extension may single out some interesting subsets of states, like $\mathcal{T}(A)$ in Sec. 2.3.4 or $\mathcal{T}(\varphi_\sigma^G)$ in Sec. 2.3.3, w.r.t. which the pair (Q, P) is informationally complete, at least in a restricted sense as indicated e.g., in Eqs. (15) and (18). However, no such extension is known to lead to informational completeness. In the most interesting cases, with $A=P^2+V(Q)$, $V(Q)\in\mathcal{L}(\mathcal{H})^+$, the triple (Q, P, A) is demonstrably informationally incomplete. Thus it seems natural, or even necessary, to seek for the informational completion of the pair (Q, P) in the set $\mathcal{E}(\mathcal{H})$ of effects by replacing the pair (Q, P) with an unsharp pair (Q_f, P_g). In studying the state distinction property of the pair (Q, P) such a replacement is justified by the fact that the pairs (Q, P) and (Q_f, P_g) are informationally equivalent, so that they have exactly the same state distinction power (see the following).

Let $f:\mathbb{R}\to\mathbb{R}$ be a probability density function, or a confidence function (i.e., $f(x) \geqslant 0$ a.e., $x \in \mathbb{R}$, $\int_{\mathbb{R}} f(x)\,dx = 1$). The pair (Q, f) defines a semispectral measure $E^{Q_f}: \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})^+$ through the formula $E^{Q_f}(X) = (f * \chi_X)(Q)$, $X \in \mathscr{B}(\mathbb{R})$, where $f * \chi_X$ is the convolution of the characteristic function χ_X with the confidence function f (see e.g. Davies[8]). The effects $E^{Q_f}(X)$ are thus obtained by "smearing" the original spectral projections $\chi_X(Q)$ with the confidence function f. E^{Q_f} is called an unsharp position observable, and it is occasionally denoted by Q_f. Similarly one defines unsharp momentum observables P_g with any confidence function g. The physical relevance of such unsharp quantities have been discussed extensively in the literature, see e.g., Ali and Doebner,[5] Prugovecki,[6] Busch and Lahti,[25] or Busch.[26]

By construction, Q_f and P_g are commutative semispectral measures. Thus Thm. 2.1.2 holds for them, too.

The informational equivalence of the pairs (Q, P) and (Q_f, P_g), where f and g are arbitrary confidence functions, can now be formulated:

$$[T]^{Q,P} = [T]^{Q_f, P_g} \qquad \text{for any} \quad T \in \mathscr{T}_s(\mathscr{H})_1^+ \tag{19}$$

As $[T]^{Q,P} = [T]^Q \cap [T]^P$, $T \in \mathscr{T}_s(\mathscr{H})_1^+$, the result of Eq. (19) follows if only

$$[T]^Q = [T]^{Q_f} \quad \text{and} \quad [T]^P = [T]^{P_g} \quad \text{for any} \quad T \in \mathscr{T}_s(\mathscr{H})_1^+ \tag{20}$$

Ali and Doebner[5] have shown this for the pure states $P[\varphi]$, $\varphi \in \mathscr{H}_1$, but the extension of their proof to all states is straightforward.

The informational equivalence of the pairs (Q, P) and (Q_f, P_g) means, in particular, that all the results concerning the pair (Q, P) hold true for the pairs (Q_f, P_g) as well. Especially, the combined measurement result sequences (Γ_T^Q, Γ_T^P) and $(\Gamma_T^{Q_f}, \Gamma_T^{P_g})$ have exactly the same state distinction power, and, in general, do not suffice to determine the initial state T of the system.

One of the important advantages of replacing the sharp pair (Q, P) with and unsharp pair (Q_f, P_g) is that the latter pair may have a natural informationally complete extension. We shall study such cases next.

Consider a pair (Q_f, P_g) of unsharp position and momentum for which the confidence functions are Fourier-related, i.e., $f = |\varphi|^2$ and $g = |\hat{\varphi}|^2$ for some $\varphi \in \mathscr{H}_1$. (Here, again, $|\hat{\varphi}|$ is the Fourier transform of φ.) In that case Q_f and P_g are obtained as the marginal observables of the joint observable

$$E: \mathscr{B}(\mathbb{R}^2) \to \mathscr{L}(\mathscr{H})^+, \qquad Z \mapsto E(Z) := (2\pi)^{-1} \int_Z |\mathrm{U}_{qp}\varphi\rangle\langle \mathrm{U}_{qp}\varphi| \, dq\, dp \tag{21}$$

where U_{qp}, $q, p \in \mathbb{R}$, is the two-parameter family of unitary operators of Sec. 2.3.3, and $\varphi \in \mathscr{H}_1$. Indeed, then $E(X \times \mathbb{R}) = E^{Q_f}(X)$ and $E(\mathbb{R} \times Y) =$

$E^{P_g}(Y)$ for any $X, Y \in \mathscr{B}(\mathbb{R})$, where now $f = |\varphi|^2$ and $g = |\hat{\varphi}|^2$ (see e.g., Davies[8]). As the ranges of Q_f and P_g are now contained in the range of E we have $[T]^{Q_f, P_g} \supset [T]^E$ for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$. But we also have

$$[T]^E = \{T\} \qquad \text{for any} \quad T \in \mathscr{T}_s(\mathscr{H})_1^+ \tag{22}$$

whenever E is the joint observable of Eq. (21) with $\varphi \in \text{dom}(Q) \cap \text{dom}(P)$ such that $\langle \varphi \mid \text{U}_{qp}\varphi \rangle \neq 0$ a.e. $(q, p) \in \mathbb{R}^2$. Indeed, if $T' \in [T]^E$, i.e., $\text{tr}(T'E(Z)) = \text{tr}(TE(Z))$ for any $Z \in \mathscr{B}(\mathbb{R}^2)$, then by Eq. (21) $\int_Z \langle \text{U}_{qp}\varphi \mid (T' - T)\, \text{U}_{qp}\varphi \rangle\, dq\, dp = 0$ for any $Z \in \mathscr{B}(\mathbb{R}^2)$. But this shows that $\langle \text{U}_{qp}\varphi \mid (T' - T)\, \text{U}_{qp}\varphi \rangle = 0$ a.e., $(q, p) \in \mathbb{R}^2$. Applying here again Eq. (21) we obtain $\int_Z \langle \varphi \mid \text{U}_{q'-q, p'-p}\varphi \rangle \langle \text{U}_{q'p'}\varphi \mid (T' - T)\, \text{U}_{qp}\varphi \rangle = 0$ for any $Z \in \mathscr{B}(\mathbb{R}^2)$ and for a.e. $(q, p) \in \mathbb{R}^2$, i.e., $\langle \varphi \mid \text{U}_{q'-q, p'-p}\varphi \rangle \langle \text{U}_{q'p'}\varphi \mid (T' - T)\, \text{U}_{qp}\varphi \rangle = 0$ a.e. (q, p), $(q', p') \in \mathbb{R}^2$. By assumption, $\langle \varphi \mid \text{U}_{q'-q, p'-p}\varphi \rangle \neq 0$ a.e. $(q' - q,\ p' - p) \in \mathbb{R}^2$ so that $\langle \text{U}_{q'p'}\varphi \mid (T' - T)\, \text{U}_{qp}\varphi \rangle = 0$ a.e. (q, p), $(q', p') \in \mathbb{R}^2$, i.e. $T' - T = 0$. (This proof is essentially due to Klauder and McKenna,[27] but see also Prugovecki,[6] Klauder and Skagerstam,[21] or Werner.[28]

We show next that this joint observable E of Q_f and P_g is totally noncommutative. As $\text{com}(\mathscr{R}(E)) \subset \text{com}(\mathscr{R}(E^{Q_f}) \cup \mathscr{R}(E^{P_g})) \subset \text{com}_{(2)}(\mathscr{R}(E^{Q_f}) \cup \mathscr{R}(E^{P_g})) = \text{com}_{(2)}(Q_f, P_g)$ it suffices to show that $\text{com}_{(2)}(Q_f, P_g) = \{0\}$. Assume that $\varphi \in \text{com}_{(2)}(Q_f, P_g)$, $\varphi \neq 0$, i.e. $(E^{Q_f}(X)\, E^{P_g}(Y) - E^{P_g}(Y)\, E^{Q_f}(X))\varphi = 0$ for all $X, Y \in \mathscr{B}(\mathbb{R})$. Due to the covariance of Q_f and P_g under the Weyl group (i.e. $E^{Q_f}(X + q) = \text{U}_{qp} E^{Q_f}(X)\, \text{U}_{qp}^{-1}$ for any $X \in \mathscr{B}(\mathbb{R})$, $q, p \in \mathbb{R}$) we then have also $\langle \text{U}_{qp}\psi \mid (E^{Q_f}(X)\, E^{P_g}(Y) - E^{P_g}(Y)\, E^{Q_f}(X))\, \text{U}_{qp}\varphi \rangle = 0$ for any $\psi \in \mathscr{H}$, and for all $q, p \in \mathbb{R}$, X, $Y \in \mathscr{B}(\mathbb{R})$. But this implies that $\text{com}_{(2)}(Q_f, P_g) = \mathscr{H}$, i.e., $E^{Q_f}(X)\, E^{P_g}(Y) = E^{P_g}(Y)\, E^{Q_f}(X)$ for all $X, Y \in \mathscr{B}(\mathbb{R})$, as the linear span of any $\{\text{U}_{qp}\xi\colon q, p \in \mathbb{R}\}$, $\xi \in \mathscr{H}$, $\xi \neq 0$, is dense in $\mathscr{H}$. The result $\text{com}_{(2)}(Q_f, P_g) = \mathscr{H}$ is known to be wrong (see, Busch, Schonbek and Schroeck[29]). Hence $\text{com}_{(2)}(Q_f, P_g) = \{0\}$.

Results of Eqs. (19) and (22) now have the following physical content: The combined statistics $\{(E_T(X \times \mathbb{R}),\ E_T(\mathbb{R} \times Y))\colon X, Y \in \mathscr{B}(\mathbb{R})\}$ are "informationally equivalent" with the combined statistics $\{(E_T^Q(X), E_T^P(Y))\colon X, Y \in \mathscr{B}(\mathbb{R})\}$. In general, they do not determine the initial state $T \in \mathscr{T}_s(\mathscr{H})_1^+$ of the system. However, the richer statistics $\{E_T(X \times Y)\colon X, Y \in \mathscr{B}(\mathbb{R})\}$ always suffice to determine the state T of the system. This result is very important as the joint observable E can be taken to describe the approximate path of e.g. an electron.[26]

2.4. Example. Spin-1/2 System

As another example of determining the past and the future of a physical system through measurements we consider the spin measurements on a spin-1/2 system. It is well known that spin measurements show up similar features as position-momentum measurements but in a much simpler form. In this subsection we shall discuss mainly the problem of determining the past of a spin-1/2 system along the ideas and results of subsections 2.1 and 2.2. The aspects of measurements which address more directly to the determination of the future of the system will again be postponed to Sec. 3.

The spin-1/2 system has been discussed in great detail in the literature. As a standard reference and introduction to the topic we shall take Chapter 4 in the monograph of Beltrametti and Cassinelli.[2] Measurements of unsharp spin quantities have also been investigated in detail, the paper of Busch[30] serving as our main reference on that topic.

Due to the importance of the spin measurements we shall collect here a number of known results in attempting to give a systematic exposition of the present problem.

2.4.1. *Preliminaries*

We consider here only the spin quantities of a spin-1/2 system. Thus the Hilbert space of the system may now be identified as $\mathscr{H} = \mathbb{C}^2$. In this case, like in any finite dimensional Hilbert space, the classification of the operators in $\mathscr{L}(\mathscr{H})$, $\mathscr{L}_s(\mathscr{H})$, $\mathscr{L}(\mathscr{H})^+$, $\mathscr{E}(\mathscr{H})$, $\mathscr{P}(\mathscr{H})$, and $\mathscr{T}_s(\mathscr{H})_1^+$ is straightforward, and they all can be constructed from the one-dimensional projection operators in a simple way. This means, in particular, that the ranges of the $\mathscr{L}(\mathscr{H})^+$-valued spectral and semispectral measures can now also easily be characterized. We shall present first these mathematical results here, applying them subsequently to the physical problems in question.

Let $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the Pauli spin matrices. Then any linear operator on $\mathscr{H} = \mathbb{C}^2$ can be written as

$$A = \mathbf{a} \cdot \boldsymbol{\sigma} + a_4 I \tag{23}$$

where $\mathbf{a} = (a_x, a_y; a_z) \in \mathbb{C}^3$, $a_4 \in \mathbb{C}$, and I is the identity operator on $\mathscr{H}$. The self-adjoint operators on $\mathscr{H}$ are thus exactly those operators A in Eq. (23) for which $(\mathbf{a}, a_4) \in \mathbb{R}^4$. The spectrum of any $A \in \mathscr{L}_s(\mathscr{H})$ is then $\mathrm{sp}(A) = \{a_4 - \|\mathbf{a}\|, a_4 + \|\mathbf{a}\|\}$, which shows that $A \in \mathscr{L}(\mathscr{H})^+$ if and only if (iff, for short) $\|\mathbf{a}\| \leqslant a_4$. In the case of positive operators $A \in \mathscr{L}(\mathscr{H})^+$ it is customary to write $A = \mathbf{a} \cdot \boldsymbol{\sigma} + a_4 I$ in the form $A = \alpha/2(\mathbf{a}' \cdot \boldsymbol{\sigma} + I) = \alpha A(\mathbf{a}')$, with $A(\mathbf{a}') = 1/2(\mathbf{a}' \cdot \boldsymbol{\sigma} + I)$, $\alpha \geqslant 0$, $\|\mathbf{a}'\| \leqslant 1$, in which case the eigenvalues of A are $\alpha/2(1 - \|\mathbf{a}'\|)$ and $\alpha/2(1 + \|\mathbf{a}'\|)$. With this convention we have:

$$A \in \mathscr{L}(\mathscr{H})^+ \quad \text{iff} \quad A = \alpha A(\mathbf{a}), \alpha \geqslant 0, \mathbf{a} \in \mathbb{R}^3, \|\mathbf{a}\| \leqslant 1 \tag{24}$$

$$A \in \mathscr{E}(\mathscr{H}) \quad \text{iff} \quad A = \alpha A(\mathbf{a}), 0 \leqslant \alpha \leqslant 2/(1 + \|\mathbf{a}\|), \|\mathbf{a}\| \leqslant 1 \tag{25}$$

$$A \in \mathscr{P}(\mathscr{H}) \quad \text{iff} \quad A = \alpha A(\mathbf{a}), \alpha = 1, \|\mathbf{a}\| = 1, \text{ or } \alpha = 0, \text{ or } \mathbf{a} = \mathbf{0}, \alpha = 2 \tag{26}$$

$$A \in \mathscr{T}_s(\mathscr{H})_1^+ \quad \text{iff} \quad A = A(\mathbf{a}), \|\mathbf{a}\| \leqslant 1 \tag{27}$$

We shall now follow this convention.

As already remarked previously, the one-dimensional projection operators $A(\hat{a})$, $\hat{a} \in \mathbb{R}_1^3 := \{\mathbf{a} \in \mathbb{R}^3 \colon \|\mathbf{a}\| = 1\}$ (which are also the pure states of the system) determine the whole class $\mathscr{L}(\mathscr{H})$; they generate the base $\mathscr{T}_s(\mathscr{H})_1^+$ $(= \mathrm{conv}(\mathrm{Ex}(\mathscr{T}_s(\mathscr{H})_1^+)))$ of the positive cone $\mathscr{L}(\mathscr{H})^+$ $(= \mathscr{T}(\mathscr{H})^+)$ of $\mathscr{L}_s(\mathscr{H}) = \mathscr{L}(\mathscr{H})^+ - \mathscr{L}(\mathscr{H})^+$ which again determines $\mathscr{L}(\mathscr{H}) = \mathscr{L}_s(\mathscr{H}) + i\mathscr{L}_s(\mathscr{H})$. Indeed, any $A(\mathbf{a})$, $\|\mathbf{a}\| \leqslant 1$, can be written as $A(\mathbf{a}) = (1 + \|\mathbf{a}\|)/2\, A(\hat{a}) + (1 - \|\mathbf{a}\|)/2\, A(-\hat{a})$. The projection operator $A(-\hat{a})$ is, in fact, the complement of $A(\hat{a})$, i.e., $I - A(\hat{a}) = A(-\hat{a})$.

Let $A(\hat{a})$, $B(\hat{b}) \in \mathscr{P}(\mathscr{H})$. As $A(\hat{a})\, B(\hat{b}) = B(\hat{b})\, A(\hat{a}) = 1/2(\hat{a} \times \hat{b}) \cdot \boldsymbol{\sigma}$, and as $(\hat{a} \times \hat{b}) \cdot \boldsymbol{\sigma} = 0$ iff $\hat{a} = \pm\hat{b}$, the only Boolean subalgebras of $\mathscr{P}(\mathscr{H})$ are $\mathscr{R}(\hat{a}) = \{0, A(\hat{a}), A(-\hat{a}), I\}$, $\hat{a} \in \mathbb{R}_1^3$, and $\{0, I\}$. Hence the range of any nonconstant spectral measure $E \colon \mathscr{A} \to \mathscr{P}(\mathscr{H})$ is of that form, i.e., $\mathscr{R}(E) = \mathscr{R}(\hat{a})$ for some $\hat{a} \in \mathbb{R}_1^3$. The spectral measure $E \colon \mathscr{B}(\mathbb{R}) \to \mathscr{P}(\mathscr{H})$ associated with the unit vector $\hat{a} \in \mathbb{R}_1^3$ and with the spectrum $\{-1/2, 1/2\}$ determines *the $\hat{a}$-component of the spin* $S_{\hat{a}} = \frac{1}{2}A(\hat{a}) - \frac{1}{2}A(-\hat{a}) = \hat{a} \cdot \boldsymbol{\sigma}$. The projection operators $A(\hat{a})$, $\hat{a} \in \mathbb{R}_1^3$, are *the spin properties* of the system. The spin property $A(\hat{a})$ and its complement property $A(\hat{a})^{\perp} = I - A(\hat{a}) = A(-\hat{a})$ thus determine the spin component $S_{\hat{a}}$.

Consider the spin components $S_{\hat{a}}$ and $S_{\hat{b}}$. If $\hat{a} \neq \pm\hat{b}$, then $\mathrm{com}(S_{\hat{a}}, S_{\hat{b}}) = \{0\}$, i.e., any two different spin components $S_{\hat{a}}$ and $S_{\hat{b}}$ are totally noncommutative.

The structure of the semispectral measures $E \colon \mathscr{A} \to \mathscr{E}(\mathscr{H})$ is not as simple as that of the spectral measures $E \colon \mathscr{A} \to \mathscr{P}(\mathscr{H})$ even in the case $\mathscr{H} = \mathbb{C}^2$. There is, however, a similar characterization for any two effects $A = \alpha A(\mathbf{a})$ and $B = \beta B(\mathbf{b})$ to belong to the range $\mathscr{R}(E)$ of some semispectral measure $E \colon \mathscr{A} \to \mathscr{E}(\mathscr{H})$, i.e., to the *coexistence* of A and B. If A and B are coexistent, i.e., $A = E(X)$ and $B = E(Y)$ for some $E \colon \mathscr{A} \to \mathscr{E}(\mathscr{H})$, $X, Y \in A$, then $E(X \cap Y)$ is an effect satisfying the conditions $E(X \cap Y) \leqslant E(X) = A$, $E(X \cap Y) \leqslant E(Y) = B$, and $A + B - E(X \cap Y) = E(X) + E(Y) - E(X \cap Y) = E(X \cup Y) \leqslant I$. On the other hand, if for given $A, B \in \mathscr{E}(\mathscr{H})$ there is an effect $C \in \mathscr{E}(\mathscr{H})$ which satisfies the above conditions, i.e. $C \leqslant A$, $C \leqslant B$, $A + B - C \leqslant I$, then A and B are coexistent. Indeed, then e.g., the map E: $i \to E_i$, $i = 1, 2, 3, 4$, with $E_1 = C$, $E_2 = A - C$, $E_3 = B - C$, and $E_4 = I - A - B + C$ defines a semispectral measure $\mathscr{B}(\{1, 2, 3, 4\}) \to \mathscr{E}(\mathscr{H})$ with $A = E(\{1, 2\})$ and $B = E(\{1, 3\})$. (Note that if $A, B \in \mathscr{P}(\mathscr{H})$, $A \neq B$, then necessarily $C = 0$, and $A + B \leqslant I$ iff $\hat{a} \cdot \boldsymbol{\sigma} + \hat{b} \cdot \boldsymbol{\sigma} = 0$, i.e., $\hat{a} = -\hat{b}$, i.e. $AB = BA$.)

This coexistence condition can be explicated further for the case

$\alpha=\beta=1$ with using the special form in Eq. (25) of the effects on $\mathscr{H}=\mathbb{C}^2$ and the fact for any two effects C and A, $C\leqslant A$ iff $\mathrm{tr}(TC)\leqslant \mathrm{tr}(TA)$ for any $T\in\mathscr{T}_s(\mathscr{H})_1^+$, where now e.g., $\mathrm{tr}(TC)=\gamma/2(\mathbf{t}\cdot\mathbf{c}+1)$, with $T=T(\mathbf{t})$ and $C=\gamma C(\mathbf{c})$. This then leads to the following geometric characterization of the coexistence (cf. Busch[30]): Effects $A(\mathbf{a})$ and $B(\mathbf{b})$ are coexistent iff

$$\|\mathbf{a}+\mathbf{b}\|+\|\mathbf{a}-\mathbf{b}\|\leqslant 2 \tag{28}$$

Moreover, any effect $C\leqslant\gamma C(\mathbf{c})$, with $\frac{1}{2}\|\mathbf{a}+\mathbf{b}\|\leqslant\gamma\leqslant 1-\frac{1}{2}\|\mathbf{a}-\mathbf{b}\|$, is an effect satisfying $C\leqslant A$, $C\leqslant B$, $A+B-C\leqslant I$.

The condition of Eq. (28) shows that a maximal violation of coexistence of $A(\mathbf{a})$ and $B(\mathbf{b})$ is expected when $\mathbf{a}\cdot\mathbf{b}=0$. There are two special cases which deserve to be mentioned separately: (i) If $A=A(\hat{a})$, then the only coexistent $B(\mathbf{b})$ are those for which $\hat{b}\cdot\hat{a}=\pm 1$; (ii) If $A=A(\mathbf{a})\in\mathscr{E}(\mathscr{H})$, $\|\mathbf{a}\|<1$, then Eq. (28) can be satisfied in any direction $\hat{b}$ for some $\mathbf{b}$.

These results do by no means characterize all the semispectral measures $E\colon\mathscr{A}\to\mathscr{E}(\mathscr{H})$. They give only necessary and sufficient conditions for some pairs of effects $A, B\in\mathscr{B}(\mathscr{H})$ to belong to the range of some $E\colon\mathscr{A}\to\mathscr{E}(\mathscr{H})$. However, with respect to sharp as well as unsharp spin quantities this characterization is sufficient. We recall (see Busch[30]) that an effect $A=\alpha A(\mathbf{a})$ is an unsharp property if its eigenvalues satisfy the conditions: $\alpha/2(1+\|\mathbf{a}\|)>1/2$ and $\alpha/2(1-\|\mathbf{a}\|)<1/2$. An unsharp property $A=\alpha A(\mathbf{a})$ is *an unsharp spin property* if $A^{\perp}=I-A=\alpha A(-\mathbf{a})$. This is the case exactly when $\alpha=1$. An unsharp spin property $A(\mathbf{a})$ and its complement property $A(-\mathbf{a})$ determine again *a simple unsharp spin quantity* $\mathscr{R}(\mathbf{a})=\{0,\ A(\mathbf{a}),\ A(-\mathbf{a}),\ I\}$. The semispectral measure $E^{\mathbf{a}}$: $\mathscr{B}(\{-1/2, 1/2\})\to\mathscr{E}(\mathscr{H})$, with $E^{\mathbf{a}}(\{1/2\})=A(\mathbf{a})$, might again be taken as the canonical representation of $\mathscr{R}(\mathbf{a})$. The condition of Eq. (28) characterizes now the coexistence of any two simple (sharp or unsharp) spin quantities $\mathscr{R}(\mathbf{a})$ and $\mathscr{R}(\mathbf{b})$, or $E^{\mathbf{a}}$ and $E^{\mathbf{b}}$, with $\|\mathbf{a}\|\leqslant 1$, $\|\mathbf{b}\|\leqslant 1$. There is also a canonical way to construct a joint observable of the coexistent unsharp spin quantities $E^{\mathbf{a}}$ and $E^{\mathbf{b}}$. Indeed, if $A(\mathbf{a})$ and $B(\mathbf{b})$ are coexistent, and if C is an effect with $C\leqslant A(\mathbf{a})$, $C\leqslant B(\mathbf{b})$, and $A(\mathbf{a})+B(\mathbf{b})-C\leqslant I$, then $E\colon\mathscr{B}(\{1,2,3,4\})\to\mathscr{E}(\mathscr{H})$, as defined previously, is a semispectral measure generated by C. The function $f\colon\{1,2,3,4\}\to\{1/2, -1/2\}\times\{1/2, -1/2\}$, with $f(1)=(1/2, 1/2)$, $f(2)=(1/2, -1/2)$, $f(3)=(-1/2, 1/2)$, $f(4)=(-1/2, -1/2)$, induces then the semispectral measure $E_f\colon\mathscr{B}(\{1/2, -1/2\}\times\{1/2, -1/2\})\to\mathscr{E}(\mathscr{H})$, $X\to E_f(X):=E(f^{-1}(X))$ which is a joint observable of $E^{\mathbf{a}}$ and $E^{\mathbf{b}}$, i.e., $E^{\mathbf{a}}$ and $E^{\mathbf{b}}$ are its marginal observables. We denote it as $E^{\mathbf{a},\mathbf{b};C}$.

2.4.2. We shall now turn to study the question of informational completeness of the spin quantities starting with the usual (sharp) spin quantities

$S_{\hat{a}}$, or $\mathscr{R}(\hat{a})$, $\hat{a} \in \mathbb{R}^3_1$. For any $T = T(\mathbf{t}) \in \mathscr{T}_s(\mathscr{H})^+_1$, $[T]^{S_{\hat{a}}} = [T]^{\mathscr{R}(\hat{a})} = [T]^{A(\hat{a})} = \{T(\mathbf{t}') \in \mathscr{T}_s(\mathscr{H})^+_1 \colon (\mathbf{t}' - \mathbf{t}) \cdot \hat{a} = 0\}$. Identifying $\mathscr{T}_s(\mathscr{H})^+_1$ with $\mathbb{R}^3_{\leqslant 1} = \{\mathbf{t}' \in \mathbb{R}^3 \colon \|\mathbf{t}'\| \leqslant 1\}$ the equivalence class $[T]^{\mathscr{R}(\hat{a})}$ appears as the section of the unit sphere $\mathbb{R}^3_{\leqslant 1}$ by the plane $\{\mathbf{t}' \in \mathbb{R}^3 \colon (\mathbf{t}' - \mathbf{t}) \cdot \hat{a} = 0\}$. Clearly, $[T]^{\mathscr{R}(\hat{a})} = \{T\}$ only when $T = A(\hat{a})$ or $T = A(-\hat{a})$ (cf. Thm. 2.1.5). Similarly, $[T]^{S_{\hat{a}}, S_{\hat{b}}} = [T]^{\mathscr{R}(\hat{a}) \cup \mathscr{R}(\hat{b})} = [T]^{\mathscr{R}(\hat{a})} \cap [T]^{\mathscr{R}(\hat{b})}$ is the interval defined by the two sections $[T]^{\mathscr{R}(\hat{a})}$ and $[T]^{\mathscr{R}(\hat{b})}$. Again, the class $[T]^{S_{\hat{a}}, S_{\hat{b}}}$ reduces to $\{T\}$ only if T is one of the eigenstates $A(\hat{a})$, $A(-\hat{a})$, $B(\hat{b})$, or $B(-\hat{b})$ of $S_{\hat{a}}$ and $S_{\hat{b}}$. Thus any two spin quantities $\mathscr{R}(\hat{a})$ and $\mathscr{R}(\hat{b})$ are informationally incomplete, even though they are totally noncommutative, whenever $\hat{a} \neq \pm\hat{b}$ (cf. Cor. 2.1.9).

The measurement statistics $\Gamma^{S_{\hat{a}}}_T$ of a single spin quantity $S_{\hat{a}}$, $\hat{a} \in \mathbb{R}^3_1$, determines, in general, only one component of the state $T(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^3_{\leqslant 1}$, the projection of $\mathbf{t}$ along $\hat{a}$. For a full determination of $\mathbf{t}$ three linearly independent directions are needed, in general. In other words, $[T]^{\mathscr{R}(\hat{a}) \cup \mathscr{R}(\hat{b}) \cup \mathscr{R}(\hat{c})} = \{T\}$ for any $T \in \mathscr{T}_s(\mathscr{H})^+_1$ iff the vectors $\hat{a}$, $\hat{b}$, and $\hat{c}$ are linearly independent. This is the well known result (see, e.g., von Weizsäcker[31]) that no single (usual) spin quantity $S_{\hat{a}}$ is informationally complete and that a minimal informational completion of $S_{\hat{a}}$ among the set of all (usual) spin quantities is $\{S_{\hat{a}}, S_{\hat{b}}, S_{\hat{c}}\}$, with $(\hat{a} \times \hat{b}) \cdot \hat{c} \neq 0$. In other words, no $\mathscr{R}(\hat{a}) \subset \mathscr{P}(\mathscr{H}) \subset \mathscr{E}(\mathscr{H})$ is informationally complete and a minimal informational completion of $\mathscr{R}(\hat{a})$ in $\mathscr{P}(\mathscr{H})$, $\mathscr{H} = \mathbb{C}^2$, is of the form $\mathscr{R}(\hat{a}) \cup \mathscr{R}(\hat{b}) \cup \mathscr{R}(\hat{c})$, with linearly independent $\hat{a}$, $\hat{b}$, and $\hat{c}$.

As in the case of the Pauli problem it is possible to find other physically relevant informational completions of $S_{\hat{a}}$, or $\mathscr{R}(\hat{a})$, with replacing the spin quantity $S_{\hat{a}}$, or $\mathscr{R}(\hat{a})$, with some of its informationally equivalent unsharp counterparts $\mathscr{R}(\mathbf{a})$, $\mathbf{a} \in \mathbb{R}^3_{\leqslant 1}$. We shall discuss these options next.

2.4.3. Consider a (sharp) spin quantity $\mathscr{R}(\hat{a})$ and any of its unsharp counterparts $\mathscr{R}(\mathbf{a})$, $\mathbf{a} = a\hat{a}$, $0 < a \leqslant 1$. Clearly, $[T]^{\mathscr{R}(\hat{a})} = [T]^{\mathscr{R}(\mathbf{a})}$ for any $T \in \mathscr{T}_s(\mathscr{H})^+_1$, i.e. $\mathscr{R}(\hat{a})$ and $\mathscr{R}(\mathbf{a})$ are informationally equivalent. The state distinction powers of $\mathscr{R}(\hat{a})$ and $\mathscr{R}(\mathbf{a})$ are the same. Consider next two spin quantities $\mathscr{R}(\hat{a})$ and $\mathscr{R}(\hat{b})$. From this we know that $\mathscr{R}(\hat{a}) \cup \mathscr{R}(\hat{b})$ is not informationally complete and it is informationally equivalent with any $\mathscr{R}(\mathbf{a}) \cup \mathscr{R}(\mathbf{b})$, with $\mathbf{a} = a\hat{a}$, $0 < a \leqslant 1$, $\mathbf{b} = b\hat{b}$, $0 < b \leqslant 1$. Take then any $\hat{c} \in \mathbb{R}^3_1$. If $\hat{a}$, $\hat{b}$, and $\hat{c}$ are linearly independent, and thus also $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \neq 0$, with $\mathbf{c} = c\hat{c}$, $0 < c \leqslant 1$, then

$$[T]^{\mathscr{R}(\hat{a}) \cup \mathscr{R}(\hat{b}) \cup \mathscr{R}(\hat{c})} = [T]^{\mathscr{R}(\mathbf{a}) \cup \mathscr{R}(\mathbf{b}) \cup \mathscr{R}(\mathbf{c})} = \{T\}$$

for any $T \in \mathscr{T}_s(\mathscr{H})^+_1$.

To determine the initial preparation (state) of a spin-1/2 system we may collect the measurement statistics of three independent (mutually exclusive) spin measurements: $\Gamma_T^{S_{\hat{a}}}$, $\Gamma_T^{S_{\hat{b}}}$, $\Gamma_T^{S_{\hat{c}}}$. The same can be achieved by the statistics ($\Gamma_T^{S_{\mathbf{a}}}$, $\Gamma_T^{S_{\mathbf{b}}}$, $\Gamma_T^{S_{\mathbf{c}}}$), as well, In the first case the spin $\hat{a}$-, $\hat{b}$-, and $\hat{c}$-measurements are always mutually exclusive and they can never be combined as a joint measurement of the spin quantities $S_{\hat{a}}$, $S_{\hat{b}}$, and $S_{\hat{c}}$. This means, in particular, that there is no combined spin ($\hat{a}$, $\hat{b}$, $\hat{c}$)-measurement to provide any information on the pre- or postmeasurement states of a spin-1/2 system. To obtain informationally complete measurement statistics $\Gamma_T^{S_{\hat{a}}}$, $\Gamma_T^{S_{\hat{b}}}$, $\Gamma_T^{S_{\hat{c}}}$ we need three mutually exclusive spin measurement programmes which cannot be combined to determine the future of the system. But to determine the past of the system via the measurements of the unsharp spin quantities $\mathscr{R}(\mathbf{a})$, $\mathscr{R}(\mathbf{b})$, and $\mathscr{R}(\mathbf{c})$ opens also the possibility to determine the future of the system, at least to some reasonable degree. Indeed, if e.g. $\mathscr{R}(\mathbf{a})$ and $\mathscr{R}(\mathbf{b})$ are coexistent, i.e.

$$\|\mathbf{a}+\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\| \leqslant 2 \tag{29}$$

then there are joint observables $E^{\mathbf{a},\mathbf{b};C}$ of $\mathscr{R}(\mathbf{a})$ and $\mathscr{R}(\mathbf{b})$ and if

$$\|\mathbf{a}+\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\| < 2 \tag{30}$$

(and if lin span$\{\mathbf{a}, \mathbf{b}\} = \mathbb{R}^2$) it is always possible to choose $E^{\mathbf{a},\mathbf{b};C}$ such that it is informationally complete (i.e., it is possible to choose $C = \gamma C(\mathbf{c})$ such that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \neq 0$). For the given directions $\hat{a}$ and $\hat{b}$, $\mathbf{a} = a\hat{a}$, $a > 0$, and $\mathbf{b} = b\hat{b}$, $b > 0$, may always be so chosen that Eq. (30) is satisfied. The quantity $\mathscr{R}(\mathbf{c})$ above may then be taken to be the one associated with the chosen lower bound C of $A(\mathbf{a})$ and $B(\mathbf{b})$ which generates the joint observable $E^{\mathbf{a},\mathbf{b};C}$. Then clearly

$$[T]^{\mathscr{R}(\hat{a}) \cup \mathscr{R}(\hat{b}) \cup \mathscr{R}(\hat{c})} = [T]^{\mathscr{R}(\mathbf{a}) \cup \mathscr{R}(\mathbf{b}) \cup \mathscr{R}(\mathbf{c})} \subseteq [T]^{E^{\mathbf{a},\mathbf{b},C}} = \{T\}$$

for any $T \in \mathscr{T}_s(\mathscr{H})_1^+$. What is gained here is, indeed, that the informationally complete measurement statistics can be obtained not only through three mutually exclusive spin measurements but also by measuring a single unsharp observable $\mathrm{E}^{\mathbf{a},\mathbf{b};C}$. A (preparatory) measurement of the observable $E^{\mathbf{a},\mathbf{b};C}$ leads to a postmeasurement state of the system (the determination of the future of the system) in which we also have probabilistic information on the three independent (unsharp) spin quantities $\mathscr{R}(\mathbf{a})$, $\mathscr{R}(\mathbf{b})$, and $\mathscr{R}(\mathbf{c})$, as well as on the three mutually exclusive (sharp) quantities $S_{\hat{a}}$, $S_{\hat{b}}$, and $S_{\hat{c}}$. Clearly, the observable $E^{\mathbf{a},\mathbf{b};C}$ is again totally noncommutative. Finally we remark that Busch[32] discusses realizable proposals of informationally complete spin measurements.

2.5. Remarks on Higher Spin

We note first that for no value $s(>0)$ of spin a pair like $\{S_x, S_y\}$ is informationally complete, although the necessary condition of Corollary 2.1.9, in always satisfied: $\text{com}(E^{S_x}, E^{S_y}) = v_s$, with $v_s = 0$ for half-integer s and $v_s = 1$ for integer s. In the following we shall specify for which values of s the triple $\{S_x, S_y, S_z\}$ is informationally complete.

2.5.1. Theorem

Let $\mathscr{H}_s$ $(= \mathbb{C}^n)$ be the n-dimensional Hilbert space hosting an irreducible unitary representation of the rotation group SO(3) $(n = 2s+1$, $s = 0, 1/2, 1, 3/2,...)$ with the self-adjoint generators S_x, S_y, S_z. The set $\{S_x, S_y, S_z\}$ is informationally complete if and only if $s \leqslant 1$.

Proof. The case $s=0$ is trivial. The case $s=1/2$ was treated in Sec. 2.4.2. For the remaining cases we proceed as follows. Informational completeness of the set $\mathscr{L} = \mathscr{R}(E^{S_x}) \cup \mathscr{R}(E^{S_y}) \cup \mathscr{R}(E^{S_z}) \subset \mathscr{L}_s(\mathscr{H}_s)$ is equivalent to the equation $\text{span}(\mathscr{L}) = \mathscr{L}_s(\mathscr{H}_s)$. In fact, if this equation holds, then clearly $\mathscr{L}$ is informationally complete. Conversely, if $\hat{\mathscr{L}} = \text{span}(\mathscr{L}) \neq \mathscr{L}_s(\mathscr{H}_s)$ then choose an orthonormal basis $\{L_i \mid i = 1,..., N\}$ of $\mathscr{L}_s(\mathscr{H}_s)$ such that $\{L_i \mid i = 1,..., N_0 < N\}$ is a basis of $\hat{\mathscr{L}}$. Recalling that now $\mathscr{T}_s(\mathscr{H}_s) = \mathscr{L}_s(\mathscr{H}_s)$ we then have for any $T = \sum t_i L_i$ and $L = \sum l_j L_j$, $\text{tr}(TL) = \sum\sum t_i l_j \,\text{tr}(L_i L_j) = \sum t_i l_i$. Thus, for any $0 \neq T \in \hat{\mathscr{L}}^{\perp}$, $\text{tr}(TL) = 0$ for all $L \in \hat{\mathscr{L}}$; in particular $\text{tr}(T) = 0$, as $I \in \hat{\mathscr{L}}$. Thus $\hat{\mathscr{L}}$, and therefore $\mathscr{L}$, is not informationally complete. Now we observe that $\mathscr{L}$ is in the span of the set

$$\tilde{\mathscr{L}} = \{E_i^v \mid v = x, y, z; i = -s, -s+1,..., s-1, s\}$$

(Here we use the notation $E_i^v = E^{S_v}(\{i\})$.) Moreover this set is already spanned by

$$\mathscr{L}_0 = \tilde{\mathscr{L}} \cup \{I\} \setminus \{E_{i_0}^x, E_{i_0}^y, E_{i_0}^z\},\ i_0 \in \{-s,..., s\} \text{ fixed}$$

$\mathscr{L}_0$ contains $N = 3n-2$ elements, and we find:

$$n=3\colon N = 7 > \dim[\mathscr{L}_s(\mathscr{H}_1)] = 6$$

$$n=4\colon N = 10 = \dim[\mathscr{L}_s(\mathscr{H}_{3/2})] = 10$$

$$n \geqslant 5\colon N < \dim[\mathscr{L}_s(\mathscr{H}_s)]$$

This shows that for $n \geqslant 5$ (i.e., $s \geqslant 2$) the set $\mathscr{L}_0$ cannot be informationally complete. We turn to the remaining two cases.

Case $n=3$. We show that the sel $\hat{\mathscr{L}}_0 = \mathscr{L}_0 \setminus \{I\} = \{E_1^x, E_{-1}^x, E_1^y, E_{-1}^y, E_1^z, E_{-1}^z\}$ is linearly independent. Assume that

$$\alpha_x E_1^x + \beta_x E_{-1}^x + \alpha_y E_1^y + \beta_y E_{-1}^y + \alpha_z E_1^z + \beta_z E_{-1}^z = 0 \tag{31}$$

This is of the form

$$\phi(\mathbf{S}) = f(S_x) + g(S_y) + h(S_z) = 0$$

Applying rotations about the x, y, and z axes by an angle π shows that the functions f, g, h are symmetric (functions on $\{-s, -s+1,..., s-1, s\}$). Thus

$$f(S_x) = \alpha_x E_1^x + \beta_x E_{-1}^x = \alpha_x(E_1^x + E_{-1}^x) = \alpha_x S_x^2$$

$$g(S_y) = \alpha_y E_1^y + \beta_y E_{-1}^y = \alpha_y(E_1^y + E_{-1}^y) = \alpha_y S_y^2$$

$$h(S_z) = \alpha_z E_1^z + \beta_z E_{-1}^z = \alpha_z(E_1^z + E_{-1}^z) = \alpha_z S_z^2$$

Applying rotations by an angle $\pi/2$ yields $\alpha_x = \alpha_y = \alpha_z = a$. Therefore,

$$\phi(\mathbf{S}) = a(S_x^2 + S_y^2 + S_z^2) = 0$$

But the commutation relations for the S_i give the well-known relation

$$S_x^2 + S_y^2 + S_z^2 = s(s+1)I \ (\neq 0 \text{ for } s > 0) \tag{32}$$

Therefore $a = 0$, and all the coefficients in Eq. (31) must vanish.

Case $n = 4$. Introducing the notation $P_1^v = E_{1/2}^v + E_{-1/2}^v$, $P_2^v = E_{3/2}^v + E_{-3/2}^v$ ($v = x, y, z$), Eq. (32) reads (note: $s = 3/2$):

$$\tfrac{9}{4}(P_2^x + P_2^y + P_2^z) + \tfrac{1}{4}(P_1^x + P_1^y + P_1^z) = \tfrac{15}{4}I$$

Substituting $P_1^v = I - P_2^v$ yields:

$$0 = 3I - 2[P_2^x + P_2^y + P_2^z]$$

which shows that the set $\{I, E_{3/2}^x, E_{-3/2}^x, E_{3/2}^y, E_{-3/2}^y, E_{3/2}^z, E_{-3/2}^z\}$ is linearly dependent; therefore its extension to any set of the form $\mathscr{L}_0$ (above) is linearly dependent, too. ∎

Referring again to Theorem 2.1.8 or Corollary 2.1.9, we remark that the informational completeness of $\{S_x, S_y, S_z\}$ for spin 1 goes along with $\dim[\mathrm{com}(E^x, E^y, E^z)] = 0$. Thus so far we encountered only instances of informational completeness in connection with total noncommutativity. (Note added: in a forthcoming paper it will be shown that total noncommutativity is a necessary condition for informational completeness.)

3. STATES OF MAXIMAL INFORMATION

3.1. General Considerations

We shall now turn to the problem of determining the future of a physical system $\mathscr{S}$ on the basis of a measurement performed on it. We remark first that the *pure states* $P[\varphi]$, $\varphi \in \mathscr{H}_1$, of $\mathscr{S}$ are its *maximal information states* in the folowing obvious sense:

$$
\begin{aligned}
P[\varphi] &= \bigwedge \{P \in \mathscr{P}(\mathscr{H}) : \langle \varphi \mid P\varphi \rangle = 1\} \\
&(= \bigwedge \{E \in \mathscr{E}(\mathscr{H}) : \langle \varphi \mid E\varphi \rangle = 1\})
\end{aligned}
\tag{33}
$$

In other words, the pure states of $\mathscr{S}$ are in one-to-one onto correspondence with the maximal sets of properties (effects) which the system may posses (with probability equal to one) at a time. (Cf., e.g., Beltrametti and Cassinelli,[2] Piron,[33] or Mittelstaedt.[34,35]) An optimal determination of the future of $\mathscr{S}$ is thus obtained whenever $\mathscr{S}$ is prepared in a pure state. It is one of the basic assumptions of quantum mechanics—consistent with the quantum theory of measurement—that $\mathscr{S}$ can be prepared in any pure state (cf. below). Here we shall be concerned with the question of preparing $\mathscr{S}$ in pure states with some particular properties. To motivate our approach we shall start with considering the sequential measurements of pairs of physical quantities A and B within the quantum theory of measurement.

Assume that a measurement $\langle \mathscr{H}_M, A_M, T_M, V \rangle$ of a quantity A on $\mathscr{S}$ in a (initial) state T_i is performed. This measurement induces a transformation of the state of $\mathscr{S}$: $T_i \mapsto I_V^A(\mathbb{R}) T_i$, where $I_V^A(\mathbb{R}) T_i$ is the state of $\mathscr{S}$ after the A-measurement but before reading of the actual measurement result. Assuming that the actual measurement result is the set $X \in \mathscr{B}(\mathbb{R})$, then $\mathrm{tr}[T_i E^A(X)] \neq 0$ and $T_f \equiv \mathrm{tr}(T_i E^A(X))^{-1} I_V^A(X) T_i$ is the final (or post-measurement) state of $\mathscr{S}$. (Here the possibility of "reading the actual measurement result" shall not be questioned, at all. Beltrametti *et al.*[1] contains a systematic analysis of this question and there can also be found references to some other relevant work.) The probability that the measured quantity A has the measured value X in the postmeasurement state T_f of $\mathscr{S}$ is now $\mathrm{tr}(T_f E^A(X)) = \mathrm{tr}(T_i E^A(X))^{-1} \mathrm{tr}(I_V^A(X)^2 T_i)$, which shows that $\mathrm{tr}(T_f E^A(X)) = 1$ if and only if $\mathrm{tr}(I_V^A(X)^2 T_i) = \mathrm{tr}(I_V^A(X) T_i)$ (as $\mathrm{tr}(T_i E^A(X)) = \mathrm{tr}(I_V^A(X) T_i) \neq 0$). Thus a measurement $\langle \mathscr{H}_M, A_M, T_M, V \rangle$ of a quantity A on $\mathscr{S}$ may or may not leave the system $\mathscr{S}$ in a state T_f in which the measured quantity has (with probability equal to one) the measured value, i.e. $\mathrm{tr}(T_f E^A(X)) = 1$. Also, T_f may or may not be a pure state.

It is a basic result of the quantum theory of measurement that any discrete quantity A, i.e., a quantity with a pure point spectrum, admits this type of pure preparatory measurements (see e.g. Beltrametti *et al.*,[1]). The von Neumann–Lüders measurement of (a discrete quantity) A is the well known prototype of such measurements. In particular, any property

$P \in \mathscr{P}(\mathscr{H})$, like $P = E^A(X)$ or $P = P[\varphi]$, $\varphi \in \mathscr{H}_1$, defines such a quantity through the range $\{0, P, P^\perp, I\}$.

Assume next that a measurement of the quantity B follows the earlier A-measurement, which lead to the state T_f. Then $\operatorname{tr}(T_f E^B(Y))^{-1} I^B_W(Y) T_f = \operatorname{tr}(I^B_W(Y) I^A_V(X) T_i)^{-1} I^B_W(Y) I^A_V(X) T_i \equiv T$ is the postmeasurement state of $\mathscr{S}$ with the condition that the B-measurement (with the instrument I^B_W) lead to the result $Y \in \mathscr{B}(\mathbb{R})$ (so that, again, $\operatorname{tr}(T_f E^B(Y)) \neq 0$). Again, $\operatorname{tr}(TE^B(Y)) = 1$ if and only if $\operatorname{tr}(I^B_W(Y)^2 I^A_V(X) T_i) = \operatorname{tr}(I^B_W(Y) I^A_V(X) T_i)$ $(\neq 0)$, and the state T may or may not be a pure state.

We come now to the following question: Under which conditions does the sequential measurement ("first the A-measurement with the result X, then the B-measurement with the result Y") lead the system to a state T in which $E^A_T(X) = E^B_T(Y) = 1$. A necessary and a sufficient condition is that

$$\overline{\mathscr{R}(T)} \subset E^A(X) \wedge E^B(Y)(\mathscr{H}) \tag{34}$$

where $E^A(X) \wedge E^B(Y)(\mathscr{H})$ is to be nonzero. If the projections $E^A(X)$ and $E^B(Y)$ are disjoint, i.e., $E^A(X) \wedge E^B(Y) = 0$, then Eq. (34) can never be fulfilled. This is most typical for complementary pairs of observables, like position and momentum or the different spin quantities of a spin-1/2 system. For such quantities we may still ask what is the maximal information that can be obtained on them e.g. with this type of sequential measurements on $\mathscr{S}$. This is exactly the question which we aim to study here. To answer this type of question a suitable "information functional" $T \mapsto \operatorname{INFO}(E^A_T(X), E^B_T(Y))$, or $T \mapsto \operatorname{INFO}(E^A_T, E^B_T)$, should be given with respect to which the posed question could be analyzed. Here we shall be concerned with the problem of information on the values of the quantities A and B so that we consider a functional of the form $T \mapsto \operatorname{INFO}(E^A_T(X), E^B_T(Y))$, the so-called local information functionals, and for that we simply choose the map

$$T \mapsto \operatorname{tr}(TE^A(X)) + \operatorname{tr}(TE^B(Y)) \tag{35}$$

A state $T_0 \in \mathscr{T}_s(\mathscr{H})^+_1$ is a *state of maximal information of A and B associated with the value sets X and Y* if $\operatorname{tr}(T_0(E^A(X) + E^B(Y))) = \sup\{\operatorname{tr}(T(E^A(X) + E^B(Y)) \colon T \in \mathscr{T}_s(\mathscr{H})^+_1\}$. If such a state T_0 exists, and if it is pure, i.e. $T_0 = P[\varphi]$, for some $\varphi \in \mathscr{H}_1$, then the positive outcome of the von Neumann–Lüders measurement of the elementary quantity $\{0, P[\varphi], P[\varphi]^\perp, I\}$ would lead the system into a (pure) state $T_0 = P[\varphi]$ in which the information on the values X and Y of A and B is maximal. If $E^A(X) \wedge E^B(Y) = 0$, then clearly the question of the existence of such a T_0 becomes relevent. Theorem 3.2.1 will give necessary and sufficient conditions for the existence of such states T_0 as well as a complete description of them. The results will then be illustrated with the position-momentum (Sec. 3.3) and the spin quantities (Sec. 3.4).

3.2. Mathematical Interlude

In this subsection we shall present some results from the operator theory which are relevant for the problem of the existence of the maximal information states. When no explicit reference is given here the results can be found in the standard texts like Halmos,[36] Dunford and Schwartz,[37] or Kato.[38]

The map $\mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{H})^*$, $A \mapsto \mathrm{tr}(\cdot A)$ is an isometric isomorphism. In particular, this means that the norm of an $A \in \mathscr{L}_s(\mathscr{H})$, $\|A\| := \sup\{\|A\varphi\|: \varphi \in \mathscr{H}_1\}$, equals the norm of the functional $T \mapsto \mathrm{tr}(TA)$, i.e.,

$$\|A\| = \sup\{|\mathrm{tr}(TA)|: T \in \mathscr{T}_s(\mathscr{H}),\ \|T\|_1 = 1\} \tag{36}$$

The norm of $A \in \mathscr{L}_s(\mathscr{H})$ can also be given e.g. by its numerical range or by its spectrum. They all are of use for us.

The numerical range of A is, by definition, the set $w(A) := \{\langle \varphi \mid A\varphi \rangle: \varphi \in \mathscr{H}_1\}$ which is a subinterval (as a convex set) of $[-\|A\|, \|A\|]$ (for $\langle \varphi \mid A\varphi \rangle \in \mathbb{R}$ and $|\langle \varphi \mid A\varphi \rangle| \leqslant \|A\|$ for any $\varphi \in \mathscr{H}_1$). Let $v(A) = \{\mathrm{tr}(TA): T \in \mathscr{T}_s(\mathscr{H})_1^+\}$, so that $w(A) \subset v(A)$ for $\mathrm{Ex}(\mathscr{T}_s(\mathscr{H})_1^+) = \{P[\varphi]: \varphi \in \mathscr{H}_1\}$ and $\mathrm{tr}(P[\varphi]A) = \langle \varphi \mid A\varphi \rangle$. From Eq. (36) we get that also $v(A) \subset [-\|A\|, \|A\|]$. Moreover, it is a σ-convex set of reals, for $\mathscr{T}_s(\mathscr{H})_1^+$ is a σ-convex subset of $\mathscr{T}_s(\mathscr{H})$ (in the $\|\cdot\|_1$-topology). Let $t \in v(A)$ so that $t = \mathrm{tr}(TA)$ for some $T \in \mathscr{T}_s(\mathscr{H})_1^+$. According to Hadjisavvas[4] (Thm. 3) T has a decomposition $T = \sum \alpha_i P[\varphi_i]$, $\alpha_i > 0$, $\sum \alpha_i = 1$, $\varphi_i \in \mathscr{H}_1$, such that $\langle \varphi_i \mid A\varphi_i \rangle = t$ for any i (cf. p. 2–2). Hence $t \in w(A)$, as well. Thus we have the result:

$$w(A) = v(A) \tag{37}$$

In particular, this result shows that $w(A)$ is not only a convex set but also a σ-convex set of reals. It is important to note that the set $v(A)$ $(= w(A))$ need not be a closed one. Indeed, if e.g. $A = E^Q((-\infty, x]) + E^P((-\infty, y])$, $x, y \in \mathbb{R}$, then $v(A) = w(A)$ is the open interval $(0, 2)$ (cf. subsection 3.3).

The spectrum of A is, by definition, $\mathrm{sp}(A) := \{\lambda \in \mathbb{C}: A - \lambda I$ is not invertible$\}$, and it now equals the support of the (real) spectral measure E^A of A, i.e. $\mathrm{sp}(A) = \mathrm{supp}(E^A) := \bigcap \{X \in \mathscr{B}(\mathbb{R}): X$ closed, $E^A(X) = I\}$. For bounded self-adjoint operators A the spectrum $\mathrm{sp}(A)$ is a compact subset of $\mathbb{R}$. Let $\mathrm{sp}_p(A)$ denote the point spectrum (the eigenvalues) of A. Then clearly $\mathrm{sp}_p(A) \subset w(A)$, but only $\mathrm{sp}(A) \subset \overline{w(A)}$, where $\overline{w(A)}$ is the closure of the set $w(A)$.

In addition to Eq. (36) the norm of $A \in \mathscr{L}_s(\mathscr{H})$ can now be expressed as

$$\begin{aligned} &(a) \quad \|A\| = \sup\{|t|: t \in v(A)\} \\ &(b) \quad = \sup\{|a|: a \in w(A)\} \\ &(c) \quad = \max\{|\lambda|: \lambda \in \mathrm{sp}(A)\} \end{aligned} \tag{38}$$

The equality of Eq. (38c) shows, in particular, that at least one of the numbers $\pm\|A\|$ belongs to $\mathrm{sp}(A)$ (, and thus $\|A\| \in \mathrm{sp}(A)$ whenever $A \geqslant 0$). We note also, that if A is compact then $\|A\| = |\lambda_1|$, where λ_1 is the maximal (in absolute value) eigenvalue of A. However, in general neither $-\|A\|$ nor $\|A\|$ need be an eigenvalue of A. (Cf. the example $A = E^Q((-\infty, x]) + E^P((-\infty, y])$, $x, y \in \mathbb{R}$, where $\|A\| = 2$, but $\mathrm{sp}_p(A) \subset w(A) = (0, 2)$.)

Assume that $|a| = \|A\|$ for some $a \in w(A)$. Then a is an eigenvalue of A. Indeed, if $a = \langle \varphi \mid A\varphi \rangle$, $\varphi \in \mathscr{H}_1$, and $\|A\| = |a| = |\langle \varphi \mid A\varphi \rangle| \leqslant \|\varphi\| \, \|A\varphi\| \leqslant \|A\|$, we have, in particular, that $|\langle \varphi | A\varphi \rangle| = \|\varphi\| \, \|A\varphi\|$. But this is known from the Cauchy–Schwarz inequality to be the case only if $A\varphi = \lambda\varphi$ for some (eigenvalue) λ. Now $\lambda = \langle \varphi \mid \lambda\varphi \rangle = \langle \varphi \mid A\varphi \rangle = a$, which shows that $a \in \mathrm{sp}_p(A)$. (Consequently, then $\|A\|$, or $-\|A\|$, is an eigenvalue of A.)

Let $\mathrm{kern}(A - \lambda I) = E^A(\{\lambda\})(\mathscr{H})$ be the eigenspace associated with the eigenvalue $\lambda \in \mathrm{sp}_p(A)$. If (φ_i) is a sequence of unit vectors in $\mathrm{kern}(A - \lambda I)$, then $\mathrm{tr}(TA) = \lambda$ for any $T = \sum \alpha_i P[\varphi_i]$, $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$. On the other hand, if $T \in \mathscr{T}_s(\mathscr{H})_1^+$ is such that $\mathrm{tr}(TA) = \lambda$ then T can be decomposed as $T = \sum \beta_i P[\psi_i]$, $\beta_i > 0$, $\sum \beta_i = 1$, such that $\psi_i \in \mathrm{kern}(A - \lambda I) \cap \mathscr{H}_1$ for any i (cf. again Hadjisavvas,[4] Thm. 3). Together with the earlier results, this now shows that for any $A \in \mathscr{L}_s(\mathscr{H})$: If $|a| = \|A\|$ for some $a \in w(A) = v(A)$, then $a = \mathrm{tr}(TA)$ exactly for those $T \in \mathscr{T}_s(\mathscr{H})_1^+$ for which $\overline{\mathscr{R}(T)} \subset \mathrm{kern}(A - aI)$.

The following theorem, based on these results, is useful in studying the existence of maximal information states of pairs of physical quantities.

3.2.1. Theorem

Let $A \in \mathscr{L}(\mathscr{H})^+$. The following four conditions are equivalent:

(a) $\mathrm{tr}(T_0 A) = \sup\{\mathrm{tr}(TA): T \in \mathscr{T}_s(\mathscr{H})_1^+\}$ for some $T_0 \in \mathscr{T}_s(\mathscr{H})_1^+$

(b) $\|A\|$ is an eigenvalue of A

(c) $\|A\| \in w(A)$

(d) $\|A\| \in v(A)$

If one of these conditions, and thus all are satisfied, then a state T_0 satisfies (a) if and only if $\overline{\mathscr{R}(T_0)} \subset \mathrm{kern}(A - \|A\| I)$.

In this work we shall apply Theorem 3.2.1 to study the existence of maximal information states for some pairs of projection operators, i.e., $A = P + R$, $P, R \in \mathscr{P}(\mathscr{H})$, and for some pairs of effects, i.e., $B = E + F$, $E, F \in \mathscr{E}(\mathscr{H})$. Before entering the physically relevant special cases we shall point out some general features of such cases. First of all, both A and B are positive operators and their norm is at most 2. Hence their numerical range is in $[0, 2]$.

Let $A = P + R$, $P, R \in \mathscr{P}(\mathscr{H})$. Clearly, $\|P + R\| = 2 \in w(A)$ if and only if $P \wedge R \neq 0$ (and $0 \in w(A)$ if and only if $P^\perp \wedge R^\perp \neq 0$). On the other hand, $\|P + R\| = 1$ if and only if ($P \wedge R = 0$ and) $P \leqslant R^\perp$. Then kern $((P + R) - I) = P(\mathscr{H}) \oplus R(\mathscr{H})$. Thus the nontrivial case occurs only if the projection operators $P, R \in \mathscr{P}(\mathscr{H})$ are disjoint (i.e., $P \wedge R = 0$) but not orthogonal (i.e., $P \nleqslant R^\perp$). In that case $\|P + R\|$ may or may not belong to $w(A)$. Nontrivial examples of both cases will appear next.

Let $B = E + F$, $E, F \in \mathscr{E}(\mathscr{H})$. Again, $\|E + F\| = 2 \in w(B)$ if and only if $E\varphi = \varphi$ and $F\varphi = \varphi$ for some $\varphi \in \mathscr{H}_1$. In that case l.b.$\{E, F\}$ $(:= \{G \in \mathscr{E}(\mathscr{H}): G \leqslant E, G \leqslant F\}) \neq \{0\}$ (recall, that $\mathscr{E}(\mathscr{H})$ is not a lattice), but the converse need not hold now since an effect G need not have the eigenvalue 1. (Similarly, $0 \in w(B)$ if and only if $\mathrm{kern}(E) \cap \mathrm{kern}(F) \neq \{0\}$.) In the case of effects $E, F \in \mathscr{E}(\mathscr{H})$, the nontrivial case thus occurs when $\mathrm{kern}(E - I) \cap \mathrm{kern}(F - I) = \{0\}$.

3.3. Example. The Position-Momentum Pairs

As the first physical application of the results of the previous section we shall study the existence of the maximal information states of the spectral projections $E^Q(X)$ and $E^P(Y)$ of Q and P, and of their unsharp counterparts $E^{Q_f}(X)$ and $E^{P_g}(Y)$, associated with the given sets $X, Y \in \mathscr{B}(\mathbb{R})$.

The degree of commutativity of the spectral projections of Q and P have been investigated in detail (see e.g. Lenard,[39] Busch and Lahti,[40] and Busch, Schonbek, and Schroeck[29]). Also the question of the existence of the maximal information states for the pairs $(E^Q(X), E^P(Y))$, $X, Y \in \mathscr{B}(\mathbb{R})$ has already been discussed (Lahti,[41] Busch and Lahti[42]) so that here we may be content with stating the relevant results, only.

Let $X, Y \in \mathscr{B}(\mathbb{R})$ be bounded. This is a nontrivial case for now $E^Q(X) \wedge E^P(Y) = E^Q(X)^\perp \wedge E^P(Y) = E^Q(X) \wedge E^P(Y)^\perp = 0$ (though, of course, all the involved projection operators are nonzero). In this case $E^Q(X)^\perp \wedge E^P(Y)^\perp \neq 0$ so that $0 \in w(A)$, with $A = E^Q(X) + E^P(Y)$. Moreover, $\|A\|$ (< 2) is an eigenvalue of A, so that $w(A) = [0, \|A\|]$ for any $A = E^Q(X) + E^P(Y)$ with bounded $X, Y \in \mathscr{B}(\mathbb{R})$. According to Thm. 3.2.1, any $T_0 \in \mathscr{T}_s(\mathscr{H})_1^+$ such that $\overline{\mathscr{R}(T_0)} \subset \mathrm{kern}(A - \|A\| I)$ is now a maximal information state for Q and P associated with the value sets X and Y.

The eigenvalue $\|E^Q(X)+E^P(Y)\|$ and the corresponding eigenspace can also be constructed explicitly e.g. via the compact operator $B=E^P(Y)\,E^Q(X)\,E^P(Y)$: $\|E^Q(X)+E^P(Y)\|=1+\lambda_0$, where λ_0^2 is the maximal eigenvalue of B, and $\mathrm{kern}((E^Q(X)+E^P(Y))-(1+\lambda_0)I)=\mathrm{kern}(B-\lambda_0^2 I)\cap E^P(Y)(\mathscr{H})$. We emphasize that this result holds true for *any* bounded value sets X and Y. Thus, though $E^Q(X)\wedge E^P(Y)=0$, and $E^Q(X)\nleq E^P(Y)^\perp$, for any bounded $X, Y\in\mathscr{B}(\mathbb{R})$, the system $\mathscr{S}$ may always be prepared in a pure state $P[\varphi_0]$, say, in which $\mathrm{tr}(P[\varphi_0]\,E^Q(X))+\mathrm{tr}(P[\varphi_0]\,E^P(Y))$ $(=1+\lambda_0,\ \lambda_0=\lambda_0(X, Y))$ is maximal for given $X, Y\in\mathscr{B}(\mathbb{R})$. (We note also that for any $\varepsilon>0$ there are bounded $X, Y\in\mathscr{B}(\mathbb{R})$ such that $1+\lambda_0>2-\varepsilon$ (Maczynski[43]).

The other interesting case are the pairs $(E^Q(X), E^P(Y))$ associated with the half-lines like $X=(-\infty, x]$, $Y=(-\infty, y]$, $x, y\in\mathbb{R}$, for such projection operators are totally noncommutative, i.e. $E^Q(X)\wedge E^P(Y)=E^Q(X)^\perp\wedge E^P(Y)=E^Q(X)\wedge E^P(Y)^\perp=E^Q(X)^\perp\wedge E^P(Y)^\perp=0$. In that case $w(E^Q(X)+E^P(Y))$ is the open interval $(0, 2)$, so that $\|E^Q(X)+E^P(Y)\|=2$ and no maximal information state exists now. It may be appropriate to note also that $w(E^Q(X)+E^P(Y))=[0, 2]$, whenever X and Y are periodic sets, $X=X+d$, $Y=Y+2\pi/d$. They are exactly those (nontrivial) spectral projections of Q and P which commute (Busch, Schonbek, Schroeck,[29] see also Ylinen[44]).

In studying the Pauli problem in Sec. 2.3 it turned out that an informational completion of the pair (Q, P) may be obtained with replacing this pair with an informationally equivalent pair (Q_f, P_g). Thus it becomes important to ask whether such an informational completion of (Q, P) also leads to an increase in the information on the values of such quantities. To answer this question we shall next attempt to compare the numbers $\langle\varphi|(E^Q(X)+E^P(Y))\varphi\rangle$ and $\langle\varphi|(E^{Q_f}(X)+E^{P_g}(Y))\varphi\rangle$, $\varphi\in\mathscr{H}_1$, of the pairs (Q, P) and (Q_f, P_g) associated with the value sets $X, Y\in\mathscr{B}(\mathbb{R})$.

Let $A=E^Q(X)+E^P(Y)$ and $B=E^{Q_f}(X)+E^{P_g}(Y)$, $X, Y\in\mathscr{B}(\mathbb{R})$. Clearly, $w(A)$ as well as $w(B)$ are contained in the interval $[0, 2]$. If $X, Y\in\mathscr{B}(\mathbb{R})$ are bounded, then $w(A)=[0, \|A\|]$, with $\|A\|=1+\lambda_0<2$. We shall demonstrate first that for bounded $X, Y\in\mathscr{B}(\mathbb{R})$ also $2\notin w(B)$. In fact, we shall show here that $w(B)\subset[0, \|A\|]$ in any case, i.e. for all $X, Y\in\mathscr{B}(\mathbb{R})$. But the case $2\notin w(B)$, with bounded $X, Y\in\mathscr{B}(\mathbb{R})$, is of interest in itself, as now $\mathrm{l.b.}\{E^{Q_f}(X), E^{P_g}(Y)\}\neq\{0\}$ (with Fourier-related confidence functions f and g), though $E^Q(X)\wedge E^P(Y)=0$. The fact that $2\notin w(B)$ implies that the system $\mathscr{S}$ cannot be prepared in a state in which it would have (with probability equal to one) the unsharp (coexistent) properties $E^{Q_f}(X)$ and $E^{P_g}(Y)$. Assume now that $2\in w(B)$. Then 1 is an eigenvalue of both $E^{Q_f}(X)$ and $E^{P_g}(Y)$, with a common eigenvector ψ, say. But 1 is an eigenvalue of $E^{Q_f}(X)=(f*\chi_X)(Q)$ if and only if the corresponding spectral projection of $(f * \chi_X)(Q)$ is nonzero, i.e. $0 \neq E^{(f * \chi_X)(Q)}(\{1\}) = E^Q((f * \chi_X)^{-1}(\{1\}) = E^Q(\{x \in \mathbb{R}: \quad (f * \chi_X)(x) = 1\})$. Similarly, $E^P(\{y \in \mathbb{R}: (g * \chi_Y)(y) = 1\}) \neq 0$. As $\int f = \int g = 1$, these relations show that $\text{supp}(f) \subset X$, and $\text{supp}(g) \subset Y$ (as e.g. $(f * \chi_X)(x) := \int_{\mathbb{R}} f(x-y)\, \chi_X(y)\, dy = \int_X f(x-y)\, dy$ a.e. $x \in \mathbb{R}$). But $\text{l.b.}\{E^{Q_f}(X), E^{P_g}(Y)\} = \{0\}$ for any bounded $X, Y \in \mathscr{B}(\mathbb{R})$ whenever f and g have compact supports (Busch[45]). Thus $2 \in w(B)$ leads to a contradictory result $0 \neq P[\psi] \in \text{l.b.}\{E^{Q_f}(X), E^{P_g}(Y)\} = \{0\}$.

Assume next that $0 \in w(B)$. Then, as shown previously, one obtains: $\text{supp}(f) \subset \mathbb{R} \backslash X$, and $\text{supp}(g) \subset \mathbb{R} \backslash Y$. If X and Y are bounded, and if $\text{supp}(f) = \text{supp}(g) = \mathbb{R}$, as was the case in Sec. 2.3.5, then $0 \notin w(B)$, though $0 \in w(A)$.

There is no general result on the order of magnitude of the numbers $\langle \varphi \mid A\varphi \rangle$ and $\langle \varphi \mid B\varphi \rangle$, with A and B as earlier. However, it turns out that the introduction of unsharpness cannot increase the maximal information on the values of the complementary position and momentum observables Q and P. In fact, $w(B) \subset [0, \|A\|]$ for any $A = E^Q(X) + E^P(Y)$ and $B = E^{Q_f}(X) + E^{P_g}(Y)$, $X, Y \in \mathscr{B}(\mathbb{R})$. Indeed, for any $\varphi \in \mathscr{H}_1$,

$$\begin{aligned}
\langle \varphi \mid B\varphi \rangle &= \langle \varphi \mid (E^{Q_f}(X) + E^{P_g}(Y))\varphi \rangle \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\, g(y) \langle \varphi \mid (E^Q(X+x) + E^P(Y+y))\varphi \rangle \, dx\, dy \\
&\leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\, g(y)\, \|A\|\, dx\, dy \leqslant \|A\|, \text{ for}
\end{aligned}$$

$$\begin{aligned}
&E^Q(X+x) + E^P(Y+y) \\
&\quad = e^{ixP} E^Q(X) e^{-ixP} + e^{iyQ} E^P(Y) e^{-iyQ} \\
&\quad = e^{ixP} e^{iyQ} [E^Q(X) + E^P(Y)] e^{-iyQ} e^{-ixP}, \; x, y \in \mathbb{R}, \text{ and} \\
&\langle \varphi \mid (E^Q(X+x) + E^P(Y+y))\varphi \rangle \\
&\quad = \langle \varphi \mid e^{ixP} e^{iyQ} [E^Q(X) + E^P(Y)]\, e^{-iyQ} e^{-ixP} \varphi \rangle \\
&\quad \leqslant \|E^Q(X) + E^P(Y)\| \text{ for any } \varphi \in \mathscr{H}_1, \text{ and for all } x, y \in \mathbb{R}
\end{aligned}$$

As an important special case of the above result we obtain the following:

$$w(A) = [0, \|A\|] \qquad \text{and} \qquad w(B) \subset (0, \|A\|]$$

with $A = E^Q(X) + E^P(Y)$, $B = E^{Q_f}(X) + E^{P_g}(Y)$, whenever X and Y are bounded value sets and $\text{supp}(f) = \text{supp}(g) = \mathbb{R}$. With this result we close our discussion of the position-momentum example.

3.4. Example. Spin-$\frac{1}{2}$ System

We continue to study the spin quantities of a spin-$\frac{1}{2}$ system of Section 2.4. Now the Hilbert space $\mathscr{H} = \mathbb{C}^2$ is finite dimensional so that the operators $A \in \mathscr{L}(\mathscr{H})^+$ are compact, and thus the maximal information states are exactly those states T_0 for which $\overline{\mathscr{R}(T_0)}$ is in the eigenspace of the maximal eigenvalue $\|A\|$ of A. In spite of this trivial character of the problem it seems worthwhile to explicitly give the maximal information states associated with the complementary spin properties $A(\hat{a})$, $B(\hat{b})$, $\hat{a}, \hat{b} \in \mathbb{R}^3_1$ and their unsharp counterparts $A(\mathbf{a})$, $B(\mathbf{b})$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3_{\leqslant 1}$.

From Sec. 2.4.1 one immediately gets that $w(A(\hat{a}) + B(\hat{b})) = \{1 + \frac{1}{2}\mathbf{t} \cdot (\hat{a} + \hat{b})\colon \ \mathbf{t} \in \mathbb{R}^3_{\leqslant 1}\}$, and $\|A(\hat{a}) + B(\hat{b})\| = 1 + \frac{1}{2}\|\hat{a} + \hat{b}\|$, which is attained by the pure state $T = \frac{1}{2}(\hat{t} \cdot \boldsymbol{\sigma} + \mathbf{I})$, with $\hat{t} = (\hat{a} + \hat{b})/\|\hat{a} + \hat{b}\|$. Clearly, $\|A(\hat{a}) + B(\hat{b})\| = 2$ only if $\hat{a} = \hat{b}$. E.g., for the two orthogonal spin components S_x and S_y, $\|A(i\hat{x}) + B(j\hat{y})\| = 1 + \sqrt{\frac{1}{2}}$, with $i, j = +, -$, and the corresponding (pure) state of maximal information is $T = \frac{1}{2}(i\sqrt{\frac{1}{2}}\sigma_x + j\sqrt{\frac{1}{2}}\sigma_y + I)$, $i, j = +, -$.

In Sec. 2.4.3 the introduction of unsharpness was again seen to lead to an informational completion of a pair of (complementary) spin quantities $\mathscr{R}(\hat{a})$ and $\mathscr{R}(\hat{b})$ (or $S_{\hat{a}}$ and $S_{\hat{b}}$), $\hat{a}, \hat{b} \in \mathbb{R}^3_1$, $\hat{a} \cdot \hat{b} \neq \pm 1$. Consider thus an unsharp (informationally equivalent) counterpart $(\mathscr{R}(\mathbf{a}), \mathscr{R}(\mathbf{b}))$ of $(\mathscr{R}(\hat{a}), \mathscr{R}(\hat{b}))$, with $\mathbf{a} = a\hat{a}$, $\mathbf{b} = b\hat{b}$, $0 < a, b < 1$. Again, $w(A(i\mathbf{a}) + B(j\mathbf{b})) = \{1 + \frac{1}{2}\mathbf{t} \cdot (i\mathbf{a} + j\mathbf{b})\colon \ \mathbf{t} \in \mathbb{R}^3_{\leqslant 1}\}$, $i, j = +, -$, so that $\|A(i\mathbf{a}) + B(j\mathbf{b})\| = 1 + \frac{1}{2}\|i\mathbf{a} + j\mathbf{b}\|$, which is again attained by the pure state along the vector $i\mathbf{a} + j\mathbf{b}$, $i, j = +, -$.

In case of the position-momentum pair we observed that with introducing unsharpness the maximal information on the values of the quantities cannot be increased. In the present case this is, however, possible. Indeed, e.g., the condition $\|A(\mathbf{a}) + B(\mathbf{b})\| > \|A(\hat{a}) + B(\hat{b})\|$ can be satisfied with an appropriate choice of the directions $\hat{a}$ and $\hat{b}$ and of the lengths of $\mathbf{a}$ and $\mathbf{b}$ even together with the coexistence condition $\|\mathbf{a} + \mathbf{b}\| + \|\mathbf{a} - \mathbf{b}\| \leqslant 2$. (For example, the case $\hat{a} \cdot \hat{b} = -0.98$, $a = 0.8$, $b = 0.6$ is such.) In the important case of orthogonal spin quantities ($\hat{a} \cdot \hat{b} = 0$) we always have $\|A(i\mathbf{a}) + B(j\mathbf{b})\| = 1 + \frac{1}{2}\sqrt{a^2 + b^2} \leqslant 1 + \sqrt{\frac{1}{2}} = \|A(i\hat{a}) + B(j\hat{b})\|$, $i, j = +, -$. The coexistence condition ($\sqrt{a^2 + b^2} \leqslant 1$) and the condition for the existence of an informationally complete joint observable of $\mathscr{R}(\mathbf{a})$ and $\mathscr{R}(\mathbf{b})$ ($\sqrt{a^2 + b^2} < 1$) then give $\|A(i\mathbf{a}) + B(j\mathbf{b})\| < \|A(i\hat{a}) + B(j\hat{b})\|$ for all $i, j = +, -$.

4. THE PAST AND THE FUTURE IN QUANTUM MEASUREMENTS — A NEW MODE OF COMPLEMENTARITY

According to Pauli[15] quantum mechanics might have been called the theory of complementarity. This was to emphasize the importance of the idea of complementarity in the foundations of quantum mechanics. Several aspects of quantum mechanics which are related to Bohr's intuitive ideas on complementarity have been discussed in detail in the literature. For example, the notion of complementary physical quantities has been analyzed carefully and its measurement-theoretical content is now well understood (see e.g. Busch and Lahti,[46] and Lahti[47] which also contain references to other relevant work). The investigations weighing the possibilities of defining the state of a physical system, on one hand, and of making observation on the system, on the other hand, formed the central theme in Bohr's analyses on the viewpoint of complementarity (Bohr[48, 49]). The determination of the past and the future of a physical system in quantum mechanics exhibit features which suggest to formulate a new quantitative mode of complementarity, and which may be related to the intuitive ideas of Bohr on the complementarity between definition and observation. In fact, defining (preparing) a physical system in a pure state implies that it is isolated from its environment. Therefore, strictly speaking, it cannot be observed, since an observation entails an interaction which amounts to suspending the system's isolation. Rather than developing here a systematic theory, we shall be content with illustrating the "complementary but exclusive features" (Bohr[48]) of the past and the future of a quantum mechanical system, referring to and summarizing the results of Sections 2 and 3.

A measurement of a physical quantity yields *values* which provide information on the *state* of the system. In classical mechanics measurements can always be devised such that they do not "disturb" the system. The observables of the classical systems assume objective values at all times. In contrast to this, quantum mechanics has taught us (a) that prior to the measurement a physical quantity need not be objective, i.e., it need not have a value (namely if the premeasurement state is no eigenstate of the measured quantity), and (b) that in general a measurement will change the state of the system. Furthermore, as we have seen in Sec. 2, a single measurement result does not generally allow to infer the (initial) state the system. This leads to the following distinction of the purpose a quantum measurement can serve: a measurement yields a *value* of a physical quantity, and it is intended to provide information on the *state* of the system; both the values as well as the inferred states may refer to either the *past* (premeasurement situation) or the *future* (post measurement situation). Therefore we distinguish between the determination of the past values (DPV) or state (DPS) and the determination of the future values (DFV) or state (DFS). We shall see that the determination of the past (DP) and the determination of the future (DF) possess complementary features.

Corresponding to the above two points (a) and (b) there are two mutually exclusive requirements to be imposed on quantum measurements: (1) the objectification of the measured quantity (which will amount to DF; DFV, DFS), and (2) the preservation of the state of the system during the measurement (so that the measured value would also refer to the past; DPV, DPS). [We recall that the objectification-requirement simply means that the measurement should objectify the measured quantity, i.e., after the measurement the measured quantity should have a well defined though possible subjectively unknown value. The objectification-requirement is related to the problem of "reading the actual measurement result" which was already mentioned in the beginning of Sec. 3. We do not apply the quantum theory of measurement to analyze the possibility of the objectification (for that see again Beltrametti *et al.*[2]) but we take it here for granted.] Both of the two requirements are necessary preconditions for the constitution of a quantum system, and as such they have been analyzed in a realistic approach to quantum mechanics (Busch,[7] Mittelstaedt[50]). They can be reconciled only if the set of premeasurement states is known to be restricted to the set of eigenstates of a (sharp) quantity A (cf. Thm. 2.1.5). In such a situation an *ideal* measurement can be performed to detect the actual (eigen) value of A—which is already objective—without changing the state of the system. In that case the aims DPV and DFV are simultaneously satisfied, too. Moreover, if the eigenvalues of A are nondegenerate, then the detected value determines also the associated unique eigenstate of A, in which case the aims DPS and DFS are satisfied as well. Clearly if the quantity A under consideration is maximal then this all is optimally satisfied. (We recall that a maximal quantity A, or equivalently a complete set of commuting quantities, determines a maximal Boolean sub σ-algebra of $\mathcal{P}(\mathcal{H})$. Hence such a quantity A may be called *Boolean complete.*) If the premeasurement state is no eigenstate of the measured quantity, then the objectification (requirement (1) for DFV) can be achieved by a repeatable measurement which then forces the system into an eigenstate of the measured quantity. The requirement (2) must then be given up; the state of the system was changed under the measurement. We refer to this as the destruction of the history of the system. (In the literature one finds the related terms "collapse," "reduction," or "quantum jump.")

In the following we investigate the possibility of realizing the three pairwise exclusive goals of the determination of the past (DP), the future

(DF), and the preservation of history (PH), and we shall point out the prices to be paid for optimizing either of them.

4.1. Determination of the Future

We are now seeking for an optimal determination of the future (DFV, DFS) of the system by means of a measurement on it. Assume thus that an E-measurement $\langle \mathscr{H}_M, A_M, T_M, V\rangle$ is performed on $\mathscr{S}$ and the result $X \in \mathscr{B}(\mathbb{R})$ is obtained. Then $T_f := \operatorname{tr}(T_i E(X))^{-1} I_V^E(X) T_i$ is the post-measurement state of $\mathscr{S}$, when T_i was its initial state. Though the state T_f is here uniquely given by the E-measurement we do not consider this as an optimal DFS (or DFV). This is because in the state T_f we have, in general, only probabilistic information on E; the condition $\operatorname{tr}(T_f E(X)) = 1$ need not hold now. If this condition would hold then the E-measurement would have led to an optimal DFV. As it was already pointed out in Sec. 3.1, $\operatorname{tr}(T_f E(X)) = 1$ if and only if $\operatorname{tr}(I_V^E(X)\, I_V^E(X)\, T_i) = \operatorname{tr}(I_V^E(X)\, T_i)$. If this condition holds for all possible value sets X and for any (initial) state T_i, then the measurement (or its instrument) is *repeatable.* Repeatable measurements are thus exactly those measurements which lead to an optimal DFV. But it is known that the repeatability condition can only be realized for discrete quantities $E\colon \omega_i \mapsto E_i$ (Ozawa[13]). However, the repeatability condition does not yet guarantee that from the measured value ω_k, say, the postmeasurement state could be inferred in a *unique* way. The condition $\operatorname{tr}(T_f E_k) = 1$ leads to a unique determination of T_f only if the 1-eigenvalue of E_k is nondegenerate, i.e. the eigenspace $\{\varphi \in \mathscr{H} \mid E_k \varphi = \varphi\}$ is one-dimensional. But then quantum theory of measurement teaches us that, in essence, only the von Neumann–Lüders measurements $T \mapsto I_{vNL}^A(\{a_i\})T = P_i T P_i$ (of discrete sharp quantities $A = \sum a_i P_i$) realize these requirements (Beltrametti *et al.*[1]). To guarantee the unique future state inference, the measured sharp quantity must also be maximal (Boolean complete) so that the spectral projections P_i associated with the eigenvalues a_i are one-dimensional.

We shall illustrate next a possibility to quantify the degree of the determination of the future values (DFV) of a physical quantity. Consider a discrete sharp quantity A with the spectral measure E^A and with the eigenvalues a_i, $i = 1, 2,\ldots$. Let $T \in \mathscr{T}_s(\mathscr{H})_1^+$. The Shannon information of the discrete probability measure E_T^A is defined as $\mathrm{INFO}_S(E_T^A) \equiv \mathrm{INFO}_S(A, T) := \sum E_T^A(\{a_i\}) \ln E_T^A(\{a_i\})$, where now $E_T^A(\{a_i\}) = \operatorname{tr}(TE^A(\{a_i\}))$. Assume that the von Neumann–Lüders measurement of A is performed on the system $\mathscr{S}$ in the state T. Then $I_{vNL}^A(\mathbb{R})T = \sum E^A(\{a_i\})\, TE^A(\{a_i\})$ is the postmeasurement (objectified) state of $\mathscr{S}$ before the reading of the actual value of A. The objectified state

("Gemenge") $I^A_{vNL}(\mathbb{R})T$ represents the ignorance about the actual value of A. Now $\text{INFO}_S(A, T) = \text{INFO}_S(A, I^A_{vNL}(\mathbb{R})T)$ which shows that the amount of information contained in E^A_T and characterized by $\text{INFO}_S(A, T)$ has been made accessible through the von Neumann–Lüders measurement of A and it can be obtained by reading the actual result. This justifies to interpret $\text{INFO}_S(A, T)$ as the potential information gain in determining the values of A in the state T. Hence we denote

$$\text{PIG}(A, T) := \text{INFO}_S(A, T) = \text{INFO}_S(A, I^A_{vNL}(\mathbb{R})T) \tag{40}$$

and we take it as a quantitive measure of the degree of DFV.

The potential information gain $\text{PIG}(A, T)$, $T \in \mathcal{T}_s(\mathcal{H})^+_1$, is the bigger the "finer" the quantity A is. Indeed, if B is another (sharp) quantity such that $\mathcal{R}(E^B) \subseteq \mathcal{R}(E^A)$ (so that $B = f(A) = \sum f(a_i)\, E^A(\{a_i\})$ for some Borel function f) then it can easily be verified that

$$\text{PIG}(B, T) \leqslant \text{PIG}(A, T) \tag{41}$$

for any $T \in \mathcal{T}_s(\mathcal{H})^+_1$. Thus the potential information gain is maximal for maximal quantities, in which case also DFS is optimal.

4.1.1. Remark

The definition of the Shannon information of a probability measure E^A_T does not require that this measure is discrete. However, the *interpretation* of $\text{INFO}_S(A, T)$ as the potential information gain $\text{PIG}(A, T)$ refers explicitly to the von Neumann–Lüders measurement of A, i.e. $\text{INFO}_S(A, T) = \text{INFO}_S(A, I^A_{vNL}(\mathbb{R})T)$. This then requires that A is discrete. E.g. in case of position Q (as well as for the momentum P) we have $\text{INFO}_S(Q, P[\varphi]) = \int |\varphi(x)|^2 \ln |\varphi(x)|^2\, dx$ for any unit vector $\varphi \in \mathcal{H} = \mathcal{L}^2(\mathbb{R}, dx)$ but $\text{INFO}_S(Q, P[\varphi])$ cannot be interpreted—along the above ideas—as $\text{PIG}(Q, P[\varphi])$. Instead of Q we may then consider "discretized" position. Let $\{X_i: i = 1, 2, \ldots\} \subset \mathcal{B}(\mathbb{R})$ be a countable disjoint partition of $\mathbb{R}$, the spectrum of Q, i.e. $\mathbb{R} = \bigcup X_i$, and $X_i \cap X_j = \varnothing$ for all $i \neq j$. Let f denote the Borel function on $\mathbb{R}$ with the property $f(x) = i$ whenever $x \in X_i$, for all i. Then $f(Q)$ is a discrete position with the eigenvalues $i = 1, 2, \ldots$ and with the eigenprojections $E^{f(Q)}(\{i\}) = E^Q(f^{-1}(\{i\})) = E^Q(X_i)$, $i = 1, 2, \ldots$. Clearly, for any such $f(Q)$ we have $\text{PIG}(f(Q), T) = \text{INFO}_S(f(Q), T) = \text{INFO}_S(f(Q), I^{f(Q)}_{vNL}(\mathbb{R})\, T)$, and $\text{PIG}(f(Q), T) \leqslant \text{PIG}(g(Q), T)$, $T \in \mathcal{T}_s(\mathcal{H})^+_1$, whenever g refers to a finer (countable disjoint) partition of $\mathbb{R}$. As none of the discrete positions $f(Q)$ is maximal we do not obtain, in general, an optimal DFS with them.

We shall next estimate the price for obtaining optimal DFS, which is two-fold. First, Boolean complete quantities A allow only rather poor DPS: such A are informationally complete only with respect to their set of nondegenerate eigenstate (cf. Thm. 2.1.5). Nevertheless, among the set of sharp discrete quantities they allow relatively optimal DPS. Indeed, if A is a maximal quantity and $B=f(A)$ for some Borel function f, then $[T]^A \subseteq [T]^B$ for any $T \in \mathcal{T}_s(\mathcal{H})_1^+$ (as $\mathcal{R}(E^{f(A)}) \subseteq \mathcal{R}(E^A)$ for any f), i.e. A is *more informative* than B (in the sense that the state distinction power of A is higher than that of B's). The second price for optimal DFS is the *maximal destruction of history.* Consider again a discrete (sharp) quantity $A=\sum a_i P_i$ (with $P_i = E^A(\{a_i\})$) and let $B=f(A)=\sum f(a_i)P_i$ (so that B can be interpreted as a coarse-grained version of A). The spectral projections of $B=f(A)$ can now be given as $\bar{P}_i = \vee\{P_j: f(a_j)=f(a_i)\}$. If $P[\varphi]$ is a premeasurement pure state of $\mathcal{S}$, then $P[\varphi_i] \equiv P_i P[\varphi] P_i / \mathrm{tr}(P_i P[\varphi])$ and $\bar{P}[\varphi_i] \equiv \bar{P}_i P[\varphi] \bar{P}_i / \mathrm{tr}(\bar{P}_i P[\varphi])$ are the potential postmeasurement pure states of $\mathcal{S}$ after the von Neumann–Lüders measurements of A and B, respectively. (Here we assume that e.g. $\mathrm{tr}(P_i P[\varphi]) \neq 0$, i.e. a_i is a possible result when an A-measurement is performed on $\mathcal{S}$ in the state $P[\varphi]$.) We observe next that for any two pure states $P[\varphi]$ and $P[\psi]$, $\|P[\varphi]-P[\psi]\|_1 = 2(1-|\langle\varphi|\psi\rangle|^2)^{1/2}$. Hence $\|P[\varphi]-P[\varphi_i]\|_1 = 2(1-\langle\varphi|P[\varphi_i]\varphi\rangle)^{1/2} = 2\|(I-P[\varphi_i])\varphi\|$ and $\|P[\varphi]-\bar{P}[\varphi_i]\|_1 = 2\|(I-\bar{P}[\varphi_i])\varphi\|$. But for any i, $P[\varphi_i] \leqslant \bar{P}[\varphi_i]$, and thus $I-\bar{P}[\varphi_i] \leqslant I-P[\varphi_i]$, which then gives

$$\|P[\varphi]-\bar{P}[\varphi_i]\|_1 \leqslant \|P[\varphi]-P[\varphi_i]\|_1 \tag{42}$$

for any $\varphi \in \mathcal{H}$, $\|\varphi\|=1$ and for all i. This result shows that the more refined the preparatory measurement is the larger the distance between the past and the future states will be. In other words, to improve the preservation of history one has to use coarse-grained quantities, i.e. give up the optimal determination of the future values.

4.2. Determination of the Past

In general, the measured quantity is not objective in the premeasurement state of the system. Hence a single measurement result does not usually provide much information on the past state of the system. Therefore one has to make use of statistical methods for determining the past state of the system. (Cf. Sec. 2.2; for examples of some practical inference procedures for determining the past state, see Busch and Lahti,[51] and Busch and Schroeck.[52])

We have seen that an optimal determination of the past state of the system is possible by means of an informationally complete set of physical quantities (Sec. 2). Such a set is necessarily strongly noncommutative (Thm. 2.1.8). Therefore, as sharp quantities, they are not jointly measurable. The price for obtaining informational completeness in this way is then obvious: one has to make independent measurements of noncommuting quantities (to get DPS) and will not achieve a simultaneous preparation of the values of all the quantities (i.e. DFV). This motivates the introduction of unsharp quantities which, in fact, allow one to reconcile an optimal DPS (informational completeness) with a reasonable, though necessarily unsharp DFV.

Let E be a (generalized) physical quantity. If E is informationally complete, i.e. it admits an optimal determination of the past state of the system (DPS) (cf. Sec. 2.1–2.2), then E is necessarily strongly noncommutative, i.e., $\mathscr{R}(E)$ is strongly noncommutative (Thm. 2.1.8). This implies that E does not admit repeatable (completely positive) instruments, or conversely: if a (generalized) physical quantity E admits a repeatable (completely positive) instrument then E is neither informationally complete nor totally noncommutative. In fact, the existence of such an instrument implies that E is discrete, i.e., Ω can be chosen as a discrete set $\Omega = \{\omega_1, \omega_2, ...\}$ (Ozawa[13]). Put $E_i = E(\{\omega_i\})$. By repeatability, each E_i has an eigenvalue 1. Let $\varphi_i \neq 0$ be an eigenvector of E_i such that $E_i \varphi_i = \varphi_i$. Any such φ_i is in $\text{com}(E)$ as $E_k \varphi_i = 0$ for $k \neq i$. Assuming that E is nontrivial ($\mathscr{R}(E) \neq \{0, I\}$) shows that there are at least two different effects E_1, E_2 in $\mathscr{R}(E)$ with corresponding $\varphi_1, \varphi_2 \in \text{com}(E)$. But $\langle \varphi_1 | \varphi_2 \rangle = \langle E_1 \varphi_1 | \varphi_2 \rangle = \langle \varphi_1 | E_1 \varphi_2 \rangle = \langle \varphi_1 | 0 \rangle = 0$, so that $\dim[\text{com}(E)] \geqslant 2$. This excludes informational completeness ($\dim[\text{com}(E)] \leqslant 1$) as well as total noncommutativity ($\text{com}(E) = \{0\}$). The following two examples will illustrate the degree of unsharpness of the values of a quantity which is needed for its informational completeness.

4.2.1. Let $E^{\mathbf{a},\mathbf{b};C}$ be an informationally complete joint observable of any two orthogonal unsharp spin quantities $\mathscr{R}(\mathbf{a})$ and $\mathscr{R}(\mathbf{b})$ (cf. Sec. 2.4). Due to the coexistence of $\mathscr{R}(\mathbf{a})$ and $\mathscr{R}(\mathbf{b})$, and due to the informational completeness of $E^{\mathbf{a},\mathbf{b};C}$ the vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy the Eq. (30) which puts a limitation on the lengths of $\mathbf{a}$ and $\mathbf{b}$. In particular, neither $\mathbf{a}$ nor $\mathbf{b}$ is a unit vector: $\|\mathbf{a}\| < 1$ and $\|\mathbf{b}\| < 1$. Thus their "reality degrees" (in the sense of maximal possible probabilities) are less than one. Also the maximal information on the values of $\mathscr{R}(\mathbf{a})$ and $\mathscr{R}(\mathbf{b})$ is not optimal, i.e. $\|A(i\mathbf{a}) + B(j\mathbf{b})\| < 1 + \frac{1}{2} < 1 + \sqrt{\frac{1}{2}} = \|A(i\hat{a}) + B(j\hat{b})\|$ for all $i, j = +, -$ (cf. Sec. 3.4).

4.2.2. In Section 2.3 we saw that position Q and momentum P admit an informational completion by means of a phase space observable E (defined by Eq. (21)), whose marginals are the unsharp position Q_f and the unsharp momentum P_g with Fourier-related confidence functions f and g. The decreasing sequence

$$[T]^{Q/P} \supset [T]^{(Q,P)} = [T]^{(Q_f,P_g)} \supset [T]^E = \{T\} \tag{43}$$

$T \in \mathscr{T}(\mathscr{H})_1^+$, then show the increasing state distinction powers of the relevant quantities. By means of the instrument

$$I^E(Z)T = \int_Z T_{qp} \operatorname{tr}(TT_{qp})\, d\mu(q, p) \tag{44}$$

where $T_{qp} = U_{qp} P[\varphi] U_{qp}^+$ and $d\mu(q, p) = (1/2\pi)\, dq\, dp$, one can achieve the preparation of "almost pure" states if one reads the "points" (q, p), i.e. "small" sets $Z = Z(q, p)$ containing (q, p):

$$I^E(Z(q, p))T \approx \mu(Z(q, p)) \operatorname{tr}(TT_{qp}) T_{qp} \tag{45}$$

(For the technical details, see Busch and Lahti[51]). We call these readings $Z(q, p)$ *maximal* determinations of the future values of E as they approximately lead to the pure state T_{qp}; they are not optimal as they are unsharp values, which implies that they cannot be reproduced with probability comparable to one. (In fact, we have $\operatorname{tr}(T_{qp}E(Z(q, p))) = (1/2\pi) \int_{Z(q,p)} \operatorname{tr}(T_{q'p'} T_{qp})\, dq'\, dp' \leqslant \mu(Z(q, p)) \ll 1$.) This shows that the price to be paid for the informational completeness (i.e. optimal DPS) is that only a rather poor DFV is possible.

The unsharpness needed for the informational completeness of the joint observable E of Q_f and P_g was specified in subsection 2.3.5; the confidence functions f $(= |\varphi|^2)$ and g $(= |\hat{\varphi}|^2)$ satisfy the uncertainty relation, i.e. $\Delta f \cdot \Delta g \geqslant 1/2$. Furthermore, the condition $\langle \varphi \mid U_{qp} \varphi \rangle \neq 0$ a.e. $(q, p) \in \mathbb{R}^2$ implies that the confidence functions f and g are nonvanishing (a.e.). In (Busch and Lahti[51]) an operational characterization of the unsharp (q, p) values is given: in order to have $\operatorname{tr}(TE(Z)) \geqslant 1 - \varepsilon$ for some small positive number ε, one needs $\mu(Z) \gg 1$. (Note that ε cannot be zero!) Furthermore, as $\varepsilon = 0$ is impossible, one can only realize (ε, δ)-repeatability: although the instrument I^E practically leads to the pure state T_{qp} upon reading the value $Z(q, p)$, this reading cannot be confirmed (with a high confidence). In order to have $\operatorname{tr}(T_{qp}E(Z)) \geqslant 1 - \varepsilon$ one has to choose a "large" set Z, $\mu(Z) \gg 1$, containing $Z(q, p)$. This shows again that an optimal determination of the past state of the system with the quantity E is connected with a poor determination of its future values. The second price for informational completeness in the present case is a strong destruction of the history, in particular in the case of "fine" readings $Z(q, p)$. In general, the measurement in question transforms a pure state $P[\psi]$ into a mixture of the states T_{qp} with the weights $\langle \psi \mid T_{qp} \psi \rangle$, i.e. $I^E(Z) P[\psi] = \int_Z T_{qp} \operatorname{tr}(P[\psi] T_{qp})\, d\mu(q, p)$.

4.3. Preservation of the History

Assume that a measurement $\langle \mathscr{H}_M, A_M, T_M, V \rangle$ of a quantity E: $\mathscr{A} \to \mathscr{L}(\mathscr{H})^+$ on the system $\mathscr{S}$ in a state T_i is performed. Then $I_V^E(\mathbb{R}) T_i$ is the state of $\mathscr{S}$ after the E-measurement but before reading the actual result. If the actual result is the set $X \in \mathscr{A}$, then $T_f = \mathrm{tr}(T_i E(X))^{-1} I_V^E(X) T_i$ is the postmeasurement state of $\mathscr{S}$. If

$$T_f = T_i \tag{46}$$

then the measurement does not change the state of the system, its history is optimally preserved. As well known, this can be achieved by an ideal measurement of a sharp discrete quantity $E = E^A$ if the system initially is in an eigenstate of A. Usually $T_f \neq T_i$, i.e. the state of $\mathscr{S}$ is changed under the measurement and, at least a part of the history is thereby destroyed. For a given quantity E one may then attempt to characterize those E-measurements which minimize the destruction of the history. Typically, the ideality of a measurement is a property which refers to a minimal state change under the measurement. (Here we do not attempt a systematic analysis of an optimal preservation of the history of the system, but we only remark on that in the connection with the determination of the past and of the future of the system.)

Let A be a discrete sharp quantity: An ideal measurement of A is then (essentially) its von Neumann–Lüders measurement (Beltrametti *et al.*[11]). It leads to a minimal destruction of the history of the system in the sense that any quantity which was objective in the premeasurement state of the system remains objective also in the postmeasurement state provided that this quantity is compatible (i.e. commutes) with the measured quantity A. If A is not Boolean complete we may refine it essentially with refining the value space of A (cf. Sec. 4.1). This then leads to an improvement in determining the future values of A. But as demonstrated in Sec. 4.1, this implies an increase in the destruction of the history of the system.

These ideas may be extended to discrete unsharp quantities E which admit almost ideal measurements. Let $\Omega = \{\omega_1, \omega_2, ...\}$, $E_k = E(\{\omega_k\})$. The state transformations $T \mapsto \phi_k T := E_k^{1/2} T E_k^{1/2}$ are almost ideal in the following sense. If $\mathrm{tr}(TE_k) \geqslant 1 - \varepsilon$ for a small positive number ε (which in general cannot be zero), then the postmeasurement state $T_k := \mathrm{tr}(\phi_k T)^{-1} \phi_k T$ is close to the premeasurement state T in the sense that $\|T - T_k\|_1$ is of the order of $\sqrt{\varepsilon}$ (cf. Busch[30]). This shows that if a value ω_k is approximately objective in the state T in the previous (ε) sense, then it can be measured with maintaining a good preservation of the history of the system. The amount of unsharpness required for a good preservation of history can be illustrated in the case of a phase space observable. Let $\{Z_k \mid k \in \mathbb{N}\} \subset \mathscr{B}(\mathbb{R}^2)$

be a partitioning of the phase space $\mathbb{R}^2$, $E_k = E(Z_k)$. The condition $\text{tr}(TE_k) \geqslant 1 - \varepsilon$ of approximate objectivity requires that $\mu(Z_k) \gg 1$. Thus, again, an improvement in determining the future values of E, i.e., a refinement of the value sets Z_k, leads to an increasing destruction of the history of $\mathscr{S}$. Finally, we remark that in the connection of the so-called Zeno paradox it has been shown that there are unsharp measurements which allow one to monitor the "trajectory" (continuous history) of the system without too much destructive influence (for that see Barchielli, Lanz, and Prosperi[53]).

These considerations illustrate that though an improvement in determining the future values of a physical quantity goes along with an increas-

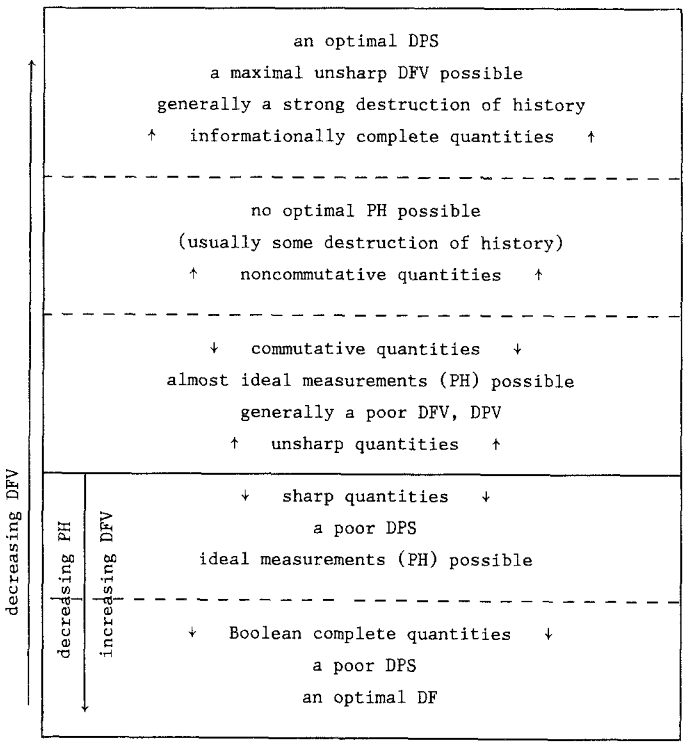

Fig. 1. The class of physical quantities splits up into five subclasses according to the associated measurement theoretical possibilities of determining the past state (DPS), future values (DFV), or of preserving the history (PH) of the system. Boolean complete quantities are a distinguished class of sharp quantities. Sharp quantities do not exhaust the class of commutative quantities as there are also unsharp commutative quantities. Among the class of noncommutative quantities there are the informationally complete quantities. The arrow shows the direction where the claimed property holds true.

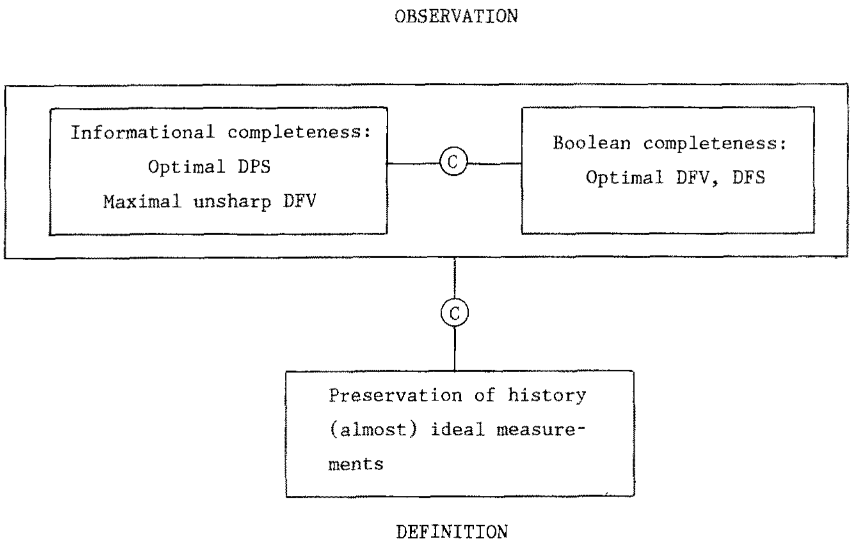

Fig. 2. The determination of the past and the future are complementary (C) aims of measurements. Observation and definition show up another mode of complementarily.

ing destruction of the history of the system, these two aspects, DFV and PH, may be reconciled, at least to some extent, by the unsharp measurements. Such measurements allow also some inference of the past state of the system (DPS). Thus "definition" and "observation," though mutually exclusive, can be reconciled in an (essentially) unsharp way. They both are needed as necessary conditions for the constitution of a physical system. "Observation" and "definition" form thus a complementary pair of concepts. Further, as a measurement usually leads to a destruction of the history of the system, one has to choose whether one wants to learn about the past (DP) or about the future (DF) of the system. These aims cannot be optimized simultaneously. But, again, they can be reconciled unsharply. Also, they both are needed to identify the physical system. In this sense the determination of the past and the determination of the future are complementary aims of measurements.

The results of this section are schematically summarized in Figs. 1 and 2.

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