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Coordinates and Covariance: Einstein's View of Space-Time and the Modern View

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Where modern formulations of relatively theory use differentiable manifolds to space-time, Einstein simply used open sets of R^4, following the then current methods of differential geometry. This fact aids resolution of a number of outstanding puzzles concerning Einstein's use of coordinate systems and covariance principles, including the claimed physical significance of covariance principles, their connection to relativity principles, Einstein's apparent confusion of coordinate systems and frames of reference, and his failure to distinguish active and passive transformations, especially in the context of his hole and point-coincidence arguments

1. INTRODUCTION

1.1. The Problem Defined

Modern readers, who approach Einstein's accounts of the fundamentals of his special and general theories of relativity, find that they often seem ambiguous, confused, or even incoherent. A major locus of these problems lies in Einstein's use of coordinate systems, coordinate transformations, and covariance principles. There are at least three specific problems:

1. Covariance and relativity principles. Einstein presented the requirement of general covariance as a fundamental physical principle which generalized the principle of relativity of the special theory of relativity. In the modern context, general covariance is taken to be a minimal requirement of mathematical coherence to be satisfied by any intelligible space-

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time theory and, therefore, essentially vacuous physically and certainly not a relativity principle. See Earman,[3] Friedman,[24] Chaps. II and V, and Torretti,[51] Sec. 5.5, for samples of an expansive literature, which also extends to the two further points below.[2] The objection must also be raised against Einstein's preferred formulations of the special principle of relativity and the principle of equivalence, for both are often stated by him as requirements of limited covariance. The former is a requirement of Lorentz covariance; the latter is an extension of it which includes transformations to accelerated coordinate systems.

2. *Coordinate systems, frames of reference and relative spaces.* When Einstein wrote of a four-dimensional space-time coordinate system or reference system—terms which he used interchangeably—it is by no means clear whether he should be read as referring to what we now call a coordinate chart of the space-time manifold, a frame of reference (a congruence of timelike curves), or a relative space (a three-space defined by a frame of reference); see Norton.[40]

3. *Passive versus active transformations.* When Einstein wrote of a coordinate transformation, it is not always clear whether he should be read as referring to what we would now call a (passive) transformation between the coordinate charts of a differentiable manifold or an (active) diffeomorphism from points of the manifold to points of the manifold. See Norton.[41]

In recent historical research into Einstein's work on general relativity, it has proved possible to circumvent at least the second and third of these problems by the simple expedient of stipulating which modern term was "really" intended by Einstein when the above ambiguities come into play. Thus I reconstructed a version of Einstein's principle of equivalence, which talks of the presence of a gravitational field in an accelerated reference system, as really talking about the presence of a gravitational field in a relative space (Norton[40]). The recent analyses of Einstein's so-called "hole argument" and "point-coincidence" argument have depended in varying degrees on reading Einstein's talk of coordinate transformations as really referring to diffeomorphisms on the manifold. See Stachel,[46,47] Torretti,[51] Sec. 5.6, and Norton.[41] At times, there is good evidence that these readings are correct construals of Einstein's intentions. At others, the readings are supported by a principle of charity: they are the only readings that save Einstein's arguments from incoherence or triviality. In either case,

[2] This objection is the modern expression of a long tradition which holds that general covariance is physically vacuous and purely a challenge to the mathematical ingenuity of the theorist; see, for example, Kretschmann[33] and Fock,[22] p. 370.

the puzzles remain. Why should such stipulations, well justified or not, be needed to clarify Einstein's writing? And, in the broader context, how was it possible for Einstein to allow what appears to modern eyes as rampant ambiguity to infect so thoroughly the foundations of his work?

1.2. The Solution Summarized: Einstein's Manifolds

In this paper, I shall argue that these problems derive in large part directly from the state of differential geometry at the turn of the century, which provided Einstein with simpler mathematical tools than are now used. We have routinely misunderstood Einstein because we have incorrectly translated his claims from his simpler to the modern, more complicated mathematical setting.

Modern space-time theories represent physically possible space-times by four-dimensional differentiable manifolds. These manifolds are sets of points with a topology that ensures that they are locally diffeomorphic to R^4. Since we are interested only in the manifolds' topological properties, there is no need to specify any further properties of the manifolds' point sets and, in particular, we do not need to designate which mathematical entities comprise the point sets.

The Einstein of the 1910s drew his mathematical techniques from a literature in which this very general concept of a differentiable manifold had not yet been developed. When this literature needed to represent multi-dimensional continua mathematically, it simply used a special case of n-dimensional differentiable manifolds, the number manifold R^n, whose point set is the set of all ordered n-tuples of real numbers.[3] Einstein naturally followed this practice so that where we represent physically possible space-times by four-dimensional differentiable manifolds, he simply used the number manifold R^4. This representation amounts to a coordination of a physically possible space-time with R^4 by a system of four variables x^1, x^2, x^3, x^4, so Einstein called the representation a "coordinate system."

Einstein's number manifolds have considerably more intrinsic structure than the differentiable manifolds of modern formulations, and this extra structure has a canonical spatiotemporal interpretation in terms of preferred positions, natural rest frames, and relative spaces. Therefore we shall see that Einstein's specification of a coordinate system automatically

[3] Or more complicated related structures like C^n. Throughout this paper I take R^n to be the point set of all n-tuples of reals endowed with the "standard topology," defined for example in Bishop and Goldberg,[(2)] pp. 11–12. I stress, however, that the distinction between this R^n, the topological space, and R^n, the set of quadruples of reals, was never clearly drawn in Einstein's work of the 1910s.

incorporates the specification of a frame of reference, a relative space, and much more.

But this extra structure creates problems for Einstein. If one coordinates a physically possible space-time with R^4, then we assert that the space-time contains a natural rest frame, coordinated to the canonical rest frame of R^4, and even a preferred center, coordinated to the origin $\langle 0, 0, 0, 0 \rangle$ of R^4. Einsteins's solution to this problem was to invoke the methods of Felix Klein's *Erlangen* program. He allowed that each physically possible space-time could be coordinated in very many different ways to the number manifold, and he accorded physical significance only to those properties of the number manifold (and structures defined on it) which remained invariant under these changes of coordinate system. Thus this principle of covariance has physical significance in Einstein's hands, since it denied, for example, that physically possible space-times have a preferred center analogous to $\langle 0, 0, 0, 0 \rangle$ of R^4. Moreover, the principle has the character of a relativity principle, since it denies, for example, that physically possible space-times have natural rest frames or, perhaps, natural inertial frames, according to the size of the covariance group in question.[4]

Einstein's covariance principles can have physical content because his coordinate systems coordinate a *physical* structure with a purely *mathematical* structure. This coordination relation looks very much like another within the modern theory of differentiable manifolds. We now define a differentiable manifold to be a topological space of points and a set of smooth, invertible maps which map its open sets onto open sets of R^n. These maps coordinate the point set with R^n, so they are naturally called "coordinate charts." Unfortunately the modern practice has been to read Einstein's coordinate systems as coordinate charts. The consequences are disatrous for the relations described by coordinate charts obtained as definitions, so that covariance principles defined in terms of them will have the status of mathematical stipulations. This mistranslation is largely responsible for the modern belief that Einstein's covariance principles are physically vacuous.

Finally, the greater mathematical complexity of the modern approach explains Einstein's apparent confusion over active and passive transformations. The modern approach has two levels of representation: first, physically possible space-times are represented by general differentiable manifolds; second, the point sets of the differentiable manifolds are coordinated to R^4 by the coordinate charts. Active transformations are defined

[4] But I shall argue in Sec. 6.3 that this result still does not allow us to characterize general relativity as the theory that extends the principle of relativity to accelerated motion.

in terms of the first level of representation and passive transformations in terms of the second. Since Einstein's approach does not have this dual level of representation—physically possible space-times are just represented by R^4—no active passive distinction needed to be drawn. Einstein just used one type of transformation, the coordinate transformation, which performed functions now distributed between the two types of transformations of the modern view, so that at times Einstein's coordinate transformations look to modern readers like active transformations and at others like passive transformations. This realization will greatly simply treatment of Einstein's so-called "hole" and "point-coincidence" arguments in Sec. 5 below.

The solution offered here is summarized pictorially in Figs. 1 and 2.

1.3. Canonical Einstein Formulation of Space-Time Theories

The recognition of these important mathematical differences between Einstein's and the modern formulation of space-time theories has been hindered by their differing styles. The modern approach is extensional in

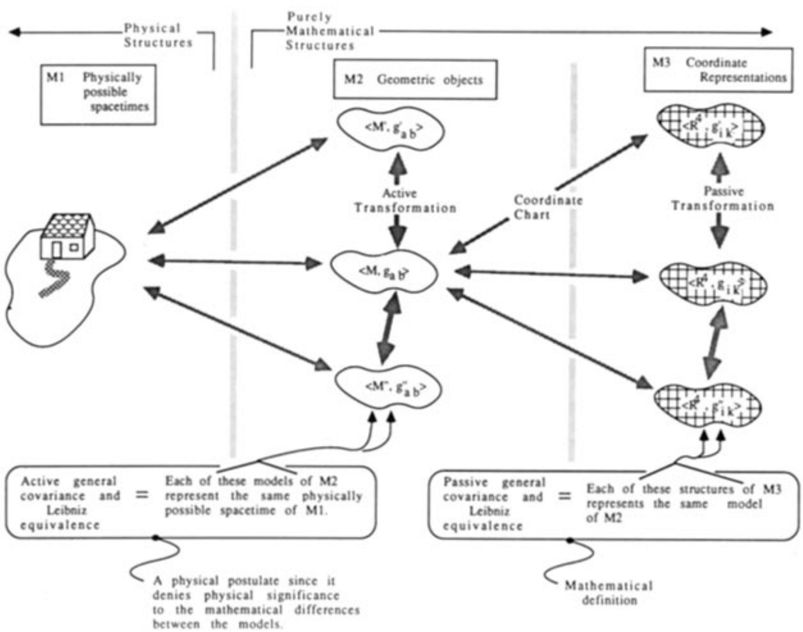

Fig. 1. The modern view of space-time theories.

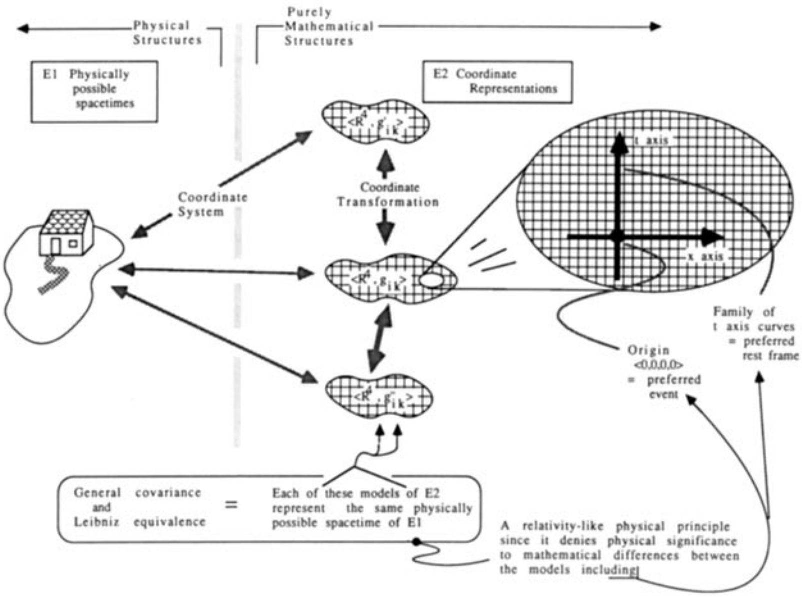

Fig. 2. The Einstein view of space-time theories.

spirit, seeking as much as possible to reduce its mathematical structures to those of intuitive set theory. Thus, broadly speaking, there are two parts in the modern formulation of a space-time theory.

> *First,* one carefully specifies the mathematical structures to be employed. This is usually achieved by specifying the form of the theories' models. General relativity, for example, will have models of the form $\langle M, g_{ab}, T_{ab}\rangle$, where M is a differentiable manifold and g_{ab} and T_{ab} tensors of appropriate type.
>
> *Second,* one formulates the conditions specifying the set of models allowed by the theory. These conditions are roughly the "laws" of the theory. They contain, for example, in the case of general relativity, the field equation $G_{ab} = \kappa T_{ab}$.

In the modern formulations, the emphasis lies squarely on the first phase; indeed the theory is often taken as synonymous with its set of models.

In Einstein's formulations of the same theories, more emphasis is placed on the second phase, with much less concern over the specification of the nature of the mathematical structures to be employed. Structures, such as a coordinate system, are often introduced without explication of their precise mathematical nature. Of course, it would be unreasonable to expect otherwise, for the mathematical literature to which Einstein turned had not yet made precise such crucial distinctions as between R^n, the point set of n-tuples of reals, and R^n, the continuum or number manifold, which is the point set endowed with the standard topology. One important outcome is that Einstein predicates the laws with theoretical properties such as general covariance, where the modern formulation predicates the set of models with such properties.

The greater complexity of the mathematical structure used in modern formulations explains why a certain amount of stipulation is needed in translating Einstein's accounts into the modern language, and it suggests that such direct translations will never be entirely satisfactory. But the above remarks point to a natural way of adding the precision of modern formulations of space-time theories to Einstein's. What is needed is an extensional style of formulating space-time theories, patterned as closely as possible after the modern one, but which only employs the mathematical structures actually used by Einstein. The canonical "Einstein" form for space-time theories of Sec. 4 attempts to achieve this. Notice that the canonical form proposed is not so much a modification of the form actually used by Einstein, but an expansion of it. Einstein's own accounts provide the laws of the theory, corresponding roughly to the second task listed above in the modern formulation of a space-time theory. The task of the canonical form corresponds to the first one listed. It is to make explicit the mathematical structures invoked implicitly by those laws and in a manner that parallels the modern formulation as closely as possible, but which is careful not to add any mathematical structures not actually used by Einstein. The result is a version of Einstein's accounts of space-time theories which closely parallels the modern accounts and whose claims can be judged with the precision now required of modern work in the philosophy of space and time.

For comparison with the canonical Einstein formulation described in Sec. 4, Sec. 2 contains a brief account of the modern view of space-time theories. It has been written with special emphasis on the status of the principles of active general covariance and Leibniz equivalence. The presence of these principles is usually not stressed elsewhere, but they are crucial to the understanding of both the modern view and the comparison with Einstein's view. Section 3 examines the tradition of work in differential geometry used by Einstein in the 1910s in order to support my basic thesis about Einstein's use of the number manifold R^n. Section 5 re-examines Einstein's accounts of the hole and point-coincidence arguments. It absolves him from any guilt in confusing active and passive transformations and shows that this pronouncements about their significance can be read both more literally and intelligibly in the context of a canonical Einstein formulation of relatively theory. Section 6 examines Einstein's accounts of covariance and relativity principles in the light of the canonical Einstein formulation of space-time theories. It shows how Einstein's covariance principles in special and general relativity can be read naturally as a sequence of relativity principles: the special principle of relativity, the principle of equivalence, and the general principle of relativity. But the equation of the general principle of relativity with a requirement of general covariance is still not vindicated.

2. THE MODERN VIEW OF SPACE-TIME THEORIES

2.1. Canonical Form

The general class of space-time theories to be discussed here are all those space-time theories that posit a differentiable manifold M with geometric object fields O_1, O_2,... defined everywhere on them. This class includes space-time versions of Newtonian mechanics and electrodynamics, special relativity and, of course, general relativity. See Friedman[24] for such space-time formulations of these theories. However, to avoid unnecessary generalization, I shall usually explicitly discuss only special and general relativity. They share the same basic space-time structure and are all that is needed for explication of the relevant parts of Einstein's writings. I shall treat these theories extensionally; that is, I shall regard each as synonymous with the set of its models. Thus special relativity is the set of all pairs of the form $\langle M, g_{ab} \rangle$, where M is a four-dimensional manifold and g_{ab} is a symmetric Lorentz signature metric field whose Riemann curvature tensor R^a_{bcd} is everywhere vanishing. At times, I shall also wish to consider the further structures of special relativistic matter theories, such as a Maxwell field F_{ab} and charge flux j^a, satisfying Maxwell's equations, or a fluid with stress energy tensor S_{ab} satisfying the usual mechanical laws. Since I do not wish to address specific issues in electrodynamics or mechanics and in order to maintain a formal similarity to general relativity, I shall represent this extra structure in extended models of the form $\langle M, g_{ab}, T_{ab} \rangle$, where T_{ab} represents whatever further structure is at hand, such as Maxwell fields or fluids. General relativity is the set of all triples of the form $\langle M, g_{ab}, T_{ab} \rangle$, where M is a four-dimensional differentiable manifold, g_{ab} a symmetric Lorentz signature metric field, and T_{ab} a stress energy tensor, such that the metric and stress energy tensor satisfy the Einstein field equation $G_{ab} = \kappa T_{ab}$, where G_{ab} is the Einstein tensor and κ

a constant.[5] All the space-times theories I shall consider will be cast into the

Canonical form for space-time theories: The theory has models of the form $\langle M, O_1,..., O_n \rangle$, where M is a differentiable manifold, optionally with specified dimensionality and global topological properties, and $O_1,..., O_n$ are n geometric objects of specified type. The set of models of the theory are exactly those which satisfy a set of conditions L, called the laws of the theory.

The specification of the laws of theory L can take many different forms. They might be relations between geometric objects, as in the above formulation of special and general relativity; or they might just be a list of the allowed members of the set.

Theories of this type deal with three types of structures:

$M1$. *Physically possible space-times*, one of which will be the physically actual space-time of our world if the theory in question is true.

$M2$. *Geometric structures*, which are mathematical objects such as $\langle M, g_{ab} \rangle$ or $\langle M, g_{ab}, T_{ab} \rangle$. They represent the space-times of $M1$.

$M3$. *Coordinate representations*, which are mathematical objects such as $\langle A, g_{ik} \rangle$ (or $\langle A, g_{ik}, T_{ik} \rangle$). They are the component representations of the structures of $M2$ in some coordinate chart. Here A is an open set of R^4, g_{ik} a 4×4 matrix of components, and the relevant coordinate chart is a diffeomorphism x^i from some neighborhood of the point set of M onto A.

The relationship between the structures of $M2$ and of $M3$ is one of mathematical definition, since the structures of $M3$ are formed from those of $M2$ by simple mathematical rules. The set A is the image set of the coordinate chart x^i; the matrix g_{ik} results from the application of g_{ab} to the matrix of pairs of basis vectors of x^i, $\langle \partial/\partial^i_x, \partial/\partial^k_x \rangle$. If we treat the g_{ab} of $M2$ as an equivalence class of matrices, rather than algebraically as a bilinear operator, then the $\langle A, g_{ik} \rangle$ of $M3$ is simply a substructure of the $\langle M, g_{ab} \rangle$ of $M2$ which it represents.

[5] I adopt the following convention with regard to indices. Indices $a, b, c, d,...$ are to be read according to the abstract index notation. See Wald,[53] Sec. 2.4. Thus g_{ab} is a second-rank covariant tensor. Indices $i, k, l, m,...$ and Greek indices designate the components of the geometric object in some coordinate chart; they take values 1, 2, 3, 4. Thus g_{ik} is the 4×4 matrix of components of g_{ab} in some coordinate chart.

By contrast, the relationship between $M1$ and $M2$ cannot be so precisely understood, since the structures of $M1$ are physically possible or even actual space-times. For example, we cannot simply point to some actual or possible object of experience which is a structure of $M1$ represented by a g_{ab} of $M2$. Rather, the essential idea is that of a similarity of *some* properties between the space-times of $M1$ and the mathematical objects of $M2$. For example, the time intervals read by a clock correspond to the length assigned to curves by the metric g_{ab}.

2.2. Physical Principles

One of the major charges against Einstein's use of covariance principles is that they are physically vacuous, at least if read literally. To enable analysis of this charge, I shall assume that

> *the physical principles of a theory are those whose truth depends at least in part on the properties of the structures of M1.*

The motivation for this definition is the idea that the properties of structures in $M1$ are matters of physical contingency and independent of any particular space-time theory which we may choose to consider.[6] This means that the question of whether the structures of $M2$ and $M3$ correctly represent those of $M1$ is in turn a matter of physical contingency. Thus the principles that deal with this representation are physical. I shall call a principle of a space-time theory purely mathematical just if it is not a physical principle. Therefore the principles that relate the structures in $M2$ to those in $M3$ are purely mathematical. This is a natural definition since the relationship between a geometric object of $M2$ and its coordinate representations in $M3$ is independent of the physically contingent properties of $M1$ and, in particular, of whether the objects in $M2$ or $M3$ represent such properties correctly.

[6] While I believe that the definition is sound, this motivation may not stand philosophical scrutiny. It is natural to define the notion of physical possibility, as used in $M1$, by means of a space-time theory, so that physically possible space-times are just those allowed by some selected space-time theory. Then a true space-time theory would be one whose set of possible space-times in $M1$ includes the actual space-time of our world. To adopt this approach in the context of the above definition of physical principle leads to the problem that the properties of structures in $M1$ become dependent by definition on those of $M2$. For the motivation to succeed, we must employ a notion of physical possibility that is theory independent and thus not so precisely defined. An instance of this notion is our belief (*pace* Kepler!) that it is physically possible for there to be one more or less planet in our solar system, or one more or less moon of Jupiter, independent of whether we work with a classical, special relativistic, general relativistic, or any other approximately adequate planetary mechanics.

2.3. Duality of Active and Passive Transformations

A natural duality obtains between transformations defined in $M2$ and in $M3$. If x^i is a coordinate chart of a manifold M, which maps the neighborhood N of M onto an open subset A of R^4, and f is a C^∞ invertible function from A onto A, then x^i and f together define dual transformations[7]:

Active: the diffeomorphism h which maps the point p of M with coordinates $x^i(p)$ in x^i to the point hp in M with coordinates $fx^i(p)$ in x^i; so that $x^i(hp) = fx^i(p)$.

Passive: the transformation from coordinate chart x^i of M to x'^i of M, such that a point p of M with coordinates x^i in the original coordinate chart is assigned coordinates $x'^i = fx^i$ in the new coordinate chart.

This duality enables the identification of a series of dual operations and principles. The first is the duality of the transformation law for the components of a geometric object under a coordinate transformation and the carry-along of the object under the dual diffeomorphism. If h is a diffeo morphism from M to M and $f: x^i \to x'^i$ its dual coordinate transformation (with respect to x^i), then the following are dual for the case of a tensor g_{ab}:

Active: h induces the carry-along map h^* which maps geometric object fields on the manifold to geometric object fields on the manifold in accord with the standard rules.[8] The carry-along of a tensor g_{ab} is the second-rank tensor h^*g_{ab}.

Passive: f defines a transformation law between the components of a geometric object in charts x^i and x'^i in the usual way. For example, for a tensor g_{ab}, its components g_{ik} in x^i to g'_{mn} in x'^m according to

$$g'_{mn} = \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} g_{ik} \tag{1}$$

[7] This duality is not exhaustive. A diffeomorphism on the manifold may have no dual coordinate transformation with respect to some coordinate chart, if the domain and range of the diffeomorphism are not contained within the domain of the coordinate chart. Conversely a coordinate transformation may have no dual diffeomorphism if the range of the new coordinate chart is not contained within the range of the original chart.

[8] Briefly, h induces a "carry-along" h^*f for real-valued scalar fields f, which is defined by requiring for all point p of the manifold that $h^*f(hp) = f(p)$. The carry along h^*V^a of a vector field V^a is defined by requiring for all real-valued scalar fields f that $h^*V^a(h^*f) = V^a(f)$. Finally the carry-along h^*g_{ab} of a second-rank tensor g_{ab} is defined by requiring for all vector fields V^c that $h^*g_{ab}(h^*V^c) = g_{ab}(V^c)$. These rules can be extended naturally to cover tensors of arbitrary type. See Wald,[(53)] Appendix C.1. By writing the above definitions with f as the four scalar functions of coordinate chart x^i, V^a as its basis vectors $(\partial/\partial x^i)^a$ and so on, one recovers the compact coordinate-based definition of the carry-along given below.

The duality of the two transformations becomes most apparent with the aid of the notion of "carried-along coordinate chart." For a given coordinate chart x^i, the coordinate chart carried along by a diffeomorphism h on the manifold is hx^i and its defining property is

$$hx^i(hp) = x^i(p)$$

This notion enables a versatile coordinate-based definition of the carry-along operator for a geometric object O under diffeomorphism h:

> The components of the carried-along geometric object h^*O at hp in the carried-along coordinate chart hx^i equal numerically the components of the original geometric object O at p in the original coordinate chart x^i.

Accordingly, for the case of a tensor g_{ab}, its carry-along is defined by

$$(h^*g)_{i'k'}(hp) = g_{ik}(p)$$

where primed indices i', k' represent components in the carried-along coordinate chart and unprimed indices represent components in the original coordinate chart. To compare the components of h^*g_{ab} and g_{ab} in the original coordinate chart, we now need only transform the matrix of components $(h^*g)_{i'k'}$ from the carried-along coordinate chart back to the original chart by applying the coordinate transformation $f: x^i \to x'^i$. Thus it follows that

> *The matrices of components* $(h^*g)_{ik}(hp)$ and $g_{ik}(p)$ *in the same coordinate chart are related by the coordinate transformation rule (1).*

This simple result tells us that the coordinate transformation rule (1) in effect covers both active and passive transformations. In the passive case it relates the components of a single tensor under coordinate transformation. In the dual active case, it relates the components of the two diffeomorphic tensors written in the same coordinate chart.

2.4. Active Principles

We can define two principles which employ active transformations and which may obtain in our space-time theories. Dual passive versions of them will be defined in the following subsection.

> *General covariance of a theory (active version):* If $\langle M, O_1, O_2, \ldots \rangle$ is any model of a space-time theory and h any diffeomorphism from M to a manifold M', then $\langle hM, h^*O_1, h^*O_2, \ldots \rangle$ is also a model of the theory.

Active general covariance usually arises automatically from the formulation of a space-time theory. Both special and general relativity, as defined at the beginning of this section, satisfy the requirement. Special relativity, for example, was defined as the theory whose models are the set of *all* pairs of the form $\langle M, g_{ab}\rangle$, where the symmetric Lorentz signature metric field g_{ab} satisfies the field equation $R^a_{bcd}=0$. If $\langle hM, h^*g_{ab}\rangle$ is any diffeomorphic copy of a model of special relativity, then its curvature tensor will be the carry-along of the curvature tensor of the original model. Thus it will vanish everywhere as well, since the carry-along of a zero tensor is itself a zero tensor. Similar remarks apply to general relativity, whose models are specified by the vanishing of the tensor $(G_{ab}-\kappa T_{ab})$. More generally, a space-time theory in canonical form will be generally covariant if its laws L are limited to relations between geometric objects. Earman and Norton[4] describe a broad subclass of this type of actively generally covariant space-time theories, called local space-time theories, whose laws L are all tensor equations.

Active general covariance is not an essential requirement of space-time theories. In older texts, special relativity is taken to be the theory of *a* Minkowski space-time, so that (in canonical terms) the laws L of the theory list just *one* model $\langle M, \eta_{ab}\rangle$, where η_{ab} is a Minkowski metric and M is usually an R^4 manifold.

Since the relation of being diffeomorphic is an equivalence relation, the models of a theory can be divided into equivalence classes of diffeomorphic models. Active Leibniz equivalence requires that each of the members of one of these classes represents the same physically possible space-time.

> *Leibniz Equivalence (active version)*[9]*:* If $\langle M, O_1, O_2,\ldots\rangle$ and $\langle hM, h^*O_1, h^*O_2,\ldots\rangle$ are diffeomorphic models of a space-time theory, then they represent the same physically possible space-time.

The following result is of great importance:

> *Active general covariance and active Leibniz equivalence are physical principles that one can choose to accept or deny in formulating a space-time theory.*

What justifies this result is that diffeomorphic models of a theory are distinct mathematical structures. Thus, in the most general case in which

[9] John Earman and I introduced the term Leibniz equivalence in Earman and Norton[4] since it is a modern formulation of Leibniz' thesis that the world would differ in no way if God had decided to place all objects in space with their positions reflected East to West. The requirement had already explicitly entered the newer general relativity texts, such as Hawking and Ellis,[25] p. 56, and Sachs and Wu,[43] p. 27, but its importance and physical content are rarely stressed.

every property of a model of $M2$ represents a unique physical aspect of a physically possible space-time of $M1$, each model must represent a *different* physically possible space-time. Active general covariance provides a recipe for taking a model from $M2$ of a physically possible space-time and generating arbitrarily many more mathematical structures, the carry-alongs under arbitrary diffeomorphism of the original model. It then asserts that all the differing mathematical structures so produced are also models of physically possible space-times. Whether this assertion is true depends on the properties of the physically possible space-times, the structures of $M1$, so, recalling Sec. 2.2, it follows that active general covariance is a physical principle. Active Leibniz equivalence asserts that two diffeomorphic models of $M2$ represent the *same* physically possible space-time of $M1$. This amounts to asserting that the properties on which the models differ represent nothing in the physically possible space-times of $M1$. Again, whether this assertion is true depends on the properties of the physically possible space-times of $M1$. So active Leibniz equivalence is a physical principle.

Active general covariance and Leibniz equivalence combined provide a systematic means of distinguishing those properties of the models of $M2$ which are physically significant, that is, which represent properties in the physically possible space-times of $M1$. Consider a space-time theory with initially just one model T in $M2$ representing just one physically possible space-time in $M1$. Asserting active general covariance introduces infinitely many new members to both $M1$ and $M2$. Then asserting active Leibniz equivalence counteracts this inflation of $M1$ and contracts the members of $M1$ back to the single original member. The important point, however, is that the two principles do *not* nullify one another's physical content. Applying them jointly alters the physical content of the above simple theory. Prior to their application, we would assume by default that each property of the model T represents a property of the physically possible space-time of $M1$, that is, is physically significant. After their application, however, we can only ascribe physical significance to the properties which T shares with all of its diffeomorphic copies. For, under active general covariance and Leibniz equivalence, T and every one of its diffeomorphic copies represent the same physically possible space-time of $M1$. So only those properties on which they agree can have physical significance. In general, the physically significant properties of a theory's models are those which are invariant under arbitrary diffeomorphism.

The mathematical differences between diffeomorphic models are meager, but there nonetheless. Consider the manifold M of one of the models $\langle M, g_{ab} \rangle$ of general relativity.[10] A fundamental property of a point

[10] For brevity I represent the model by its first two members, ignoring here the stress energy tensor T_{ab}.

p of M is the property of being a member of the point set of M. If M' is a different manifold, i.e., its point set is disjoint from that of M, then the property of set membership distinguishes the point p of M from any point p' of M'. Thus it follows that even two *diffeomorphic* models $\langle M, g_{ab} \rangle$ and $\langle hM, h^*g_{ab} \rangle$ of general relativity do not share all the same mathematical properties, if M and hM are different manifolds.[11]

Because these mathematical differences are so meager, there is correspondingly little physical content in the principles of active general covariance and Leibniz equivalence. But all I wanted to establish at this point is that they are physical principles.[12] Notice that the physical content of the principles would become very important if we constructed our models from manifolds, such as R^4, which have considerably more structure than is now usual. Then active general covariance and Leibniz equivalence would provide us a precise and systematic method of denying physical significance, for example, to the fact that $\langle 0, 0, 0, 0 \rangle$ is distinct from all other elements of R^4.

In actively generally covariant theories it is customary also to assert active Leibniz equivalence. But since Leibniz equivalence is a physical postulate, one must give physical arguments for or against its adoption. John Earman and I[(4)] offer two arguments for it, the first due essentially to Leibniz and Einstein and the second essentially to Einstein. Stated very briefly, in an actively generally covariant theory, if one denies active Leibniz equivalence then two undesirable consequences follow:

1. "*Point coincidence argument.*" One must accept that there are physically distinct space-times which no possible observation could distinguish. The reason is that the observables of a theory correspond to relations between structures defined on the manifold, not simply to the location of those structures on the manifold.[13] In two diffeomorphic models, all

[11] The result still holds if h is an automorphism so that M and hM are the same manifold, unless h is the identity or a symmetry of all the fields defined on M. For the point p and hp will be distinct mathematical objects. If they lie in the domain of the same coordinate chart, they will be distinguished by the difference of their coordinate values. If they do not both lie in the domain of at least some coordinate chart, then this difference distinguished them. It now follows that the two models cannot agree on all mathematical properties. For example, if the set of points $\{p\}$ is the image of a geodesic of g_{ab}, then the carried-along set $\{hp\}$ will not in general be the image of a geodesic of g_{ab}, but it will be the image of a geodesic of h^*g_{ab}.

[12] Active Leibniz equivalence does have at least one important application in the Cauchy problem of general relativity. See the "hole argument" below.

[13] Einstein's,[(9)] pp. 117–118, classic example is that the world line of a particle by itself does not correspond to an observable; what does, is the intersection of two such world lines, which corresponds to the observable collision of two particles.

those relations are preserved. Thus the models represent observationally indistinguishable space-times. But the denial of active Leibniz equivalence forces the assumption that the two models, in general, represent different physically possible space-times.

2. "*Hole argument.*" One must accept that the physically significant properties associated with a given space-time neighborhood—no matter how small—remain radically underetermined by the theory, even when all the fields on the space-time outside that neighborhood are fully specified. Let T be a model of a space-time theory and H (the "hole") any neighborhood of its manifold M; then we can define arbitrarily many diffeomorphisms h from M to M which are the identity outside H but smoothly come to differ from it inside H. The carry-along model h^*T will still be a model of the theory due to active general covariance and it will agree with T everywhere outside H; but it will differ from T within H. Denial of active Leibniz equivalence forces the conclusion that, in general, these differences are physically significant and establishes the radical indeterminism claimed.

2.5. Passive Principles

Recall that in space-time theories a coordinate chart x^i is a diffeomorphism from the point set S of a differentiable manifold M to R^4. Under x^i, the restriction of a space-time model $\langle M, O_1,..., O_n\rangle$ of $M2$ to the domain of x^i is given the coordinate representation $\langle A, (O_1)_{ik...},..., (O_n)_{ik...}\rangle$ in $M3$, where $A = x^i(S)$ and $(O_1)_{ik...}$ are the components of O_1 in x^i, etc. A coordinate transformation, a map f from R^4 to R^4 which assigns x^i to x'^m, induces a map from structures in $M3$ to structures in $M3$ so that $\langle A, (O_1)_{ik...},..., (O_n)_{ik...}\rangle$ is mapped onto $\langle fA, (O_1)'_{mn...},..., (O_n)'_{mn...}\rangle$, where the matrices $(O_1)_{ik...}$ and $(O_1)'_{mn...}$ (etc.) are related by the usual transformation rules, such as (1).

> *General Covariance of a theory (passive version):* If $\langle A, (O_1)_{ik...},..., (O_n)_{ik...}\rangle$ is a coordinate representation of a space-time model $\langle M, O_1,..., O_n\rangle$ of the theory and f any coordinate transformation with domain A, then the transform $\langle fA, (O_1)'_{mn...},..., (O_n)'_{mn...}\rangle$ is also a coordinate representation of a space-time model of the theory.
>
> *Leibniz Equivalence (passive version):* These two coordinate representations $\langle A, (O_1)_{ik...},..., (O_n)_{ik...}\rangle$ and $\langle fA, (O_1)'_{mn...},..., (O_n)'_{mn...}\rangle$ represent the same model $\langle M, O_1,..., O_n >$.

Unlike their active counterparts, these principles are purely mathematical, not physical, principles, for their truth is independent of the structures of

$M1$. It is a mathematical stipulation on our part that every relevant coordinate chart generate a coordinate representation of a space-time model and that each (obviously!) represents the same model. Usually the two requirements are combined and labelled "general covariance." I have separated them here only to display their duality with the active versions.[14] That the two passive principles are purely mathematical, but their active versions are physical, is crucial to my story. For Einstein's requirement of general covariance is now usually read as a passive principle. I shall argue that he intended something much closer to the active version.

3. MANIFOLDS IN THE ABSOLUTE DIFFERENTIAL CALCULUS

The distinction between the two levels of mathematical structure $M2$ and $M3$ depends upon the concept of the differentiable manifold of modern point set topology. (See, for example, Bishop and Goldberg.[(2)]) For it is crucial to the distinction to distinguish a differentiable manifold, as a general type of mathematical object upon which $M2$ is based, from R^n, a special case of a differentiable manifold upon which $M3$ is based. But the Einstein of 1912, standing on the threshold of his general theory of relativity, found no such distinction drawn clearly in the mathematical literature to which he then turned.

For the development of general relativity, Einstein turned to a branch of mathematics, which was that based upon Gauss' and Riemann's theory of surfaces. Riemann,[(49)] pp. 412–415, laid the concept of the "n-fold extended manifold" at the foundation of his classic inaugural lecture. But his description of it fell short of the mathematical precision soon to be achieved. Briefly he took an n-fold extended manifold to be a structure whose "modes of determination [allow] a continuous transition from one to another" (p. 412) in such a way that "the fixing of position in an n-dimensional manifold is reduced to n determinations of quantities" (p. 415). His examples were more physical than mathematical: "the position of objects of sense, and the colors" (p. 413). Felix Klein,[(32)] p. 289, later explained what Riemann "really" meant:

> At the foundations of his researches, Riemann laid n variables $x_1, x_2, \dots, x_n$, each of which can take all real values. Riemann denoted the totality of their value systems as *manifold of n dimensions*; by a fixed value system $x'_1, x'_2, \dots, x'_n$, he meant a point in this manifold. (Emphasis in original.)

[14] Thus in Norton[(41)] and Earman and Norton[(4)] active and passive versions of Leibniz equivalence are not distinguished at all. What is called active Leibniz equivalence here is just Leibniz equivalence there.

In modern language, this manifold is just R^n. This understanding of n-dimensional manifold was the standard for a long period. As early as 1873, Klein had already defined an n-dimensional manifold as the totality of values of the variables $x_1, x_2,..., x_n$, noting that this definition was "in agreement with usual terminology."[15] This definition survived well into the 1920s. Levi-Civita began his 1923 treatise on the absolute differential calculus with the observation that frequently "comlicated algebraic relationships represent simple geometrical properties." To take advantage of this conceptual simplification, he urges that "it is necessary to adopt the fundamental convention of using the term *point of an abstract n-dimensional manifold* (n being any positive integer) to denote a set of n values assigned to any n variables $x_1, x_2,..., x_n$." (Levi-Civita[(34)], p. 1; emphasis in original.)

The work of Levi-Civita figures prominently in the story of Einstein's introduction to the mathematical tools needed to construct his general theory of relativity. That story is now well known. (See Pais,[(42)] Sec. 12b.) Einstein returned to Zürich in August 1912 believing that the key to furthering his work on extending the principle of relativity to accelerated motion lay in Gauss' theory of surfaces, on which he had been lectured by C. F. Geiser during his student days at the ETH. He approached his mathematician friend Marcel Grossmann for mathematical assistance. Grossmann introduced Einstein to a tradition which embraced the work of Gauss, Riemann, Christoffel, Ricci, and Levi-Civita. Its culmination was a lengthy 1900 article written on the so-called "absolute differential calculus" by Ricci and his pupil Levi-Civita,[(48)] which contained the mathematical machinery needed for Einstein to construct his first outline of the general theory of relativity. That outline was completed in early 1913 and was published as a two-part paper by Einstein and Grossmann.[(20)] The first part—the "physical part"— contained Einstein's development of the theory; the second part—the "mathematical part"—contained Grossmann's summary of the mathematical techniques drawn from the absolute differential calculus and the proof of a number of results required for Einstein's physical part. The two review articles written by Einstein over the ensuing three years followed the pattern set in 1913. Both Einstein[(6)] and Einstein[(9)] contain a complete if compact development of the mathematical methods required for the theory. The motive, as Einstein[(6)] noted on p. 1030, was to "enable a complete understanding of the theory without the need to read other pure mathematical treatises."

[15] Klein,[(29)] p. 315. See Torretti,[(50)] p. 138, for some of the intricasies of Klein's own definition of manifold. Also see Toretti,[(50)] Sec. 2.2.6, for further discussion of Riemann's concept of manifold.

The development of the absolute differential calculus found in these three articles is characterized by its failure to discuss the nature of the mathematical manifold upon which the fields of the theory are defined. Rather the notion of a coordinate system is simply taken as a primitive and the starting point of the development is the study of coordinate transformations and the transformation laws for contravariant and covariant vectors. (This, of course, became the tradition for a generation of texts on general relativity.) This order of development is essentially the one which Grossmann found in Ricci and Levi-Civita's[48] article of 1900. Discussion of the manifold or space in which the structures of the paper are defined is limited by Ricci and Levi-Civita to a few brief remarks in that paper's short preface (pp. 481–482), which is largely devoted to tracing the ancestry of the results to be displayed:

> But these same method [of the absolute differential calculus], and the advantages which they present, have their *raison d'être* and their origin in the intimate connections, which bind them to the concept of manifold of n dimensions, which we owe to the genius of Gauss and Riemann.—
>
> According to this concept, a manifold V_n is defined intrinsically in its metrical properties by n independent variables and by all of a class of quadratic forms of the differentials of these variables, such that any two of them are transformable from one to the other by a point transformation.—As a result, a V_n remains invariant with respect to all transformations of its coordinates.

The structure defined here is, of course, a special form of the manifolds described above by Klein and Levi-Civita, in his later text. There, Levi-Civita,[34] p. 119, calls V_n a *metric* manifold and defines it as the aggregate of the values of the variables $x_1, x_2, \ldots, x_n$ (i.e., manifold $= R^n$) in conjunction with the specification of a differential form

$$ds^2 = \sum_{ik}{}_{1}^{n} a_{ik}\, dx_i\, dx_k$$

The 1900 definition is actually more sophisticated for it amounts to defining V_n as that part of the 1923 V_n which is invariant under arbitrary transformations.

Thus the Einstein of 1912 inherited the mathematical tools of a tradition which placed little emphasis on the question of just what were the mathematical objects which were being used to represent the real or possible real physical spaces of geometry. Insofar as it answered the question at all, that answer was to provide R^n, perhaps with extra structure, as the manifold. This was precisely the attitude that the Einstein of 1912 would have found in the contemporary literature which was then transforming special relativity into a four-dimensional space-time theory. Thus Minkowski,[(37)] pp. 56–57, introduced the requisite mathematical machinery in his less formal 1908 address as[16]:

> We will try to visualize the state of things by the graphic method. Let x, y, z be rectangular coordinates for space, and let t denote time... A point of space at a point of time, that is, a system of values x, y, z, t, I will call a *world-point*. The manifold of all thinkable x, y, z, t systems of values we will christen *the world*.

The notion of a rectangular coordinate system is introduced here as an unexplicated primitive. Four-tuples of reals do not denote world-points; they *are* world-points. And Minkowski's four-dimensional world *is*, by his definition, R^4. Coordinate systems and "space-time points" are introduced in a similar way in the opening pages of the more technical Minkowski,[(38)] but no mention is made of "the world" or of the nature of the four-dimensional manifold upon which the entire work is based. This lack of concern for the nature of coordinates and the manifold is exmplified by Sommerfeld's[(44)] resource articles which gives an exposition of the "four-dimensional vector algebra" needed to work with Minkowski's four-dimensional formulation of special relativity. The definition of coordinates, manifolds, and the like is skipped entirely; the first definitions given are for four- and six-vectors.

I can offer two reasons for this lack of emphasis.

First, the methods of the absolute differential calculus, especially, as its name betrayed, were developed in the context of real analysis, which was concerned primarily with the properties of real-valued functions on n-tuples of reals. There was a similar connection with real and complex analysis for geometry in the tradition of Klein's Erlangen program. It was often remarked how geometric ideas could elucidate nongeometric problems in analysis and *vice-versa*. (For example, see Levi-Civita,[(34)] p. 1, and F. Klein,[(31)] p. 1.) Presumably it was felt that this commerce between geometry and analysis could be encouraged if the geometric notions did not stray too far from those of real and complex analysis, which in turn encouraged the identification of the manifold of geometry with such structures as R^n.

[16] The standard translation in Minkowski,[(39)] p. 76, translates Minkowski's *Mannigfaltigkeit* in this passage as "multiplicity" where I, as elsewhere in this paper, translate it as "manifold."

Second, Felix Klein's Erlangen program directed geometricians to look to the invariants of groups as the true intrinsic geometric structures. The structures on which the groups themselves act—be it R^n or something more abstract—were of less interest. Klein,[(30)], p. 539, provided and stressed the following compact summary of his viewpoint towards special relativity[17]:

> What the modern physicists calls *relativity theory*, is the theory of invariants of a four-dimensional space-time region, x, y, z, t (the Minkowski "world") with respect to a particular group of collineations, namely the "Lorentz group";—or, more generally, and turned round the other way:
>
> If one wants to make a point of it, it would be alright to replace the phrase "theory of invariants relative to a group of transformations" with the words "relativity theory with respect to a group."[18]

(Notice that Klein, like Minkowski, takes his four-dimensional space-time to be a "region" of R^4.) Of course Einstein's work on general relativity played a large part in the demise of the Erlangen program, for that theory directed, to the forefront of mathematical attention, semi-Riemannian spaces whose symmetry groups contained in the general case just the identity map. Thus they could not be used nontrivially to define intrinsic geometric structure. Other methods were needed to plumb the arcane depths of these geometries. These methods were found in point set topology, which was in its infancy in 1910s. That decade saw the emergence of the modern notion of a topological space in the work of such mathematicians as Weyl and especially Hausdorff, with his 1914 *Grundzüge der Mengenlehre.*[19]

4. EINSTEIN'S VIEW OF SPACE-TIME THEORIES

4.1. Canonical Form

In modern philosophy of physics, the standard practice has been to read Einstein's "coordinate systems" as coordinate charts relating the mathematical structures in $M2$ to those in $M3$ of the modern formulation of space-time theories; to read his matrices g_{ik} as component representa-

[17] Einstein also came to feel that the name "*Invariantentheorie*" was a more accurate title for relativity theory. See Holton,[(28)] p. xv.

[18] Man könnte, wenn man Wert darauf legen will, den Namen "Invariantentheorie relativ zu einer Gruppe von Transformationen" sehr wohl durch das Wort "Relativitätstheorie bezüglich einer Gruppe ersetzen.

[19] I rely here on the account given in Manheim,[(35)] Chap. VI.

tions in $M3$ of geometric objects g_{ab} in $M2$; and to assume that the structures of $M2$ are present by tacit implicatin, even though they find no symbolic expression in Einstein's treatises. My basic thesis in this paper is that this reading of Einstein's work is mistaken and is responsible for a sustained and systemic misunderstanding of his account of the foundations of relativity theory.

Differential geometry of the 1910s, in the tradition of Klein, Minkowski, Levi-Civita and Einstein, provided a much simpler class of mathematical structures to represent physically actual or possible space-times than the $M2$ and $M3$ of the modern view. In the modern view, the properties of a physically actual or possible space-time of $M1$ are specified by requiring that it be representable by a differentiable manifold of point set topology of $M2$; then, as a part of the definition of a differentiable manifold, the topology of its point set is specified (partially) by requiring that its open sets be isomorphic to open sets of R^4 in $M3$. Summarizing the results of the last section, the older tradition eliminates the mediating structures of $M2$ in the sequence of representations of physically possible space-times ($M1$) point sets of general differentiable manifolds ($M2$), and R^4 ($M3$). The properties of actual or possible spaces or space-times are specified by requiring that their parts be representable by open sets of R^4 or R^n. This representation or *coordination* is effected by maps from actual or possible space or space-time points to n-tuples of reals, which maps are the systems of variables $x_1, x_2, ..., x_n$ of Klein and Levi-Civita and x, y, z, t of Minkowski and are called coordinate systems for short; their range, an open set of R^n, *is* the manifold. This was also Einstein's practice so that:

> *Where the modern approach represents a space-time by a differentiable manifold with a point set of unspecified elements. Einstein used open sets of $R^4$and called the representation a "coordinate system."*[20]

Thus an extensional formulation of space-time theories, which reflects this simpler approach, constructs its models from mathematical structures used only in $M3$. This formulation, which I urge should be used when reading Einstein's accounts, is:

> *Canonical form for space-time theories* (*Einstein*): The theory has models of the form $\langle A, (O_1)_{ik...}, ..., (O_n)_{ik...} \rangle$, where the manifold/coordinate system A is an open set of R^4, and $(O_1)_{ik...}, ..., (O_n)_{ik...}$ are n matrices with transformation laws of specified type. The set of models

[20] Throughout the paper I have adopted the convention of using the term "coordinate chart" in its modern sense only, whereas "coordinate system" refers to what are now revealed as the number manifolds of Einstein's models of special and general relativity, coordinated to the space-times of $E1$.

of the theory are exactly those which satisfy a set of conditions L, called the laws of the theory.

Where the modern view deals with three types of structures $M1$, $M2$, and $M3$, the canonical Einstein formulation deals with only two[21]:

E1. *Physically possible space-times*, one of which will be the physically actual space-time of our world if the theory in question is true.

E2. *Coordinate representations*, which are mathematical objects such as $\langle A, g_{ik} \rangle$ (or $\langle A, g_{ik}, T_{ik} \rangle$), which represent the space-times of E1.

The sets $E1$ and $M1$ for a given theory are the same. The set $E2$ contains all the mathematical structures of the theory and therefore is analogous to $M2$ and $M3$ combined. When it comes to reading Einstein's work, the immediate virtue of this "Einstein" canonical formulation over the modern view is that it employs only mathematical structures which appear explicitly in symbolic form in his accounts of space-time theories. To use the modern view, one must in effect assume the presence by tacit implication of the entire class structures of $M2$.

Einstein's formulation of space-time theories can be cast into Einstein canonical form with the minimum of modification, as the examples of special and general relativity show. His space-time formulation of special relativity, as for example in Einstein[(13)] and later writings, defines the properties of a Minkowski space-time by specifying that it has the line element

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2 \tag{2}$$

with then coordinate system x_i an unexplicated primitive notion or specified as given by natural measuring operations. Notice that the line element is always given in the simple diagonal form. In canonical form, this amounts to specifying that special relativity has a single model $\langle R^4, \eta_{ik} \rangle$, where η_{ik} is the 4×4 matrix diag$(-1, -1, -1, +1)$. (Multiple models would arise if we consider special relativistic matter theories. Using the same convention as in Sec. 2, we end up with a theory in canonical formulation whose models have the form $\langle R^4, \eta_{ik}, T_{ik} \rangle$. Each special relativistic matter theory contains all models of the appropriate form, which are related by transformations from the extended Lorentz group. Note that in all models, η_{ik} *always* has the diagonal form indicated.)

[21] I presume that the distinction between $E1$ and $E2$ is essentially the one Torretti,[(51)] pp. 302–303, note 10, introduces to negotiate an ambiguity in Minkowski's discussion of his "world."

Einstein defines the properties of the space-times of general relativity by specifying that they have a line element

$$ds^2 = g_{ik}\, dx_i\, dx_k$$

whose coefficients g_{ik} satisfy the field equation $G_{ik} = \kappa T_{ik}$. (Summation over repeated indices is implied.) So canonically, the theory has models of the form $\langle A, g_{ik}, T_{ik} \rangle$ or just $\langle A, g_{ik} \rangle$, where g_{ik} and T_{ik} are 4×4 matrices of reals such that both are symmetric and g_{ik} has Lorents signature. All such structures satisfying the following field equation: $G_{ik} = \kappa T_{ik}$ are models of the theory.[22]

Modern readers will probably suffer a powerful urge to try to interpose a class of structures between $E1$ and $E2$—say "$E^{1/2}$"—whose role would be to mediate the relationship between $E1$ and $E2$ much as $M2$ mediates between $M1$ and $M3$. Such readers will think of the matrices η_{ik} or g_{ik} as two of many possible coordinate representations of some other abstract *mathematical* objects η_{ab} or g_{ab}, and the coordinates x_i as one of many coordinate charts of another abstract differentiable manifold M. I urge against this interposition for three reasons.

First, the interpositon is unnecessary for the mathematical coherence of the formulation of the theory.[23] For a given theory, the structures of $E2$ of the Einstein formulation are formally identical to those of $M3$ of the modern formulation. But these latter structures encode all the formal properties of the structures of $M2$ so that all the results of the theory can be recovered from them.

Second, there is no clear evidence that Einstein intended such an interposition. What we do have is ambiguous evidence. For example, in the expositions of the absolute differential calculus of Einstein and Grossmann,[(20)] Einstein,[(6)] and Einstein,[(9)] the terms "vector" and "tensor" are reserved for equivalence classes of 4-tuples or matrices defined by the appropriate tensor transformation law. (Therefore I try to refrain from using the terms "vector" and "tensor" for the 4-tuples V^i and the matrices g_{ik} of the Einstein canonical formulation.) There is the strong suggestion that these classes in turn represent coordinate-free structures, and Einstein insists to Besso in a letter of October 31, 1916, that this is the case. (See

[22] A basic inadequacy of this formulation of special and especially general relativity is that it makes manifold topologies other than R^4 difficult to work with. Thus an Einstein universe can only be represented by more than one model which are appropriately connected to one another. Since this point is largely irrelevant to my concerns here, I shall not pursue the problem.

[23] Notice that the need for a "patchwise" treatment of space-times which are not globally R^4 is awkward, but not incoherent.

Toretti[51] p. 316n, Note 1.) But I see no way to inflate such remarks about vectors and tensors into the intermediate level of structures "$E1^{1/2}$." Such a level would have to have similar mathematical richness to $E2$ and considerably more structures than just vectors and tensors. For example, it would need differentiable manifolds distinct from R^4. But Einstein gives no systematic discussion of such a level and, for that matter, does not even reserve separate symbols for his tensors to distinguish them from their component matrices. I surmise that Einstein's terms "vector" and "tensor" either name thephysical structures *of E1* represented by tensorial matrices in $E2$ or name abstractions constructed from these matrices of $E2$. Such abstractions could have been used to begin construction of a new level of structures, but, in the event, *were not so used* by Einstein.

Third, if we insist that Einstein really did intend the interposition of this extra class of mathematical structures, then we will have to forgo many of the results of the remainder of the paper and concede to Einstein's critics, for example, that his treatment of the requirement of general covariance did involve a quite simple confusion of a mathematical and a physical principle.

4.2. Coordinate Transformations

The canonical Einstein formulation of a theory provides for just one type of transformation between the models of $E2$, the coordinate transformation. A coordinate transformation is defined by a map $f: x^i \to x'^i$ from R^4 to R^4 which relates a model $\langle A, (O_1)_{ik...}, ..., (O_n)_{ik...}\rangle$ of $E2$ to another model $\langle A, (O'_1)_{mn...}, ..., (O'_n)_{mn...}\rangle$. The relationship between each matrix $(O_r)_{ik...}$ and $(O'_r)_{mn...}$ will depend on the transformation law specified for the matrices in the canonical description of the theory's models and will follow the rules of standard developments of tensor analysis such as Einstein's own Einstein,[9] Part B. For example, the coordinate transformation $f: x^i \to x'^i$ transforms the model $\langle A, g_{ik}\rangle$ of $E2$ into $\langle A', g'_{mn}\rangle$, if the matrices g_{ik} and g'_{mn} transform as covariant tensors, when f maps A onto A' and g_{ik} and g'_{mn} stand in relation (1).

The modern formulation of space-time theories provides a richer set of mathematical structures than the Einstein formulation, for it provides two distinct mathematical levels $M2$ and $M3$, where the Einstein formulation provides just $E2$. The two levels $M2$ and $M3$ enable defining the dual active and passive transformations of Sec. 2. No such duality is available in the Einstein formulation. One just has transformations between the models of $E2$.[24] The existence of dual levels of transformations in the modern for-

[24] Notice that another duality of tranformation does remain. The transformation $f: x^i \to x'^i$ can be read as a simple coordinate transformation or as mapping the point with coor-

mulation means that one must proceed with extreme caution in translating results invoking a coordinate transformation in an Einstein formulation of a theory to corresponding results in the modern formulation of the theory. Take the above example of $f: x^i \rightarrow x'^i$ transforming the model $\langle A, g_{ik} \rangle$ of $E2$. Assuming that A covers the entire space-time modelled, the Einstein model $\langle A, g_{ik} \rangle$ will correspond to some model $\langle M, g_{ab} \rangle$ in $M2$ of the modern formulation, which, in some coordinate chart, will be represented by the structure $\langle A, g_{ik} \rangle$ in $M3$. The map f induces two transformations in the modern formulation:

—the passive transformation in $M3$ that relates the two component representations $\langle A, g_{ik} \rangle$ and $\langle A', g'_{mn} \rangle$ in $M3$ of the model $\langle M, g_{ab} \rangle$ in $M2$.

—the active transformation h, which maps the model $\langle M, g_{ab} \rangle$ of $M2$ onto $\langle hM, h^*g_{ab} \rangle$.

The standard practice has been to read Einstein's transformations as equivalent to the passive transformation. This reading is certainly the natural one, since the mathematical matrix expressions involved in both are *exactly* the same. But this reading fails in at least one significant aspect. The structures $\langle A, g_{ik} \rangle$ and $\langle A', g'_{mn} \rangle$ within $M3$ represent the same model $\langle M, g_{ab} \rangle$ *by definition*. But, as I shall urge below, whether the two structures $\langle A, g_{ik} \rangle$ and $\langle A', g'_{mn} \rangle$ of $E2$ represent the same physically possible space-time of $E1$ is a matter of *physical contingency*. In this regard, the Einstein transformations are just like the active transformations of the modern formulation, for whether the models $\langle M, g_{ab} \rangle$ and $\langle hM, h^*g_{ab} \rangle$ represent the same physically possible space-time depends on the physically contingent hypothesis of Leibniz equivalence. Note finally that there is a certain formal naturalness in relating Einstein transformations to the modern active transformations. It follows from the concluding result of Sec. 2.3 tat, if we represent the two models $\langle M, g_{ab} \rangle$ and $\langle hM, h^*g_{ab} \rangle$ in the same coordinate system, then we simply recover exactly the mathematical structures of the Einstein transformation, the structures $\langle A, g_{ik} \rangle$ and $\langle A', g'_{mn} \rangle$ related by (1).

This behavior resolves the third problem of the introduction: that Einstein appears to confuse active and passive transformations. The Einstein formulation provides just one type of transformation, the coor-

dinates x^i to the point with coordinates x'^i in the *same* coordinate system. I take this to be the traditional point/coordinate transformation duality described, for example, in Klein,[31] pp. 136–137. The point transformation is not defined on mathematical structures but on the very vaguely defined structures of $E1$, physically possible space-times, so its status is not entirely clear to me. (I am grateful to Don Howard for this reference to Klein's work.)

dinate transformation. But some of its properties are like the active transformations of $M2$ and others like the passive transformations of $M3$. A problem really only arises when modern readers try to restate Einstein's claims in the modern formulation of space-time theories. For then they must decide whether to translate a particular coordinate transformation in one of Einstein's works into an active or a passive transformation, when neither alone is appropriate. No such problem arises, of course, if one reads Einstein in terms of the canonical Einstein formulation offered here. This point will be illustrated in some detail in Sec. 5 with Einstein's discussion of the hole and point-coincidence arguments.

4.3. Covariance Principles

In the Einsteins formulation, we can also define covariance principles and Leibniz equivalence principles, but without dual active and passive versions. In formulating these principles, it will be convenient to allow the groups of transformations involved to vary in size.

> *Covariance of a theory under a group G of transformations:* If $\langle A, (O_1)_{ik\ldots}, \ldots, (O_n)_{ik\ldots} \rangle$ is a model of a space-time theory, then every other tuple $\langle A, (O'_1)_{mn\ldots}, \ldots, (O'_n)_{mn\ldots} \rangle$ related to it by a member of G is also a model of the theory.

Thus Einstein's formulation of special relativity is covariant just under the Lorentz group of transformations, for its Einstein formulation requires that the matrix η_{ik} of every $\langle R^4, \eta_{ik}, T_{ik} \rangle$ have the diagonal form $\mathrm{diag}(-1, -1, -1, 1)$. General relativity is covariant under the general group, however, since every triple $\langle A, g_{ik}, T_{ik} \rangle$ which satisfies the generally covariant field equation $G_{ik} = \kappa T_{ik}$ is a model of the theory.

As in the case of the modern view, I shall treat separately the natural postulate that two models, related by a member of the theory's covariance group, represent the same space-time in $E1$.

> *Leibniz equivalence:* If two models $\langle A, (O_1)_{ik\ldots}, \ldots, (O_n)_{ik\ldots} \rangle$ and $\langle A, (O'_1)_{mn\ldots}, \ldots, (O'_n)_{mn\ldots} \rangle$ are related by a member of the covariance group G of the theory, then they represent the same physically possible space-time.

Under the assumption of Leibniz equivalence, the physical space-times of $E1$ are represented by equivalence classes of intertransformable models of $E2$. The status of an Einstein covariance principle and of Leibniz equivalence is almost exactly the same as that of *active* general covariance and Leibniz equivalence in the modern view. In both cases, their truth depends upon the properties of $E1 = M1$, so that following precisely the argument for the modern case

> *a covariance principle and Leibniz equivalence in the Einstein formulation of space-time theories are physical principles that one can choose to accept or deny.*

For, as before, a covariance principle asserts that a new mathematical structure, generated by an allowed transformation from one of the theory's models, will itself represent a physically possible space-time of $E1$. Leibniz equivalence amounts to assuming that the mathematical differences between two intertransformable models have no physical counterpart in the space-times of $E1$ that they represent. But the truth of these assumptions depends on the physically contingent properties of the space-times of $E1$.

4.4. Coordinate Systems, Frames of Reference, and Relative Spaces

The second problem of the introduction was Einstein's failure, to modern eyes, to distinguish clearly between what we would now call a coordinate chart, a frame of reference, and a relative space. I have described elsewhere in Norton[40] how Einstein's term "coordinate system" can be read, in the modern context, as referring to any of the three. The fact that Einstein used R^4 or its open sets where the modern view uses an abstract differentiable manifold provides a solution to this problem.

The differentiable manifolds used to represent space-times in the modern view are relatively impoverished as far as intrinsic structures are concerned. If we wish to represent frames of reference and the like on them, the manifolds' intrinsic structure is not sufficient; we must *add* further structure. Thus a frame of reference is introduced in standard practice as a congruence of timelike curves defined on the manifold (with metric). The frame, if smooth, assigns a velocity, its tangent vector, to every event in the manifold. Associated with any frame is a three-dimensional manifold, the relative space, which is just the manifold whose point set is the set of curves of the frame. Once frames and relative spaces are defined, we can introduce more structure. We might seek to foliate the manifold into hypersurfaces intersecting the frame, which we can think of as instantaneous "snapshots" of the relative space. We might seek a real-valued parametrization of these hypersurfaces which could play the role of a frame time. In each case, we find ourselves adding further structure to the largely featureless background canvas of the manifold. Finally, we generally expect some natural adaptation of frames, relative spaces, and the like to the geometric structure posited by the theory. Thus an inertial frame in special relativity is a congruance f parallel *timelike geodesis* and is therefore adapted to the

Minkowski metric. A foliation of a Minkowski space-time is usually defined as *orthogonal* to an inertial frame, where orthogonality is defined by the Minkowski metric. The frame time usually coincides withthe metrically defined proper time along some timelike curve; and so on.

In the Einstein formulation, however, this problem looks very different. Einstein's manifolds are open sets of R^4. Thus they have considerably more structure than the manifolds of the modern view, and the Einstein formulation provides a canonical physical interpretation for this extra structure. I list some of this structure and its interpretation:

(a) Inhomogeneity (individuation of points): each point of R^4 is distinct, so every space-time event is intrinsically different from every other.

(b) Anisotropy: each direction in R^4 is distinct, so every direction in space-time is intrinsically different from every other.

(c) Absolute simultaneity: the x_4 coordinate is a time coordinate, so the hypersurfaces of constant x_4 represent instantaneous snapshots of a three-dimensional space, corresponding to a relative space of the modern view.

(d) Absolute rest: the natural rest frame of the space-time is represented by the congruence of x_4 curves.

(e) Set of inertial frames: each parallel congruence of straights with constant velocity dr/dx_4, where $r^2 = x_1^2 + x_2^2 + x_3^2$, represents a frame of reference of uniform velocity.

(f) Coordinate lengths and times: the x_1, x_2, and x_3 coordinates provide measures of length; the x_4 coordinate, a measure of time.

Moreover, one does not adapt these structures to the further geometric structure posited by the theory, as is the case in the modern approach. One works the other way around in the Einstein approach. The geometric structures are adapted to the manifold's structure whereever possible. Thus the line element of special relativity is always adapted to the coordinate system (manifold) insofar as it can be written as (2), which ensures that the natural frame of the manifold is inertial and that the coordinate differences correspond to times and distances measured directly on idealized rods and clocks.

Thus, in the Einstein formulation, the specification of a coordinate system is very different from the specification of a coordinate chart of the modern view. It requires the coordination of an open set of R^4 with a physically possible space-time, combined with the automatic interpretation of the intrinsic structure of the open set according to (a)–(f) above. Thus confusion only arises when the modern reader asks whether one of Einstein's coordinate systems should be read as a coordinate chart *or* a frame of reference *or* a relative space, for they ought to read as all of them at once, and more.

5. EINSTEIN'S HOLE AND POINT COINCIDENCE ARGUMENTS

In this section, I offer the example of these two arguments, which were central to Einstein's understanding of general covariance, as an illustration of the advantages of reading Einstein's work in terms of canonical Einstein formulations. A major difficulty in recent treatments of these arguments has been the establishment of whether Einstein intended his coordinate transformations to be read as active or passive transformations. If they are read in terms of the Einstein canonical formulation, the problem disappears entirely and much of what he says can be read literally. A detailed account of these arguments, including Einstein's statements of them, is in Norton.[(41)] So I shall review the arguments here with the briefest of sketches.

5.1. The Hole Argument

By mid 1913, Einstein, with the mathematical assistance of his friend Marcel Grossmann, has discovered all the major components of the general theory of relativity, with one major exception. They had chosen gravitational field equations which were not generally covariant. By the end of 1913, Einstein has developed the "hole" argument to justify this choice. It purported to show that generally covariant field equations would be physically uninteresting since they would lead to a serious violation of determinism. There were four version of the argument. In order of dates of publication, they were: Einstein and Grossmann,[(20)] pp. 260–261,[25] Einstein,[(8)] p. 178, Einstein and Grossmann,[(21)] pp. 217–218, and Einstein,[(6)] pp. 1066–1067. The first three versions were essentially the same and were presented as follows.

We consider a solution g_{ik} in some coordinate system x^i of any generally covariant gravitational field equations for a space-time which contains a matter-free region, the "hole," in which $T_{ik} = 0$. We transform to a new coordinate system x'^i which agrees with the original outside the hole

[25] The argument appeared in the addendum to the journal printing of the article but not in the original separatum.

but smoothly comes to differ from it within the hole. The transformed g'_{ik} differs from the original g_{ik} within the hole, so that we have

$$g_{ik} \neq g'_{ik} \qquad (3)$$

A footnote to the second version made the, at first glance, curious stipulation that the inequality (3) be read so that the variables x^i and x'^i have the same values. Since a tensor whose components are all zero remains so under all transformations, we have for the entire space-time

$$T_{ik} = T'_{ik}$$

It followed, Einstein concluded, that the single matter distribution of $T_{ik} = T'_{ik}$ is compatible with two different gravitational fields within the hole, g_{ik} and g'_{ik}. This result would constitute a serious failure of determinism for, according to it, even the most complete specification of the matter distribution *and* the gravitational field outside the hole could not determine the field within it. (To see how very serious is the form of indeterminism considered, notice that the hole is a neighborhood of space-*time* and can be arbitrarily small. It could, for example, be less than a millimeter in spatial size and less that a microsecond in temporal duration.)

The fourth version of the argument is more complex. The matrix of functions of the coordinates representing g_{ik} in x^i is written compactly but nonstandardly as "$G(x)$," and those representing g'_{ik} in x'^i *as* "$G'(x')$." I surmise that the purpose of tis new notation is to make sure the reader understands the stipulation on how the inequality of (3) is to be read. "$G'(x')$ and $G(x)$ describe the same gravitational field," he notes. But he proceeds to construct $G'(x)$ which must describe a different field since it assigns different matrices of components in the same coordinate system to the same sets of values of x^i. The inequality of $G(x)$ and $G'(x)$ is formally identical to the inequality (3), provided the latter inequality is read according to the stipulation in Einstein's footnote.

The standard account—the "passive account"—of Einstein's argument has been that he committed a beginner's blunder in differential geometry.[26] In that account,the transformation between g_{ik} and g'_{ik} was read as a passive transformation of the modern view, so that the differing forms of the matrices g_{ik} and g'_{ik} did not entail that they represented physically different fields. They were just different coordinate representations of the same mathematical object g_{ab} which, in turn, represents just one physically possible field.

Agreeing with a perceptive analysis by John Stachel of the fourth version of the hole argument, with Stachel I[(41)] urged an "active account" in

[26] For example, Pais,[(42)] pp. 221–222.

which the transformation of the argument must be read actively. For only that reading could explain the curious stipulation on how the inequality (3) was to be read. Under it the two matrices g_{ik} and g'_{ik} of the inequality (3) are coordinate representations of the fields g_{ab} and h^*g_{ab}, where h is the diffeomorphism dual to the coordinate transformation from x^i to x'^i. Then the point of Einstein's arguments becomes clear. He is considering theories that satisfy active general covariance but not active Leibniz equivalence. Thus he correctly concludes that, in such theories, if the field g_{ab} is allowed, then so will h^*g_{ab}, but without Leibniz equivalence—whose possibility does not seem to arise in Einstein's statements of the hole argument—one generates the extremely undesirable form of indeterminism claimed.

The virtue of the passive account is that it enables an essentially literal use of Einstein's wording in the first three version of the argument. Its vices are that it cannot make sense of the fourth version or of the footnote to the second version and that it portrays the Einstein of 1913–1915 as confused by a beginner's mistake on an issue that attracted his exhaustive attention for several years. The virtues of the active account are that it is compatible with all four versions of the argument and that it portrays Einstein's argument as exploiting a property of actively generally covariant theories of such importance that it warrants inclusion in even the brief gloss of the modern view of space-time theories of Sec. 2 above. The great vice of the active account, one that it shares to a lesser extent with the passive account, is that it supposes more levels of mathematical structure than are explicitly invoked in any of Einstein's four accounts of the hole argument or in his later accounts of its resolution. For it requires us to assume that Einstein had a reasonably clear picture of the three levels $M1$, $M2$, and $M3$ of the modern view and that his argument really manipulated structure in $M2$, even though his explicit formulas only contain structures that belong in $M3$.

If, however, we read Einstein's presentations of the hole argument in terms of canonical Einstein formulations, then we can retain *all* of the above virtues and dispense with *all* of the above vices. The resulting reading of the argument agrees with the general picture of the active account. The argument demonstrates that a generally covariant gravitation theory of the type envisaged by Einstein in 1913 and 1914 suffers from the very undesirable form of indeterminism if one fails to require Leibniz equivalence as well. The reading is compatible with all four versions of the argument. It requires only one level of mathematical structure, the level $E2$ of the canonical Einstein formulation, and therefore does not need to read Einstein's matrices g_{ik} as representing another suppressed level of mathematical structure mediating between it and physically possible space-times. The argument now becomes:

Hole Argument (Canonical Einstein Form)

Thesis: In a generally covariant gravitation theory with models $\langle A, g_{ik}, T_{ik} \rangle$, without Leibniz equivalence, a radical indeterminism arises, for if "the hole" L is a matter-free neighborhood of a physically possible space-time, then a complete specification of the field and source mass distribution outside L will not determine the field within L.[27]

Proof:

1. Let $T = \langle A, g_{ik}, T_{ik} \rangle$ be a model of the theory in which $T_{ik} = 0$ for the neighborhood L of A. Let f be a map from A to A which is the identity for points outside L and smoothly comes to differ from it within L. Then, from general covariance, the transform of T under f, $T' = \langle A, g'_{ik}, T'_{ik} \rangle$, is also a model of the theory.
2. The two models T and T' establish the indeterminism claimed, for they both represent the same source mass distribution and gravitational field outside L, but different gravitational fields within L. To see this, note that we have

   $$T_{ik} = T'_{ik}$$

   everywhere and $g_{ik} = g'_{ik}$ outside L; but inside L

   $$g_{ik} \neq g'_{ik} \tag{3}$$

3. The differing functional forms of g_{ik} and g'_{ik} in L do not automatically entail that they represent different physically possible gravitational fields. Whether they do depends on how the models T and T' of $E2$ are taken to be related to physically possible space-times in $E1$. Consider two ways in which we can relate the models T and T' to physically possible space-times in $E1$:
   (a) Each 4-tuple x^i of T and each 4-tuple $fx^i = x'^i$ of T' represent the same space-time point in $E1$. Then g_{ik} and g'_{ik} do represent the same field.[28]

[27] Notice that I state the less general form of the argument as given by Einstein. It deals with the determination of the gravitational field by source mass distributions and boundary conditions in a gravitation theory of the type of general relativity. The more general form, sketched in Sec. 2, applies to generally covariant theories of any type. Note also that the modern version argues for Leibniz equivalence in a generally covariant theory, where Einstein's version argues for the unacceptability of general covariance without considering the possibility of Leibniz equivalence as an escape.

[28] Why Einstein readily accepted this is not entirely clear to me. Certainly in this case both models will attribute identical observables to the same space-time points of $E1$. Perhaps, this fact, with a dash of verificationism, is sufficient for the claim.

(b) Each 4-tuple x^i of T and each 4-tuple x^i of T' (both having the same *numerical* values) represent the same space-time point. Then g_{ik} and g'_{ik} represent different physically possible gravitational fields. For under this specification both models T and T' assign the same 4-tuples of reals to the same points of the physically possible space-time of $E1$; that is, they employ the same coordinate system. But now, *in this same coordinate system*, T and T' assign different matrices of values g_{ik} and g'_{ik} to each point of the physically possible space-time.

Part 3 of the proof contains the most subtle part of the argument. Part 3b explains why Einstein required the curious reading of the inequality (3) in the footnote to the second version. Part 3a corresponds to the claim of the fourth version that $G(x)$ and $G'(x')$ describe the same gravitational field. Part 3b corresponds to his claim that $G(x)$ and $G'(x')$ are different gravitational fields represented in the same coordinate system.

5.2. The Point-Coincidence Argument

The natural escape from the hole argument is Leibniz equivalence. It requires that the two matrices g_{ik} and g'_{ik} represent the same field and defeats the indeterminism. It forces the models T and T' above to be coordinated to the space-times of $E1$ by 3a and not 3b or any other coordination scheme. Loosely speaking, Leibniz equivalence asserts that those properties of points of the number manifolds relevant to this coordination are *only* those inherited directly from the metric and other fields defined on the number manifolds. Thus when we change those fields by a transformation, the coordination scheme must be altered to compensate exactly in accord with 3a. Perhaps a result like this is what Stachel,[(47)] Sec. 4, has in mind when he notes that Einstein's "main difficulty here was to see that the points of the space-time manifold (the 'events' in the physical interpretation) are not individuated *a priori* but inherit their individuation, so to speak, from the metric field."

The availability and need for this escape from the hole argument's indeterminism may not have dawned on Einstein until as late as November 1915, when he was deeply embroled in a renewed search for generally covariant gravitational field equations for general relativity. By the end of the month, he had published the generally covariant field equations of the modern theory and this escape had been described in his correspondence the following month. The escape came in the form of the "point-coincidence" argument, whose classic statement is given in Einstein,[(9)]

pp. 117–118. There, no explicit connection is made to the hole argument. But Einstein's discussion of the point-coincidence argument in his contemporary correspondence makes very clear that this escape was its purpose. See Einstein to P. Ehrenfest, 26 Dec. 1915 and 5 Jan. 1916, as quoted in Norton,[41] and Einstein to M. Besso, 3 Jan. 1916 and Speziali,[45] pp. 63–64.

The starting point of the argument, as stated in the letter to Besso, was the assumption: "*Reality* is nothing but the totality of space-time point coincidences"; or, more cautiously, in the published 1916 version: "All our space-time verifications invariably amount to a determination of space-time coincidences." His example included the coincidence of moving material points for a universe in which everything could be built from the motion of such points and also measuring operations. These coincidences are preserved under arbitrary coordinate transformation, so that the physical referent of any structure like the g_{ik} of the hole argument must be the same as that of the transformed g'_{ik}. In the words of Einstein's December letter to Ehrenfest:

> If two systems of $g_{\mu\nu}$ (or [more] gen.[erally], variables used for describing the world) are so constituted that one can obtain the second from the first merely through a space-time transformation, then they refer to exactly the same thing [völlig gleichbedeutend]."

The argument has been read passively, corresponding to the passive account of the hole argument, and actively, corresponding to the active account of the hole argument. The passive account takes the thesis to be that the matrices g_{ik} and g'_{ik} of $M3$ both represent the same coordinate-free geometric object g_{ab} in $M2$. As before, its virtue is that it stays fairly close to a literal reading of Einstein's words. Its vice is that it makes very little sense. Why does Einstein need to introduce controversial assertions about reality consisting of space-time coincidences to argue for a thesis that states a mathematical definition? The active account takes the thesis of the point-coincidence argument to be that both g_{ab} and h^*g_{ab} of $M2$ represent the same field in $M1$. That is, its thesis is just active Leibniz equivalence. Its virtue is that it makes the thesis of the point-coincidence argument an insightful response to the hole argument and one which requires justification which might well run along the lines Einstein offers. Its vice, one that it again shares with the passive reading to a lesser degree, is that it requires us to assume that Einstein was indirectly manipulating the structures g_{ab} and h^*g_{ab} of a suppressed mathematical level $M2$ by manipulating the only mathematical structures explicitly present, those of $M3$.

Once again, reading the point coincidence argument in terms of canonical Einstein formulations allows us to retain all the above virtues and dispense with all the vices. The argument is simply for Leibniz equivalence and is:

Point-Coincidence argument (*Canonical Einstein form*)

Thesis: Leibniz equivalence, which asserts, in the case of general relativity: if $T = \langle A, g_{ik}, T_{ik} \rangle$ and $T' = \langle A, g'_{ik}, T'_{ik} \rangle$ are models in $E2$ of general relativity related by some coordinate transformation, then they both represent the same space-time in $E1$.

Justification: Our ontology should be limited to observables. The observables are just space-time coincidence which are preserved under coordinate transformation between T and T'. Therefore T and T' represent the same observables and thus the same space-time.[29]

Reading Einstein's accounts of these two arguments in terms of canonical Einstein formulations provides some relief from another puzzle of this episode. John Earman and I[(4)] have urged that one conclusion to be drawn from the two arguments concerns traditional questions in philosophy of space and time. They can be construed as strong arguments against that version of space-time substantivalism which urges the independent existence of the physical structure represented by each of the manifolds of $M2$. In his correspondence, Einstein urged that the resolution of the hole argument depended in part on the insight that "the reference system signifies nothing real" (to Ehrenfest, 26 Dec. 1915) and "the [coordinate] system K has no physical reality" (to Besso, 3 Jan. 1916). Read canonically, this assertion now is much closer to the denial of space-time substantivalism, for it really amounts to the denial of physical significance to a coordination of any individual number manifold of $E2$ to the structures of $E1$ where the number manifolds of $E2$ play an analogous role to the manifolds of $M2$.[30]

Finally, I recall that the distinction between a covariance principle and Leibniz equivalence as maintained throughout this paper is not one that Einstein himself used, even though it was the crucial issue for his hole and

[29] I have pointed out with John Earman[(4)] that the extreme verificationism of this argument might have found favor in 1915 and 1916. But modern readers will surely find that the hole argument, with its threat of radical local indeterminism, carries more weight, and thus it is now unfortunate that the hole argument was not even mentioned in the much read Einstein.[(9)]

[30] The other point stressed by Einstein in this context is: "There is no physical content in two different solutions $G(x)$ and $G'(x)$ existing with respect to the *same* coordinate system." (to Besso, 3 Jan. 1916). I take this as a reminder that Leibniz equivalence forbids coordinate scheme 3b in the hole argument.

point-coincidence arguments. Rather, Einstein's requirement of general covariance after 915 is the same as the *conjunction* of my versions of the requirements of general covariance and Leibniz equivalence. Thus throughout the episode, Einstein takes the issue to be the acceptability of general covariance. The point-coincidence argument is never presented as an argument for Leibniz equivalence but as showing that "this requirement of general covariance, which takes away from space and time the last remnant of physical objectivity, is a natural one" (Einstein,[9] p. 117).

6. COVARIANCE PRINCIPLES AND RELATIVITY PRINCIPLES

6.1. The Modern View

Einstein's standard accounts of special and general relativity theory stress the fundamental importance of relativity principles, which are statad as covariance principles. But, as I pointed out in the introduction, the modern view of these two theories finds it difficult to see any physical content in covariance principles, let alone characterize them as relativity principles or, for that matter, even to find a sense in which general relativity as a theory generalizes the principle of relativity of inertial motion of the special theory.

Passive general covariance is the case usually discussed, such as in the references cited in Sec. 1. It is a purely mathematical property whose satisfaction arises automatically from the use of the modern formulation. As I mentioned in Sec. 2.4, *active* general covariance and active Leibniz equivalence have only recently become a standard feature of newer treatments of general relativity, although their importance and significance are rarely stressed. Often they are ignored. Neither active nor passive principles seem to have much to do with relativity principles, for they can be properties of *any* space-time theory, formulated in the modern way, including those with absolute states of rest, in clear violation of even the relativity of inertial motion.

The principle of relativity of inertial motion in special relativity is the only unproblematic relativity principle of modern formulations of relativity theory. It does not arise as a fundamental postulate of special relativity, but as an important theorem dependent on the symmetries of the Minkowski metric. To state the principle, we recall that an inertial frame of reference in special relativity is a congruence of parallel timelike geodesis of the Minkowski metric. Then:

Principle of Relativity (Special Relativity): If $\langle M, g_{ab}, T_{ab} \rangle$ is a model of a special relativistic theory and F and F' are any two inertial frames, then the theory satisfies the principle of relativity only if there exists a member L of the symmetry group of the Minkowski metric g_{ab} such that

(a) L maps F onto F' and

(b) $\langle M, g_{ab}, L^*T_{ab}\rangle$ is also a model of special relativity.

The symmetry group of a Minkowski metric g_{ab} is the group of all automorphisms which preserve g_{ab} under the carry-along; that is, each member L satisfies $L^*g_{ab} = g_{ab}$. This group, of course, is just the extended Lorentz group if M is topologically R^4. Note that condition (a) is satisfied automatically due to the properties of the Minkowski metric. Whether condition (b) is satisfied depends on the laws of the special relativistic theory in question.

What motivates the identification of the above principle as a princip e of relativity is the picture of a Minkowski space-time $\langle M, g_{ab}\rangle$ prividng a passive background space-time against which physical processes unfold. Condition (a) reminds us that the Minkowski space-time itself designates no inertial frame as preferred. For any property endowed upon an inertial frame F by the Minkowski metric must also be endowed by the metric on any other inertial frame F', since F and F' are related by a symmetry of the metric. Condition (b) stipulates that the additional structures defined on the space-time, such as Maxwell fields or mechanical fluids, likewise do not distinguish any inertial frame as preferred. For if an allowed structure T_{ab} distinguishes the frame F as preferred, then for any other inertial frame F', there exists a corresponding allowed structure L^*T_{ab} which distinguishes F' as preferred. (To illustrate, if T_{ab} represents a homogeneous dust cloud whose particles are moving inertially, then the preferred frame is just the cloud's rest frame.)

The currently most popular vehicle for extending the definition of relativity principles to other theories, in particular general relativity, is the notion of absolute object, best known through the work of Anderson.[(1)] This notion is the best candidate for providing precise expression for Einstein's[(13)] own claim (p. 54) that the crucial aspect of the transition to general relativity is the elimination of any property of space-time absolute in the sense that it is "independent in its physical properties, having a physical effect, but not itself influenced by physical conditions."[31] One divides the geometric objects of a space-time theory into the absolute objects $A_1, A_2,...$ and the dynamic objects $D_1, D_2,...$. One pictures the

[31] See also, Einstein,[(11)] Chap. XXI; Einstein,[(15)] p. 260.

absolute objects in conjunction with the space-time manifold as providing the background space-time canvas $\langle M, A_1, A_2, ...\rangle$ against which dynamical processes unfold, the latter represented by the dynamic objects. In special relativity, the Minkowski metric is an absolute object and such structures as the Maxwell field are dynamic objects. Intuitively speaking, the defining property of absolute objects is that they remain unaffected by the dynamical processes of the theory. This is usually now rendered by the requirement that they be the same (i.e., diffeomorphic) in all models of the theory, and much of the recent work on absolute objects focuses on explication of this notion. See, for example, Friedman,[(24)] Chap. 2, and Hiskes.[(27)] But whether this work has succeeded in giving precise expression to the notion of absolute object remains undecided. See, for example, Friedman,[(24)] p. 58, footnote 9, and Toretti.[(52)] In any case, with absolute and dynamic structure identified, one then defines a relativity principle analogous to the one defined above, but using the symmetry group of the background space-time $\langle M, A_1, A_2, ...\rangle$. The case of general relativity is degenerate, since it has no absolute objects, so that the background space-time is just the manifold M. The symmetry group of M is the group of all automorphisms, which is far larger than the Lorentz group of special relativity. In this sence it is claimed that general relativity realizes an extension of the principle of relativity.

6.2. The Einstein View

The situation looks very different if we approach special and general relativity through canonical Einstein formulations. They provide a precise sense in which Einstein's covariance principles have physical content and a means of connecting them with relativity principles. But, as I shall note in Sec. 6.3, I do not think that they provide a complete vindication of Einstein's claims.

We saw in Sec. 4.4 that the number manifolds of the Einstein formulation—open sets of R^4—contain significantly more structure than the manifolds of the modern view. That additional structure, under its canonical physical interpretation, provides for preferred states of motion and the like, whose reality we seek to deny in relativity principles. The natural mechanism for denying physical significance to this structure is a combination of covariance principles and Leibniz equivalence principles. Under these principles, the models of a space-time theory are divided into equivalence classes each of which represents the same physically possible space-time. It then follows that the physically significant properties of a model and, in particular, its number manifold must be those that are common to every member of the equivalence class to which it belongs. That is:

> *The physically significant properties of a model's number manifold are exactly those which are invariant under the theory's covariance group.*

Figuratively speaking, a theory's covariance and Leibniz equivalence principles "wash out" from its model those properties with no physical significance. The strategie is precisely the one employed in Sec. 2.4 where it was noted that the combination of active general covariance and active Leibniz equivalence provides a precise and systematic method of distinguishing the physically significant properties of a space-time theory's models. Here, however, the "washing out" is applied to structures of potentially greater physical interest, which include preferred frames of reference. Notice that the strategy is by no means new. It is the essential idea of F. Klein's *Erlangen* program in which geometric structures are characterized as the invariants of groups.

We shall now see that the application of this mechanism generates a sequence of covariance principles—the special principle of relativity, the principle of equivalence, and the general principle of relativity—which are also relativity principles and which take us belong Einstein's own pathway from special to general relativity. The sequence begins with special relativity.

Special Relativity

Under the canonical physical interpretation, the number manifold R^4 of a typical model $\langle R^4, \eta_{ik}, T_{ik} \rangle$ of special relativity automatically asserts the existence of many undesirable physical properties in the space-time it represents. The space-time is homogeneous; each of its points are intrinsically different, since, for example, $\langle 0, 0, 0, 0 \rangle$ is intrinsically different from $\langle 15, 0, 27, 1 \rangle$. The space-time is spatially anisotropic; each spatial direction in the space-time is intrinsically different, since, for example, the direction of increasing x_1 in R^4 is intrinsically different from that of increasing x_2. Finally, the space-time has a preferred rest frame, represented by the congruence of x_4 curves in R^4, in direct violation of the principle of relativity.

The Lorentz covariance of the theory, with Leibniz equivalence, "washes out" the physical significance of all this unwanted structure. Since the extended Lorents group contains translations and rotations, the inhomogeneity and spatial anisotropy of each model is denied physical significance. Correspondingly the rest frame of each model is not a Lorentz invariant and thus is not physically significant. But the set of inertial frames of reference of each model (i.e., the set of congruences of parallel timelike geodesics) is preserved under Lorentz transformation and thus has physical significance. In this way the requirement of Lorentz covariance in conjunction with Leibniz equivalence becomes a physical principle which embodies the principle of relativity of special relativity. Einstein, of course, commonly states this relativity principle as a requirement of Lorentz covariance.[32]

General Relativity

Throughout the corpus of his writings on relativity theory, Einstein portrays the great advance in the transition from special to general relativity as the elimination of the "inertial system," an advance that was achieved by general covariance.[33] Recalling the interpretation of coordinate system in Einstein's sense, this claim has a natural reading within the canonical formulation of relativity theory. Lorentz covariance has washed out the physical significance of much of the intrinsic structure of the number manifolds of special relativity. But the Lorentz-invariant set of inertial frames of each number manifold retain physical significance. The covariance group of general relativity, in the canonical Einstein formulation, is the general group of smooth transformations. Under this group the number manifolds of the theory's models have essentially no invariants beyond their topological properties. Thus the theory's general covariance, in conjunction with Leibniz equivalence, deprives essentially all the intrinsic structures of the theory's number manifolds of physical significance; most notably this includes its inertial structure. I take this result to be the one Einstein had in mind in his well-known representation of the generalized principle of relativity as the principle of general covariance,[34] which implicitly must contain Leibniz equivalence as well.

Principle of Equivalence

In Einstein's standard developments, he halts at an intermediate in the transition to general covariance and general relativity, the principle of equivalence. The version of the principle which is relevant here is the one which represents it as an extension of the covariance group of special relativity from the Lorentz group to one that includes uniform proper acceleration.[35] This extension can be defined more precisely in the follow-

[32] More precisely, he states it as the requirement of Lorentz covariance for the *laws* of the theory, which entails Lorentz covariance, as defined in Sec. 4.3, when embedded in the canonical Einstein formulation of special relativity. Leibniz equivalence is also obviously intended. See, for example, Einstein,[(6)] p. 340, Einstein,[(11)] Chap. XIV, and Einstein,[(16)] p. 283.

[33] One of the earliest is Einstein,[(5)] p. 1260 (footnote) and one of the latest Einstein,[(18)] p. xv.

[34] For example, in Einstein[(9)] and Einstein,[(11)] Chap. 23.

[35] He writes in Einstein,[(10)] p. 641, "The requirement of general covariance of equations embraces the principle of equivalence as a quite special case."

ing way. One can represent Lorentz transformations as boosts to uniform motion since, under a Lorentz transformation, the natural rest frame of a number manifold, its congruence of x^4 curves, is mapped onto a congruence of curves in the new model which always represents an inertial frame, but not necessarily the new model's rest frame. Under transformations introduced in the extension, the natural rest frame of a model T is mapped onto a frame in the new model T' that has at most uniform proper acceleration.

The heuristic value to Einstein of the principle was the following. If the rest frame of the original model T was inertial, then the rest frame of the transformed model T' would not be. Thus free bodies would fall with uniform acceleration with respect to the natural rest frame, just as though they were under the influence of a homogeneous gravitational field. This showed Einstein that gravitation was already intimately connected with the space-time structure of special relativity and that the further generalization of the principle of relativity would automatically involve a theory of gravitation. See Norton[(40)] for further discussion.

The term "equivalence" of the principle of equivalence is used in precisely the sense of Leibniz equivalence. For it tells us that the models T and T' above both represent the same physical space-time, even though one appears to contain a homogeneous gravitational field. In Einstein's[(13)] words (p. 56), speaking of an inertial coordinate system K and uniformly accelerated coordinate system K': "The assumption of the complete physical equivalence of the systems of coordinates, K and K', we call the 'principle of equivalence'...."

The major elements discussed above of the transition from Lorentz to general covariance are summarized by Einstein[(17)] in his *Autobiographical Notes*, pp. 71–73, where he conjectures on the prospects of attempts to discover satisfactory gravitation theories within special relativity and, in particular, the hopelessness of arriving at a theory equivalent to general relativity:

> If one had stopped with the special theory of relativity, i.e., with the invariance under the Lorentz group, then the field law $R_{ik} = 0$ [vanishing of the Ricci tensor] would remain invariant also within the frame of this narrower group. But, from the point of view of the narrower group, there would be no offhand grounds for representing gravitation by a structure as involved as the symmetric tensor g_{ik}. If, nonetheless, one would find sufficient reasons for it, there would then arise an immense number of field laws out of quantities g_{ik}, all of which are covariant under Lorentz transformations (not, however, under the general group). Even if, however, of all the conceivable Lorentz-

invariant laws, one had accidentally guessed precisely the law belonging to the wider group, one would still not have achieved the level of understanding corresponding to the general principle of relativity. *For, from the standpoint of the Lorentz group, two solutions would incorrectly have to be viewed as physically different if they can be transformed into each other by a nonlinear transformation of coordinates, i.e., if from the point of view of the wider group they are merely different representations of the sam- field.* (My emphasis.)

What is particularly striking about this passage is that it shows clearly that Einstein's covariance principles automatically assume Leibniz equivalence. He allows that in a Lorentz-covariant gravitation theory, two models ("solutions") represent the same physical field only if they can be Lorentz-transformed into one another. In a generally covariant theory, the same holds of models which can be transformed into one another by arbitrary coordinate transformations.

6.3. Problems

Use of the canonical Einstein formulation shows us that Einstein's covariance principles are physical principles akin to the active principles of te modern view, even though formally they look like the physically vacuous passive principles. Moreover, they have the character of relativity principles, for they deny physical significance to intrinsic manifold structure taken canonically to represent preferred states of motion. But there are two serious problems for the account offered.

First, the accound does not establish a sufficiently strong connection between the covariance and relativity principles for us to be able to characterize general relativity as the theory that extends the principle of relativity to accelerated motion. Both Newtonian space-time theory and special relativity admit generally covariant Einstein formulations. So, if general covariance is all there is to the generalized principle of relativity, then both these space-time theories satisfy it. It would seem that if the various space-time theories are to have their own characterize relativity principles, they must be defined in terms of the properies of the structures defined on the manifold in each theory, much as the modern view introduced the principle of relativity to special relativity by means of the symmetries of the Minkowski metric. The best attempt at defining a relativity principle characteristic of general relativity uses the notion of absolute object, but resorting to Einstein canonical formulations does contribute some elucidation to the situation described in Sec. 6.1.

Another way to see the difficulty is to note that the relativity principles of the Einstein formulations are not robust under reformulation to the modern view, where active covariance principles no longer have the character of relativity principles. This stands in contrast to such results as energy conservation and the various field equations which have essentially the same meaning and importance in both Einstein and modern formulations. It suggests that the connection between covariance and relativity principles of the Einstein formulation is an idiosyncrasy of that formulation. To modern eyes, the problem lies in the use of number manifolds rather than general differentiable manifolds in the Einstein formulation's models. This unfortunate choice, pressed on Einstein by historical circumstances, forced him to grapple with a problem we no longer need address. His manifolds contained much intrinsic structure of no physical significance and the coherence of the theory demanded a systematic denial of physical significance to that structure. Einstein completely solved that problem with general covariance, but it is not a problem we even need now address, since the modern approach is to use differentiable manifolds without such superfluous intrinsic structure.

The second problem concerns the claim that Einstein's covariance principles are physical principles. On weveral occasions, under pressure from critics, Einstein did guardedly allow the physical vacuity of general covariance. Einstein,(12) p. 242, for example, concedes the point to Kretschmann(33) but insists nonetheless on the heuristic value of general covariance by claiming that the complexity of a generally covariant formulation of Newtonian gravitation theory would render it unworkable practically. (The point was unfortunate in the light of Cartan's later relatively simple generally covariant formulation of Newtonian gravitation theory.) Einstein,(14) pp. 90–91, also declared: "That there is, in general relativity, no preferred space-time coordinate system uniquely bound to the metric, is more a characteristic of the mathematical form of this theory than its physical content." Such remarks contradict the overwhelming importance Einstein attributed to general covariance elsewhere. But if they represent his true beliefs, then the account I offer here of the physical significance of his covariance principles cannot accord with his own account. For further discussion of Kretschmann's objection and Einstein's response, see Norton.(54)

7. EINSTEIN THE NON-MATHEMATICIAN?

I have portrayed Einstein's treatment of general relativity as employing a much simpler set of mathematical tools than is presently used. But now recall that Einstein's deprecation of his own mathematical abilities is legend.[36] Is it possible that his use of simpler mathematical tools is a symptom of his supposed lack of facility with mathematics? Would not another physicist with a modicum of mathematical ability, in all likelihood have treated the theory much more as we do today, interposing explicitly a second mathematical level between $E1$ and $E2$, and thereby shrugging off the hole argument as a trivial confusion? I answer emphatically, no. On the contrary I maintain that the devices attributed here to Einstein represented the "state of the art" application of differential geometry to physical problems in the 1910s.

A claim about what constitutes "state of the art" in the 1910s is difficult to substantiate. What we would hope to find is a treatment of identical material written in the 1910s by a mathematician who is at least competent, but ideally eminent. And that is precisely what we have! David Hilbert, the mathematician who proclaimed that physics is too difficult to be left to physicists,[37] at the height of his powers in Göttingen, then the center of the mathematical universe, presented a two-part communication to the Göttingen Academy of Science in 1915 and 1916 on the foundation of general relativity (Hilbert[(26)]). Of course, Hilbert's work outshines Einstein's in its mathematical brilliance. Hilbert amost effortlessly allows an action principle to generate the same gravitational field equations that cost Einstein three years of agonizing labor.

But Hilbert's and Einstein's work do not differ in basic mathematical machinery, and his paper can be read as dealing with structures in $E1$ and $E2$ alone. Like Einstein, Hilbert (p. 395) introduces four space-time coordinates as primitives, with no mention of a manifold akin to those of $M2$. He introduces the ten "gravitational potentials" $g_{\mu\nu}$, which are not said to represent another level of mathematical structure; rather they have "symmetric tensor character with respect to an arbitrary transformation of the world-parameters [coordinates] w_s." Hilbert assigns a canonical physical interpretation to his coordinate systems, just as Einstein did. For example, his x_4 coordinate is a *time* coordinate, so hat he urges on p. 57 that the set of coordinate transformations be restricted to those that maintain the temporal order induced by the x_4 coordinate. This would ensure that "two world-points, lying on the same time-line, can stand in the relation of cause and effect to one another, and that it should then not be possible to transform such world-points so that they are simultaneous."

Finally, on pp. 58–63, Hilbert sets up and resolves his own version of the hole argument, using only mathematical structures of $E2$. He considers the four electrodynamic and then gravitational field potentials q_s and $g_{\mu\nu}$,

[36] See, for example, McCormmach.[(36)]

[37] According, for example, to Freudenthal.[(23)]

generated by a lone electron at rest, under a coordinate transformation that is the identity for x_4 less than or equal to zero, but smoothly differs from it otherwise. The field quantities $g_{\mu\nu}$ and q_s agree with the transformed quantities $g'_{\mu\nu}$ and q'_s in the past of the hypersurface $x_4 = 0$, but disagree in the future. This does not represent a violation of the "causality principle" under which the past would not determine the future, for he makes a stipulation equivalent to Leibniz equivalence: "in physics we must designate as *physically meaningless* an assertion which does not remain invariant under each arbitrary transformation of the coordinate system" (p. 61). His paradigm example of an invariant assertion is one made with respect to a Gaussian coordinate system, which is specially adapted to the fields present, as opposed to assertions relating to arbitrary coordinate systems. What he does *not* do is shrug off the threatened indeterminism of the hole argument by saying that the pairs of field quantities $g_{\mu\nu}$ and $g'_{\mu\nu}$ represent by definition a coordinate-free geometric object of another level of mathematical structure.

8. CONCLUSION

The realization that Einstein used open sets of R^4 where we now use abstract differentiable manifolds in the formulation of space-time theories solves a number of outstanding puzzles surrounding his use of coordinate systems and covariance principles. He accorded the extra structure of his number manifolds a canonical physical interpretation which included frames of reference and relative spaces. So we need no longer try to decide whether Einstein intended the term coordinate system, in a particular context, to refer to what we now call a coordinate chart, frame of reference, or relative space. He intended them all at once. We also need no longer worry whether the transformations he invoked should be read passively or actively in the sense of Sec. 2. In the simpler mathematical structure of his space-time theories no such distinction could be made or was needed. We saw that this greatly simplifies analysis of his hole and point-coincidence arguments.

Finally we saw that covariance/Leibniz equivalence principles display a remarkable tenacity in claiming physical content. I showed how Einstein's original covariance principles were physical principles, for they made, for example, the contingent assertion that there is no physical counterpart to the mathematical differences between such structures as the matrix of values g_{ik} and its transform g'_{ik} in general relativity. Later work reconstrued such assertions as mathematical definitions by interposing a new level of mathematical objects between the matrices g_{ik} and g'_{ik} and physically possible space-times and then requiring that the two matrices by definition be just different representations of the same mathematical object g_{ab} in the new level. While this maneuver deprived covariance/Leibniz equivalence principles at one level of physical significance, what was overlooked until quite recently was that the maneuver generated the need for a physically significant covariance/Leibniz equivalence principle to apply to the structures of the new mathematical level. And the very same considerations, the hole an point-coincidence arguments, which guided Einstein in his original analysis of these principles now guide us again at the new level.

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REFERENCES

1. J. L. Anderson, *Principles of Relativity Physics* (Academic Press, New York, 1967).
2. R. L. Bishop and S. I. Goldberg, *Tensor Analysis on Manifiolds* (Dover, New York, 1980).
3. J. Earman, "Covariance, Invariance and the Equivalence of Frames," *Found. Phys.* **4**, 267–289 (1974).
4. J. Earman and J. Norton, "What Price Space-time Substantivalism? The Hole Argument," *Br. J. Philos. Sci.*, forthcoming.
5. A. Einstein, "Zum gegenwärtigen Stande des Gravitationsproblems," *Phys. Z.* **14**, 1249–1266 (1913).
6. A. Einstein, "Die formale Grundlage der allgemeinen Relativitätstheorie," *Preuss. Akad. Wiss. Sitz.*, pp. 1030–1085 (1914).
7. A. Einstein, "Zum Relativitäts-problem," *Scientia* **15**, 337–348 (1914).
8. A. Einstein, "Prinzipielles zur verallgemeinerten Relativitätstheorie," *Phys. Z.* **15**, 176–180 (1914).
9. A. Einstein, "Die Gurndlage der allgemeinen Relativitätstheorie," *Ann. Phys.* (Leipzig) **49**, 769–822 (1916); translated without p. 769 as "The Foundation of the General Theory of Relativity," pp. 111–164 in *The Principle of Relativity* (Dover, New York, 1952).
10. A. Einstein, "Uber Friedrich Kottlers Abhandlung 'Über Einsteins Äquivalenzhypothese und die Gravitation," *Ann. Phys.* (Leipzig) **51**, 639–642 (1916).
11. A. Einstein, *Relativity: the Special and the General Theory* (1917; Methuen, London, 1977).
12. A. Einstein, "Prinzipielles zur allgemeinen Relativitätstheorie," *Phys.* (Leipzig) **55**, 240–244 (1918).

13. A. Einstein, *The Meaning of Relativity* (1922; Chapman and Hall, London, 1976).
14. A. Einstein, "Über den Äther," *Schweiz Nat. Gesell. Ver.* **105**, 85–93 (1924).
15. A. Einstein, "The Mechanics of Newton and their Influence on the Development of Theoretical Physics" (1927), pp. 253–261 in Einstein.[19]
16. A. Einstein, "The Problem of Space, Ether, and the Field in Physics" (1929), pp. 276–285 in Einstein.[19]
17. A. Einstein, "Autobiographical Notes," in *Albert Einstein: Philosopher-Scientist*, P. A. Schilpp, ed. (1949; Open Court, La Salle, 1979).
18. A. Einstein, Foreword (1953) to M. Jammer, *Concepts of Space* (Harvard University Press, Cambridge, 1969).
19. A. Einstein, *Ideas and Opinions* (Crown, New York, 1954).
20. A. Einstein and M. Grossmann, *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation* (Teubner, Leibzig, 1913) (separatum); with addendum by Einstein in *Z. Math. Phys.* **63**, 225–261 (1913).
21. A. Einstein and M. Grossmann, "Kovarianzeigenschaften der Feldgleichungen der auf die verallgemeinerte Relativitätstheorie gegründeten Gravitationstheorie," *Z. Math. Phys.* **63**, 215–225 (1914).
22. V. Fock, *The Theory of Space-Time and Gravitation* (Pergamon, London, 1959).
23. H. Freudenthal, "David Hilbert," pp. 388–395, in *Dictionary of Scientific Biography* (Scribner's, New York, 1972).
24. M. Friedman, *Foundations of Space-Time Theories* (Princeton University Press, Princeton, 1983).
25. S. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Space-time* (Cambridge University Press, Cambridge, 1973).
26. D. Hilbert, "Die Grundlagen der Physik," *Akad. Wiss. Gött. Nachr.* 1915, pp. 395–407; 1917, pp. 53–76.
27. A. Hiskes, "Space-time Theories and Symmetry Groups," *Found. Phys.* **14**, 307–332 (1984).
28. G. Holton, "Introduction: Einstein and the Shaping of Our Imagination," in *Albert Einstein: Historical and Cultural Perspectives*, G. Holton and Y. Elkana, eds. (Princeton University Press, Princeton, 1982), pp. vii–xxxii.
29. F. Klein, "Über die sogenannte nich-Euklidische Geometria" (1873, 2te. Aufsatz), in *Gesammelte Mathematische Abhandlungen*, Vol. 1 (Springer, Berlin, 1921–1923), pp. 311–343.
30. F. Klein, "Über die geometrischen Grundlagen der Lorentz-gruppe" (1910), in *Gesammelte Mathematische Abhandlangen*, Vol. 1 (Springer, Berlin), pp. 533–552.
31. F. Klein, *Einleitung in die höhere Geometrie, I.* Vorlesung, gehalten im Wintersemester 1892–1893, transcribed by Fr. Schilling (Göttingen, 1893).
32. F. Klein, *Vorlesungen über nicht-Euklidische Geometrie* (Chelsea, New York, 1927).
33. E. Kretschmann, "Über den physikalischen Sinn der Relativitätspostulat, A Einsteins neue und seine ursprügliche Relativitätstheorie," *Ann. Phys.* (Leipzig) **53**, 575–614 (1917).
34. T. Levi-Civita, *The Absolute Differential Calculus* (1926; Dover, New York, 1977).
35. J. H. Manheim, *The Genesis of Point Set Topology* (Pergamon, New York, 1964).
36. R. McCormmach, "Editor's Foreword," *Hist. Stud. Philos. Sci.* **7**, xi–xxxv (1976).
37. H. Minkowski, "Raum und Zeit," in *Das Relativitätsprinzip*, H. A. Lorentz *et al.*, eds. (Teubner, Leipzig, 1908), pp. 56–68.
38. H. Minkowski, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern," *Gött. Königl. Ges. Wiss. Nachr.*, pp. 53–111 (1908).
39. H. Minkowski, "Space and Time" (1908), in *The Principle of Relativity*, A. Einstein *et al.*, eds. (Dover, New York, 1952).

40. J. Norton, "What was Einstein's Principle of Equivalence?", *Stud. Hist. Philos. Sci.* **16**, 203–246 (1985).
41. J. Norton, "Einstein, the Hole Argument and the Reality of Space," in *Measurement, Realism and Objectivity*, J. Forge, ed. (Reidel, Dordrecht, 1987).
42. A. Pais, *Subtle is the Lord...: The Science and the Life of Albert Einstein* (Clarendon, Oxford, 1982).
43. R. K. Sachs and H. Wu, *General Relativity for Mathematicians* (Springer, New York, 1977).
44. A. Sommerfeld, "Zur Relativitätstheorie. Vierdimensionale Vektoralgebra," *Ann. Phys.* (Leipzig) **32**, 749–776; **33**, 649–689 (1910).
45. P. Speziali, ed., *Albert Einstein–Michele Besso: Correspondence 1903–1955* (Paris, 1972).
46. J. Stachel, "Einstein's Search for General Covariance," in *Proceedings of the Ninth International Conference on General Relativity and Gravitation* (Jena, 1980), E. Schmutzer, ed. (Cambridge University Press, Cambridge, 1983).
47. J. Stachel, "What can a Physicist Learn from the Discovery of General Relativity?" *Proceedings, 4th Marcel Grossmann Meeting on Recent Developments in General Relativity*, Rome, Italy, 17–21 June, 1985.
48. G. Ricci and T. Levi-Civita, "Méthodes de Calcul Différentiel Absolu et leurs Application" (1900), in *Opere Matematiche*, T. Levi-Civita, ed., Vol. 1 (Bologna, 1954).
49. B. Riemann, "On the Hypotheses Which Lie at the Foundations of Geometry," (1854), in *A Source Book in Mathematics*, D. E. Smith, ed. (Dover, New York, 1959).
50. R. Torretti, *Philosophy of Geometry from Riemann to Poincaré* (Reidel, Dordrecht, 1978).
51. R. Torretti, *Relativity and Geometry* (Pergamon, New York, 1983).
52. R. Torretti, "Space-Time Physics and the Philosophy of Science," *Br. J. Philos. Sci.* **35**, 280–292 (1984).
53. R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
54. J. Norton, "The Physical Content of General Covariance," in preparation.