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Quantum Measurements, Sequential and Latent

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The results of a hypothetical experiment requiring a sequence of quantum measurements are obtained retrospectively, after the experiment has been completed, from a single reading of an "apparatus register." The experiment is carried out reversibly and Schrödinger's equation is satisfied until the terminal reading of the register. The technique is illustrated using a feasible method of measuring photon spin as the quantum "object" observable and using the photon energy as the "apparatus register." The technique is used to discuss the "watchdog" effect, the effect of repeated measurements inhibiting quantum jumps.

1. INTRODUCTION

Following von Neumann's quantum measurement procedure,[1,2,3] a sequence of quantum measurements of an object's observable results in a sequence of "reductions of the wave function" of the object. Each measurement is interpreted prospectively and the "reduction" is a sudden change of the wave function to the eigenfunction determined by the measurement. The reduction is represented by a projection operator acting on the wave function. It is irreversible and is incompatible with the Schrödinger equation. It halts the unitary (Hamiltonian) development of the wave function which is then immediately restarted with the new "reduced" wave function.

An alternative "latent measurement procedure" for making a sequence of measurements is examined here, and photon polarization measurements are used to provide a concrete and doable illustration of the technique. Instead of a sequence of completed measurements, the results of a sequence of incomplete or latent measurements are stored as a single sum in an

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"apparatus" register. The measurements are completed and their results are obtained retrospectively at the end of the sequence from a single reading of the contents of this register. This terminal reading results in a single concluding reduction of the wave function.

As discussed later, the latent measurements are represented by unitary operators acting on the wave function of a quantum system composed of the "object" and the "apparatus". Until this terminal reading of the apparatus register, the whole sequence of latent measurements is reversible. To simplify the discussion it is assumed that the eigenvalues of the "object" and "apparatus" are discrete and nondegenerate.

Von Neumann's procedure for making a single measurement consists of two parts. First, the latent measurement is made. The quantum system is enlarged to contain both the "object" and the measuring "apparatus." The wave function of this enlarged quantum system is initially $\Psi_0 = \psi v_0$ where the wave function of the "object", $\psi = \Sigma c_k u_k$, is expanded in eigenfunctions of the observable of interest, and the wave function of the "apparatus" in its "null" state is v_0. A short, intense interaction between the object and the apparatus induces a unitary transformation, U, of the combined wave function and generates a one-to-one correlation of the eigenstates u_k with v_k, the "pointer" eigenstates of the apparatus.

$$\Psi = U\Psi_0 = U\Sigma c_k u_k v_0 = \Sigma c_k u_k v_k \tag{1}$$

This completes the 1st part of the procedure, the latent measurement.

The second part of von Neumann's procedure is to read the "pointer" eigenvalue p_k. This determines the eigenstates v_k and u_k and the "object" eigenvalue. The measurement is now completed, i.e. "developed," and there is the reduction of the wave function to the projected function

$$P_k \Psi = c_k u_k v_k \tag{2}$$

where P_k is the projection operator for the u_k state and $|c_k|^2$ is the probability of obtaining the kth result. After the completion of the measurement by reading the "pointer" the "apparatus" is useless and is eliminated from the quantum system. Until the "pointer" is read the latent measurement is reversible, at least in principle. It is assumed that the "pointer" can be read macroscopically.

2. SEQUENTIAL MEASUREMENTS, TWO METHODS

To simplify the discussion, consider a 2-level quantum object with the observable S and the eigenvalues $S_{1,2} = +1, -1$. The eigenstates are $|1, 2\rangle$

and the eigenvalue equation is $S|k\rangle = S_k|k\rangle$. Assume an "apparatus" with the state function $v(q)$ and the null state $v_0 = \text{constant}$. The initial state of the combined quantum system is

$$\Psi_0 = \Sigma c_k v_0 |k\rangle \tag{3}$$

The first and second terms of the Hamiltonian $H = H_0 + H_a + H_{\text{int}}$ act on the object and apparatus variables and the last term,

$$H_{\text{int}} = -CSq/\delta t, \tag{4}$$

is an interaction which is switched on for a short time δt. The coupling C is great enough so that H_{int} is the dominant term in H when the interaction is turned on. The unitary operator

$$U = \exp[-iH_{\text{int}}\,\delta t/\hbar] = \exp[iCSq/\hbar] \tag{5}$$

transforms Ψ_0 into

$$\Psi = U\Psi_0 = U\Sigma c_k v_0 |k\rangle = \Sigma c_k v_k |k\rangle$$

where

$$v_k = v_0 \exp[iCS_k q/\hbar] \tag{6}$$

The "momentum" conjugate to the variable q, $p = (\hbar/i)\,\partial/\partial q$, serves as the "pointer" or "register." Its eigenvalues are

$$p_k = CS_k \tag{7}$$

At this point a difference between the two measurement techniques begins to develop. We first consider von Neumann's procedure. The pointer p_k is observed and a reduction of the state function to the projected function

$$P_k\Psi = c_k v_k |k\rangle \to |k\rangle \tag{8}$$

takes place, where $P_k = |k\rangle\langle k|$ is the projection operator for $|k\rangle$. After the measurement the obsolete apparatus is rejected and the v_k is eliminated from Eq. (8). Also c_k is eliminated to renormalize the reduced state vector. The probability of obtaining the eigenvalue S_k is $|c_k|^2$.

The next step in von Neumann's technique is to use the Schrödinger equation to evolve $|k\rangle$ over a time interval, Δt, into

$$\psi = T|k\rangle = \Sigma d_{kj}|j\rangle \tag{9}$$

where

$$T = \exp[-iH_0 \Delta t/\hbar] \tag{10}$$

is unitary and d_{kj} is the matrix element $\langle j| T |k\rangle$. Equation (9) has the same form as ψ_0 before the first measurement. The next measurement requires the introduction of a new "apparatus" in its null state and gives the result S_j with the probability $|d_{kj}|^2$ if the result of the first measurement was S_k. Thus the joint probability for obtaining the sequence S_k, S_j is $|c_k|^2 |d_{kj}|^2$. After another iteration, the probability of obtaining the sequence of results S_k, S_j, S_i is

$$\text{prob} = |c_k|^2 |d_{kj}|^2 |e_{ji}|^2 \tag{11}$$

The "latent measurement technique" requires the following procedural changes: (1) Except for the last measurement in the sequence the projection operation is omitted, (2) the "apparatus" is not rejected after the first measurement but is used for the whole sequence, (3) the constant C in (4) is replaced by $C2^n$ for the nth measurement and (4) H_a is a function of p only.

The change (3) encodes the whole sequence of results S_k, S_j, S_i... as a sum in the "apparatus register." The requirement (4) makes the "momentum" p a constant of the motion in the intervals between measurements and preserves the memory of the previous measurements.

Under the "latent measurement method" the combined object-apparatus system is evolved under a series of unitary transformation as follows

$$\Psi = ...U_3 T' U_2 T U_1 \Psi_0 \tag{12}$$

where

$$U_n = \exp[i2^n CSq/\hbar] \tag{13}$$

induces the latent measurement transformation and the Hamiltonian time displacement transformations (T, T',...) have the form

$$T = \exp[-i(H_0 + H_a) \Delta t/\hbar] \tag{14}$$

The apparatus Hamiltonian H_a and the "register" observable p are constants of the motion between the measurements. As discussed later, the resulting accumulated phase shift after the sequence of latent measurements does not affect the probabilities and the term H_a can be dropped from Eq. (14).

If the results from a sequence of observations were S_k, S_j, S_i,..., then the terminal value of $p_{kji...}$ would be

$$p_{kji...} = C(2S_k + 4S_j + 8S_i + \cdots) \tag{15}$$

and vice versa. Thus the numerical content of the register determines the sequence of the eigenvalues of S. For n measurements there are 2^n possible histories joining Ψ_0 and Ψ but $p_{kji...}$ determines a unique history and there are no interference effects between histories if $p_{kji...}$ is read from the register. Owing to the uniqueness of the history, the term H_a in Eq. (14) is dropped for, as noted earlier, it only generates a phase shift.

Prior to reading the apparatus register, the state vector in Eq. (12) can be written

$$\begin{aligned} \Psi &= ...U_3 T' U_2 T U_1 \Psi_0 \\ &= \Sigma c_k \Sigma d_{kj} \Sigma e_{ji...} v_{kji.}(q) \, |i.\rangle \end{aligned} \tag{16}$$

where

$$v_{kji.}(q) = v_0 \exp[ip_{kji..} q/\hbar] \tag{17}$$

and $p_{k,j,i...}$ is given by Eq. (15).

Ψ in Eq. (16), is a superposition, a sum over histories, each of which represents a possible sequence of eigenvalues S_k, S_j, $S_{i,...}$ After reading the register and obtaining the eigenvalue $p_{k,j,i...}$, there is a reduction of the apparatus function to the single term $v_{kji.}(q)$ induced by a projection operator $P_{k,j,i..}$ acting on v. The projection operator inserts zeros in all the histories but one and the reduced function becomes

$$P_{k,j,i...} \Psi = c_k d_{kj} e_{ji...} v_{kji.}(q) \, |i.\rangle \tag{18}$$

The result of interest is the probability of obtaining the reading $p_{kji..}$, i.e. the sequence of eigenvalues S_k, S_j, $S_{i,...}$. This is

$$\text{Prob} = |c_k d_{kj} e_{ji...}|^2 \tag{19}$$

in agreement with Eq. (11).

One important difference between the two procedures concerns reversibility. With the "latent measurement technique" the measurements are reversible until the terminal reading of the register. Ψ can be converted into Ψ_0 by operating on Eq. (12) with the Hermitian adjoint of the series of unitary operators in Eq. (12).

A second important difference concerns interpretation. With the latent measurement technique the results of an experiment, consisting of a sequence of measurements, are obtained retrospectively after the experiment is completed. With the von Neumann technique the measurements are interpreted prospectively, measurement by measurement.

3. PHOTON POLARIZATION MEASUREMENTS

As an example of sequential measurements of an observable of a two-level "object" we shall consider the spin of a photon. The direction of propagation of the photon (z) is chosen as the reference direction for measuring the photon spin angular momentum. The states of angular momentum $\pm\hbar$ are designated $|+\rangle$ and $|-\rangle$. They are also called $\pm$ circular polarization states.

We shall let the photon energy play the role of the "apparatus register." The photon is generated by a laser in a well defined energy state, E_0, the "apparatus null state."

Disks cut from an anisotropic medium along an optic axis and oriented normal to the z direction are used to transform the polarization of a photon passing through the disks. A half-wave plate is such a disk which retards the phase of a correctly oriented plane polarized wave by π radians relative to the phase of the orthogonally polarized wave.

If a circular polarized photon passes through a half-wave plate, the sign of the polarization is reversed. The change in the photon's angular momentum is transferred from the half-wave plate. If the disk is rotating about the z axis with an angular velocity $-\omega$, the energy $\pm 2\hbar\omega$ is transferred along with the angular momentum $\pm\hbar$ from the disk to the photon. The use of a rotating half-wave plate to generate a frequency shift may have been first discussed as a microwave technique.[4]

A combination of two half-wave plates, one rotating and one fixed, Fig. 1, plays the role of the photon spin/energy correlater. This is represented by a unitary operator similar to U in equation (1). A spin eigenstate of the photon is left unchanged by the correlater but the photon energy is shifted by $\pm 2\hbar\omega$ depending upon the spin state. The rotating disk in Fig. 1 need not be physically rotating. Electro-optical methods can be used to generate equivalent or higher frequency rotations of an anisotropic optic axis.

4. PHOTON POLARIZATION MEASUREMENT THEORY

It is convenient to use Pauli spin matrices in constructing a quantum measurement theory for photon polarization. The technique is essentially that of Poincare and predates quantum mechanics.

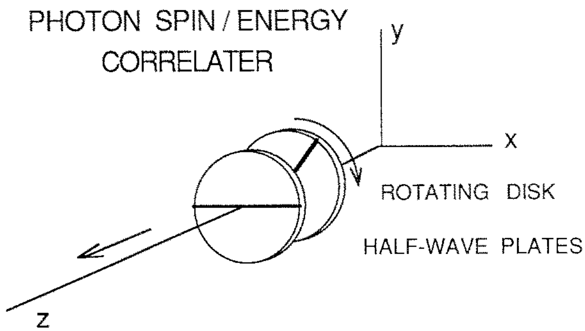

Fig. 1. Mechanism for producing a latent spin measurement of a photon. The transparent disks are cut from an anisotropic material and have their thicknesses adjusted to make them half-wave plates. One is rotating with an angular velocity $-\omega$ and the other is stationary. This mechanism couples the photon spin with a change in the photon's energy.

The x, y, z coordinate axes are sometimes designated 1, 2, 3 and the corresponding Pauli matrices are σ_1, σ_2 and σ_3. The 2 spin states $|\pm\rangle$ are eigenstates of σ_3, and $\sigma_3 |\pm\rangle = \pm|\pm\rangle$. The 2 plane polarized states $|a\rangle$ and $|b\rangle$ are defined to have their electric vectors along the x and y axes respectively, and phases are so chosen that

$$|\pm\rangle = \{|a\rangle \pm i\,|b\rangle\}/\sqrt{2} \tag{20}$$

The plane polarization states $|a\rangle$ and $|b\rangle$ are eigenvectors of σ_1.

The unitary operator

$$D(\delta, 0) = \cos\delta + i\sigma_1 \sin\delta \tag{21}$$

characterizes an anisotropic disk oriented with its retardation axis in the x direction. It operates on the state vector to represent the change in polarization induced by the disk. The thickness of the disk is proportional to δ and $\delta = \pi/2$ for a half-wave plate. For a quarter-wave plate, $\delta = \pi/4$. The state function of a circularly polarized photon is transformed into a plane polarized state by $D(\pi/4, 0)$.

If the disk is rotated about the z axis by the angle α the operator $D(\delta, 0)$ becomes transformed into

$$\begin{aligned} D(\delta, \alpha) &= R(\alpha)\, D(\delta, 0)\, R(\alpha)^* \\ &= \cos\delta + i\sin\delta[\cos 2\alpha - i\sigma_3 \sin 2\alpha]\,\sigma_1 \end{aligned} \tag{22}$$

where $R(\alpha) = \exp[-i\alpha\sigma_3]$ is the rotation operator and $R(\alpha)^*$ is the Hermitian adjoint of $R(\alpha)$.

For a half-wave plate rotating with an angular velocity $-\omega$

$$D(\pi/2, -\omega t) = i[\exp(2i\omega t\sigma_3)]\,\sigma_1 \tag{23}$$

This operator correlates the photon spin with the energy shift, but it also flips the photon spin. To avoid the spin flip induced by σ_1 the fixed half-wave plate characterized by $D(\pi/2, 0) = i\sigma_1$ is added to the rotating disk and the correlating operator (5) representing the latent spin measurement takes the form

$$\begin{aligned} U = D(\pi/2, 0)\, D(\pi/2, -\omega t) &= -\exp(-2i\omega t\sigma_3) \\ &= -[P_-\exp(2i\omega t) + P_+\exp(-2i\omega t)] \end{aligned} \tag{24}$$

where $P_\pm = |\pm\rangle\langle\pm|$ are projection operators for the two-spin states. Note that

$$Uv\,|+\rangle = -v\exp(-2i\omega t)\,|+\rangle \tag{25}$$

The photon energy is increased (decreased) by $2\hbar\omega$ for the spin states $+$ $(-)$.

Unitary transformations of U can be used to transform it into operators representing latent polarization measurements of plane polarized photons, or measurements of general elliptical polarization.

Following Eq. (13) we define

$$U_n = -\exp(-2^n i\omega t\sigma_3) \tag{26}$$

and encode the results of a sequence of measurements in the photon energy shift

$$\Delta E = E - E_0 = \hbar\omega[(\pm)_1 2 + (\pm)_2 4 + (\pm)_3 8 + \ldots] \tag{27}$$

where $(\pm)_1, (\pm)_2, (\pm)_3, \ldots$ are the results of the sequence of spin measurements.

A photospectrometer employing photon counters can be used to measure the photon energy. This represents the reading of the "apparatus register." The photon is destroyed in the process but the experiment has already been completed and the results are retrospectively available.

One important requirement is the consistency of repeated measurements. Note that $U_2 U_1 = \exp(-6i\omega t\sigma_3)$ and $\Delta E = \pm 6\hbar\omega$ which implies that $(\pm)_1 = (\pm)_2$ and the two-spin measurements are in agreement.

5. THE WATCHDOG EFFECT, AN EXAMPLE

The watchdog effect[5,6] is the inhibiting effect of repeated observations in suppressing quantum transitions. The transition to be examined is the photon spin flip induced by a half-wave plate see Eq. (21).

We assume that the half-wave plate is composed of a stack of n thin wafers. The unitary transition operator for each wafer is

$$D = D(\pi/2n, 0) = \cos(\pi/2n) + i\sigma_1 \sin(\pi/2n) \tag{28}$$

If the incident photon spin state is $|+\rangle$ the spin state after passing through m wafers is

$$\begin{aligned} \psi &= D(\pi/2n, 0)^m \,|+\rangle \\ &= \cos(\pi m/2n)\,|+\rangle + i \sin(\pi m/2n)\,|-\rangle \end{aligned} \tag{29}$$

The probability that the spin has not been changed is

$$\text{prob} = |\cos(\pi m/2n)|^2 \tag{30}$$

This expectation can be tested using this technique by inserting a latent measurement, and the operator U in Eq. (24), after the mth wafer. The photon energy is read after all n wafers have been passed and the wave function is

$$\Psi = D^{n-m} U D^m v_0 \,|+\rangle \tag{31}$$

If the energy measurement implies that the spin is $(+)$, the term in Eq. (24) with P_- does not contribute to Ψ and is dropped from U. Using the relation $P_+\sigma_1 = \sigma_1 P_-$,

$$P_+\sigma_1\,|+\rangle = 0 \tag{32}$$

and only the cosine term in D^m [see Eqs. (31) and (29)] contributes to $|+\rangle$ in Ψ. The probability in Eq. (30) is confirmed. The situation is quite different when a sequence of spin measurements is made.

Motivated by the desire to observe the spin flip which must occur when a photon transits a half-wave plate, a latent spin measurement is made after passing each of the n wafers, a total of n measurements. The combined effect of the wafers and measurements on the wave function is given by

$$\Psi = U_n D_{\dots} U_2 D U_1 D v_0 \,|+\rangle \tag{33}$$

We first calculate the probability that $(+)$ spin is obtained for each of the n measurements. For this result to hold the term with the projection operator P_- can be dropped from each U_k [see Eq. (24).] None of them can contribute to the probability amplitude of the proper photon energy. From Eq. (32) only the cosine term in each of the D's contributes. The probability is

$$\text{prob} = (\cos(\pi/2n))^{2n} \tag{34}$$

Instead of our naive expectation that one and only one spin flip should occur, the probability for no flip with $n = 10$ is prob $= 0.781$.

In similar fashion the probability that a single spin flip occurs after m wafers is

$$\text{prob} = (\cos(\pi/2n))^{2(n-1)} (\sin(\pi/2n))^2 \tag{35}$$

For $n = 10$, prob $= 0.196$. Note that this result is independent of m.

The watchdog effect is primarily due to the sequence of latent measurements. The completion of the measurements is not essential. To show this we compute the probability of obtaining $(+)$ spin after the nth wafer while ignoring all the intermediate results. To do this we omit the photon energy measurement and compute the expectation value of P_+ using the wave function in Eq. (33). For $n = 10$ this gives prob $= 0.803$, only slightly greater than the result, 0.781, obtained when all 10 $(+)$ spins are observed.

The reason for the watchdog effect is the disturbance of the phase relation between the $(+)$ and $(-)$ parts of a superposition. The spin measurement leaves the spin state undisturbed, but only if the photon is in a spin eigenstate.

6. SUMMARY

Using the latent measurement technique, a quantum experiment containing a sequence of quantum measurements is carried out reversibly, Schrödinger's equation being satisfied. After the experiment is finished it is terminated irreversibly and the results of the measurements are obtained and reported retrospectively from a single reading of an "apparatus register." This "reading" and the subsequent "reduction of the wave function" is irreversible and violates Schrödinger's equation. This is unsatisfactory but less so than the multiple violations which occur with the standard von Neumann procedure for which each measurement is irreversible.

There is a further possible difficulty with the standard von Neumann procedure. It has been suggested that it may not be possible to devise an "apparatus" for which a pointer reading can be made without disturbing the "object".[7] This is not an issue with the latent measurement technique for the experiment is finished before the "register" is read. The experimental results are retrospectively available and the "object" is no longer needed.

The simplifying assumption that there are only two-object states can be eliminated and this formalism can be generalized to any finite number of discrete objects states.

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REFERENCES

1. J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932), Chaps. V and VI.
2. J. A. Wheeler and J. A. Zurek, eds., *Quantum Theory and Measurements* (Princeton University Press, Princeton, 1983).
3. D. Greenberger, ed., "New Techniques and Ideas in Quantum Measurement Theory," *Ann. N.Y. Acad. Sci.* **480**, 1-633 (1987).
4. C. G. Montgomery, R. H. Dicke, and E. M. Purcell, eds. *M.I.T. Radiation Laboratory Series*, **8**, p. 355. (McGraw-Hill, New York, 1948).
5. B. Misra and E. C. G. Sudarshan, *J. Math. Phys.* **18**, 756 (1977).
6. K. Kraus, *Found. Phys.* **11**, 547 (1981).
7. B. van Fraassen, private communication.