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Scaling relationships in a constitutive equation with one structure variable

F. POVOLO
Comisión Nacional de Energia Atómica, Dto. de Materiales, Av. del Libertador 8250, (1429) Buenos Aires, Argentina

Fortes and Emilia Rosa [1] have published recently complete constitutive equations containing a single structure variable. The authors claimed to have derived the general equation compatible with the scaling relationship, based on the following arguments: assume first, that one parameter, p, family of $y(x, p)$ curves have a scaling relationship with $\Delta x/\Delta y = 0$, that is, the curves can be made to coincide by a translation parallel to the y-axis. The general form of a family with this property is

$$y(x, p) = p + f(x) \qquad (1)$$

where f denotes an arbitrary function. In addition, by making a change of the coordinate axes the authors have shown that the general form of the family of curves which superimpose under a translation along the direction with slope $\Delta y/\Delta x = 1/M$ is

$$y = p + f(x - My) \qquad (2)$$

Furthermore and always according to these authors, on taking $y = \log \sigma$, $x = \log \dot{\epsilon}$ and $p = \log \sigma_i$, Equation 2 leads to

$$\sigma/\sigma_i = g(\dot{\epsilon}/\sigma^M) \qquad (3)$$

or, inverting

$$\dot{\epsilon} = \sigma^M F(\sigma/\sigma_i) \qquad (4)$$

where g and F are arbitrary functions, σ is the applied stress, $\dot{\epsilon}$ is the plastic strain rate and σ_i is a structure variable. Then, Equation 4 should be the general form of a mechanical constitutive equation with one structure variable, compatible with the scaling relationship.

Povolo and collaborators [2–5] have studied in detail the scaling relationships observed frequently in the $\log \sigma$–$\log \dot{\epsilon}$ curves, obtained either during creep or stress relaxation experiments. Furthermore, it was discussed if the scaling relationship was a sufficient condition to ensure the existence of a state variable dependent on σ and $\dot{\epsilon}$. In addition, the scaling relationships were applied in a normalized diagram [4, 6] of the type

$$f(\alpha\sigma, \dot{\epsilon}/\dot{\epsilon}^*, \beta) = 0 \qquad (5)$$

where α, $\dot{\epsilon}^*$ and β are parameters that depend on the particular model considered and f is a general function, to study the restrictions imposed on the theoretical model by the scaling property observed in the $\log \sigma$–$\log \dot{\epsilon}$ diagram. It should be pointed out that Equation 2 is a particular case of the function considered by Povolo and Rubiolo [2].

It is the purpose of this paper to give the general form of a family of curves which superimpose under a translation along a given direction. It will be demonstrated that Equation 2 is not the general form of this family of curves and, consequently, that Equation 4 is not the general form of a mechanical constitutive equation with one structure variable, compatible with the scaling relationship.

In order to study the general conditions that a scalar field must satisfy to present a scaling behaviour, it is convenient to change the notation and denote all coordinates by x distinguishing the axes by subscripts, as for example:

$$x \to x_1$$

$$y \to x_2$$

$$z \to x_3$$

Let us assume the function

$$g(A_i x_i) = a_i x_i + d \qquad i = 1, 2, 3 \qquad (6)$$

where g is a real function, continuous, single-valued and differentiable and A_i, a_i, d are real constants. The Einstein convention that summation over repeated indices within a given term is implied, was used in Equation 6 and will be implied in what follows, unless otherwise stated. Furthermore, all subscripts will take the values 1, 2 and 3.

Equation 6 can be written, in general, as

$$F = g(A_i x_i) - a_i x_i - d = 0 \quad (7)$$

and, for example, set of curves in the x_i, x_j plane, at different x_k levels, etc., can be defined. On taking increments of Equation 7 leads to

$$F_i \Delta x_i = -g'(u)(A_i \Delta x_i) + (a_i \Delta x_i) = 0 \quad (8)$$

where $u = A_i x_i$, $g'(u) = dg/du$ and $F_i = \partial F/\partial x_i$. For fixed increments Δx_i Equation 8 can only be satisfied if

$$A_i \Delta x_i = 0 \quad (9)$$

$$a_i \Delta x_i = 0 \quad (10)$$

By combining Equation 9 with Equation 10 it is easy to show that

$$\Delta x_k/\Delta x_i = (A_i a_j - a_i A_j)/(A_j a_k - a_j A_k) \quad (i \neq j \neq k) \quad (11)$$

This equation gives the three scaling relationships that can be defined in the three different planes. If the translation path is parallel to the x_k-axis, then $\Delta x_i = 0$ and Equation 11 leads to

$$A_j a_k - a_j A_k = 0 \quad (12)$$

or

$$A_j/a_j = A_k/a_k = K = \text{constant} \quad (13)$$

Then, if the function defined by Equation 6 shows a scaling relationship in the x_i, x_k plane, between curves at different x_j levels, with a translation path parallel to the x_k-axis, Equation 13 must be satisfied, i.e. Equation 6 must be of the form

$$g(Ax_i + Ka_j x_j + Ka_k x_k) = a_i x_i + a_j x_j + a_k x_k + d \quad (i \neq j \neq k) \quad (14)$$

In the particular case where Equation 6 reduces to

$$g(A_i x_i) = d \quad (15)$$

that is $a_i = 0$, on taking increments leads to Equation 9 and the scaling conditions reduce to

$$\Delta x_k/\Delta x_i = -(1/A_k)[A_j(\Delta x_j/\Delta x_i) + A_i] \quad (i \neq j \neq k) \quad (16)$$

This equation means that any translation path is possible, since only a relationship between scaling in different planes is established.

By a transformation of the coordinate axes, the translation path inclined with respect to the coordinate axes can always be transformed to a translation path parallel to one axis. Let us assume that the cartesian coordinate system x_i is changed to another cartesian system x'_i with a common origin. Then

$$x'_i = \beta_{ij} x_j \quad (17)$$

$$x_i = \beta'_{ij} x'_j \quad (18)$$

$$i, j = 1, 2.3$$

where β_{ij} gives the direction cosine between axis x'_i in the new system and axis x_j in the old system; similarly for β'_{ij}. Since both coordinate systems are cartesian, the linear transformations described by Equations 17 and 18 are orthogonal transformations [7] and

$$\beta'_{ij} = \beta_{ji} \quad (19)$$

On taking into account Equation 19 it is easy to show that Equation 6 can be written, in the new coordinate system, as

$$g(A'_i x'_i) = a'_i x'_i + d \quad (20)$$

where

$$A'_i = \beta'_{ji} A_j = \beta_{ij} A_j \quad (21)$$

Furthermore, it is easy to show that the scaling relationships, expressed by Equation 11 in the old system, are transformed in the new system to

$$\Delta x'_k/\Delta x'_i = \frac{\sum_{l,n} (\beta_{il}\beta_{jn} - \beta_{in}\beta_{jl})(A_l a_n - a_l A_n)}{\sum_{l,n} (\beta_{jl}\beta_{kn} - \beta_{jn}\beta_{kl})(A_l a_k - a_l A_k)} \quad (22)$$

$$(i \neq j \neq k,\ l \neq n,\ l, n = i, j, k)$$

Equation 22 cannot be simplified further. For the purpose of this paper the important transformations of coordinates are the two-dimensional ones, i.e. a rotation of the axes in the plane defined by two coordinate axes. The coordinate corresponding to the axis perpendicular to this plane is left unchanged by the transformation. If a clockwise rotation is performed in the x_i, x_k plane, through an angle ϕ, then

$$\beta_{jj} = 1,\ \beta_{ij} = \beta_{ji} = \beta_{kj} = \beta_{jk} = 0$$

$$\beta_{ii} = \cos\phi, \quad \beta_{ki} = -\sin\phi \quad (23)$$

$$\beta_{ik} = \sin\phi, \quad \beta_{kk} = \cos\phi$$

and Equation 22 reduces to

$$\Delta x'_k/\Delta x'_i = \frac{\cos\phi(A_i a_j - a_i A_j) - \sin\phi(A_j a_k - a_j A_k)}{\sin\phi(A_i a_j - a_i A_j) + \cos\phi(A_j a_k - a_j A_k)} \quad (i \neq j \neq k) \tag{24}$$

which, on taking into account Equation 11, can be written as

$$\Delta x'_k/\Delta x'_i = \frac{(\Delta x_k/\Delta x_i)\,\mathrm{cotg}\,\phi - 1}{(\Delta x_k/\Delta x_i) + \mathrm{cotg}\,\phi} \quad (i \neq j \neq k) \tag{25}$$

If in the new system the translation is parallel to the x'_k-axis, then $\Delta x'_i = 0$ or

$$\mathrm{cotg}\,\phi = -\Delta x_k/\Delta x_i \tag{26}$$

This result should be expected, since Equation 26 means an anticlockwise rotation of the coordinate axes by an angle equal to the angle defining the translation path.

In the particular case where Equation 15 holds, with the scaling conditions given by Equation 17, it is easy to show that in the new coordinate system

$$\Delta x'_k/\Delta x'_i = \frac{A_i\cos\phi + A_k\sin\phi + A_j(\Delta x'_j/\Delta x'_i)}{(A_i\sin\phi - A_k\cos\phi)} \quad (i \neq j \neq k) \tag{27}$$

Following the analysis outlined by Povolo and Rubiolo [2], it can be seen when Equation 6 is an equation of state, interpreted as defining curves in the x_i, x_k plane, at different x_j levels. In fact, even if $g(A_i x_i)$ is unknown explicitly, it can be assumed that Equation 7 defines a function $x_j = \phi(x_i, x_k)$, where $i \neq j \neq k$, and a differential

$$dx_j = (\partial\phi/\partial x_i)_{x_k} dx_i + (\partial\phi/\partial x_k)_{x_i} dx_k \quad (i \neq j \neq k) \tag{28}$$

The necessary and sufficient condition for this to be a perfect differential is

$$\frac{\partial}{\partial x_k}(\partial\phi/\partial x_i)_{x_k} = \frac{\partial}{\partial x_i}(\partial\phi/\partial x_k)_{x_i} \tag{29}$$

and is continuous. By using the change of variables $u = A_i x_i$ and the theorems for derivatives of implicit functions [8], it can be shown that

$$(\partial\phi/\partial x_i)_{x_k} = -\frac{[g'(u)A_i - a_i]}{[g'(u)A_j - a_j]} \tag{30}$$

and

$$(\partial\phi/\partial x_k)_{x_i} = -\frac{[g'(u)A_k - a_k]}{[g'(u)A_j - a_j]} \tag{31}$$

By taking into account Equations 30 and 31, Equation 29 can be written as

$$\phi_{ik} = \phi_{ki} = -g''(u)\frac{(A_i A_k + A_k A_j \phi_i + A_i A_j \phi_k + A_j^2 \phi_i \phi_k)}{[g'(u)A_j - a_j]} \tag{32}$$

where $\phi_i = (\partial\phi/\partial x_j)_{x_k}$, $\phi_k = (\partial\phi/\partial x_k)_{x_i}$, $\phi_{ki} = (\partial/\partial x_i)(\partial\phi/\partial x_k)_{x_i}$ and $\phi_{ik} = (\partial/\partial x_k)(\partial\phi/\partial x_i)_{x_k}$. According to Equation 32, Equation 29 is continuous if $g'(u)A_j - a_j \neq 0$ or

$$g'(u) \neq a_j/A_j \tag{33}$$

Then, the scalar fields described by Equation 6, with the scaling relationships described by Equation 11, will be a consequence of an equation of state unless the condition implied by Equation 33 is not satisfied. If this is the case, the scalar field will be an equation of state only in a restricted domain of the variables, where Equation 33 is obeyed. Furthermore, if Equation 33 does not hold, two curves, at different x_j, will have a crossing point in the x_i, x_k plane given by

$$g(A_i x_i + A_k x_k + A_j x_{j1}) - g(A_i x_i + A_k x_k + A_j x_{j2}) = c(x_{j1} - x_{j2}) \quad (i \neq j \neq k) \tag{34}$$

where x_{j1} and x_{j2} are two fixed values of x_j.

Equation 6 describes the general function leading to a scaling behaviour when curves are considered in the plane determined by two coordinate axes, parameterized in the third coordinate. An important point to be noticed is that a scaling behaviour in one of the planes leads to scaling in the other two planes, with different translation paths, defined by a pair of coordinate axes. Let us consider, as an example, the very well known state equation for ideal gases

$$pV = nRT \tag{35}$$

As pointed out in [2], written in this way Equation 35 does not generate a scalar field with a scaling behaviour. However, if the equation is written as

$$\log p + \log V = \log(nR) + \log T \quad (36)$$

and the variables changed to

$$x_1 = \log p, x_2 = \log V, x_3 = \log T \quad (37)$$

and on assuming, without loss of generality, that the mass is constant, i.e.

$$\log(nR) = \text{constant} = d \quad (38)$$

the result is

$$x_1 + x_2 - x_3 = d \quad (39)$$

Equation 39 has the form of Equation 15 with $A_1 = A_2 = 1$, $A_3 = -1$. According to Equation 16 the scaling conditions are

$$\Delta x_2/\Delta x_1 = \Delta\log V/\Delta\log p = 1 + (\Delta\log T/\Delta\log p) \quad (40)$$

and

$$\Delta x_3/\Delta x_2 = \Delta\log T/\Delta\log V = 1 + (\Delta\log p/\Delta\log V) \quad (41)$$

$$\Delta x_3/\Delta x_1 = \Delta\log T/\Delta\log p = 1 + (\Delta\log V/\Delta\log p) \quad (42)$$

In this particular case, Equations 40 to 42 could have been obtained, in a straightforward way, by taking increments of Equation 36. Any translation path can be chosen in the $\log p$–$\log V$, $\log T$–$\log V$ and $\log p$–$\log T$ planes, since parallel straight lines are obtained on parameterizing with respect to the third variable. Furthermore, Equation 36 is an equation of state, in any one of the planes, since $g'(u) = 1$ and Equation 33 is always satisfied. The function

$$y = g(Ax + Bz) + az + b \quad (43)$$

considered in [2] is a particular case of Equation 6 with $x = x_1$, $y = x_2$, $z = x_3$, $A = A_1$, $A_2 = 0$, $A_3 = B$, $a_1 = 0$, $a_2 = 1$, $a_3 = -a$ and $b = -d$. According to Equation 11 the scaling conditions are

$$\Delta x_2/\Delta x_1 = \Delta y/\Delta x = -aA/B \quad (44)$$

$$\Delta x_3/\Delta x_1 = \Delta z/\Delta x = -A/B \quad (45)$$

$$\Delta x_3/\Delta x_2 = \Delta z/\Delta y = 1/a \quad (46)$$

Equation 44 has been already given in [2]. The function described by Equation 2 is a particular case of both Equation 6 and of Equation 43, with $x = x_1$, $y = x_2$ and $x_3 = p$. It is clear that Equation 2 is not the general form of the family of curves which superimpose under a translation along the direction with slope $\Delta y/\Delta x = 1/M$, as stated in [1]. In fact, the general form is given by Equation 6 with $x_1 = x$, $x_2 = y$ and $x_3 = p$, i.e.

$$g(A_1x + A_2y + A_3p) = a_1x + a_2y + a_3p + d \quad (47)$$

and with the scaling relationships given by Equation 11. If $\Delta y/\Delta x = 1/M$, then

$$\Delta y/\Delta x = \frac{A_1a_3 - a_1A_3}{A_3a_2 - a_3A_2} = 1/M \quad (48)$$

There are several functions of the form of Equation 47 which satisfy Equation 48, for example

$$g(x - My) = a_1x + a_3p + d \quad (49)$$

$$g(x - My) = a_2y + a_3p + d \quad (50)$$

$$g(My + p) = a_1(x + p) + d \quad (51)$$

$$g(x + Mp) = a_2(y + p) + d \quad (52)$$

$$g(x + p) = My + p + d \quad (53)$$

$$g(y + p) = x + Mp + d \quad (54)$$

$$g(y - p) = x - Mp + d \quad (55)$$

On taking $y = \log\sigma$, $x = \log\dot{\epsilon}$, $p = \log\sigma_i$ and $d = \log K$ Equations 49 to 55 can be written as

$$f(\dot{\epsilon}/\sigma^M) = K\dot{\epsilon}^{a_1}\sigma_i^{a_2} \quad (56)$$

$$f(\dot{\epsilon}/\sigma^M) = K\sigma^{a_2}\sigma_i^{a_3} \quad (57)$$

$$f(\sigma^M\sigma_i) = K(\dot{\epsilon}\sigma_i)^{a_1} \quad (58)$$

$$f(\dot{\epsilon}\sigma_i^M) = K(\sigma\sigma_i)^{a_2} \quad (59)$$

$$f(\dot{\epsilon}\sigma_i) = K\sigma_i\sigma^M \quad (60)$$

$$f(\sigma\sigma_i) = K\dot{\epsilon}\sigma_i^M \quad (61)$$

$$f(\sigma/\sigma_i) = K\dot{\epsilon}\sigma_i^{-M} \quad (62)$$

Equations 57 to 62 illustrate some of the possible mechanical constitutive equations with one structure variable, compatible with a scaling relationship $\Delta\log\sigma/\Delta\log\dot{\epsilon} = 1/M$. It should be pointed out that Equation 4, which is of the form of Equation 62, was obtained in [1] by inverting Equation 3, which is of the form of Equation 57 with $a_2 = 1$ and $a_3 = -1$. This inversion is not always possible as, for example, in the case where $f(\dot{\epsilon}/\sigma^M)$ is of the form $f(\dot{\epsilon}/\sigma^M) = A_1 + A_2(\dot{\epsilon}/\sigma^M) + A_3(\dot{\epsilon}/\sigma^M)^2 + \ldots$

Finally, the general discussion on possible forms of mechanical constitutive equations and on the meaning of the scaling relationship, observed in the experimental $\log \sigma - \log \dot{\epsilon}$ curves, viewed within the framework of a theoretical model, will be presented in a forthcoming paper [9].

The general form of a family of curves which superimpose under a translation along a given direction has been given. It has been shown that scaling relationships must exist in any one of the three planes defined by two variables, for family of curves parametrized in the third variable.

The necessary and sufficient condition that a scalar field with a scaling behaviour must fulfil in order to be a consequence of an equation of state have been given. The results have been applied to the very well known state equation for ideal gases.

Finally, some results reported previously in the literature for mechanical constitutive equations have been shown to be particular cases of the formalism developed in this paper.

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