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# Relation between limiting drawing ratio and plastic strain ratio

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The dependence of the limiting drawing ratio (LDR) on the plastic strain ratio or the $R$ value has been well known. The LDR increases with the $R$ value. Increases of the LDR with the $R$ value are predicted by several theoretical analyses of deep drawing. Whiteley [1] employed simplifying assumptions of a nonhardening material, and plane strain deformation in the flange to show that

$$\ln(\text{LDR}) = \eta\beta \quad (1)$$

where $\eta$ is a deformation efficiency to account for frictional and bending work and $\beta$ is the ratio of two plane-strain flow stresses corresponding to $\epsilon_y = 0$ and $\epsilon_z = 0$, i.e.

$$\beta = \sigma_{\text{wall}(\epsilon_y=0,\,\sigma_z=0)}/\sigma_{\text{flange}(\epsilon_z=0)} \quad (2)$$

where $y$ and $z$ are the circumferential and thickness directions, and $\sigma_{\text{wall}}$ and $\sigma_{\text{flange}}$ are the flow strengths of the wall and flange. He then used Hill's anisotropic plasticity theory [2], which for planar isotropy of the sheet predicts that

$$\beta = [(R+1)/2]^{1/2} \quad (3)$$

so that

$$\text{LDR} = \exp \eta\,[(R+1)/2]^{1/2} \quad (4)$$

There have been more rigorous analyses of deep drawing which allow for both work hardening and thinning or thickening of the flange. These also rely on the Hill theory to characterize the anisotropic behaviour. All the theories mentioned above predict more dependence of LDR on $R$ than experimentally observed.

Hosford and Kim [3] have suggested that the major problem with these analyses lies in the use of the Hill theory. They calculated the $R$ and $\beta$ values for sheets of cubic metals bases on assumptions of equal strains in all grains, slip restricted to $\{111\}\langle 110\rangle$ or $\{110\}\langle 111\rangle$ systems, and textures characterized by a single $\langle h\,k\,l\rangle$ sheet normal with rotational symmetry about that normal. The calculation indicated a much lower variation of $\beta$ with the $R$ value than that predicted by the Hill theory.

Recent works [4, 5] indicate that the shape of yield loci for crystallographically textured fcc and bcc metals could be better represented by a generalization of Hill's yield criterion of the form.

$$af(\sigma_{ij}) = F|\sigma_y - \sigma_z|^a + G|\sigma_z - \sigma_x|^a + H|\sigma_x - \sigma_y|^a = 1 \quad (5)$$

where the exponent $a$ is much larger than the 2 in Hill's criterion, and $F$, $G$, and $H$ are constants which characterize the anisotropy.

The purpose of this paper is to calculate the expression for the LDR based on Equation 5. The constants $F$, $G$, and $H$ in Equation 5 may be evaluated from simple tension tests. Let $X$ be the tensile yield stress in the $x$ direction. At yielding, $\sigma_x = X$, $\sigma_y = \sigma_z = 0$, so Equation 5 becomes $(G+H)X^a = 1$ or $X^a = 1/(G+H)$. Similarly, if $Y$ and $Z$ are the tensile yield stresses in the $y$ and $z$ directions,

$$X^a = \frac{1}{G+H}, \qquad Y^a = \frac{1}{H+F}, \qquad Z^a = \frac{1}{F+G} \quad (6)$$

The flow rules may be developed using the following equation

$$\mathrm{d}\epsilon_{ij} = \mathrm{d}\lambda \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \quad (7)$$

Differentiation of Equation 5 results in the flow rules:

$$\mathrm{d}\epsilon_x = \mathrm{d}\lambda[-G|\sigma_z - \sigma_x|^{a-1} + H|\sigma_x - \sigma_y|^{a-1}]$$
$$\mathrm{d}\epsilon_y = \mathrm{d}\lambda[F|\sigma_y - \sigma_z|^{a-1} - H|\sigma_x - \sigma_y|^{a-1}]$$
$$\mathrm{d}\epsilon_z = \mathrm{d}\lambda[-F|\sigma_y - \sigma_z|^{a-1} + G|\sigma_z - \sigma_x|^{a-1}] \quad (8)$$

Note that for Equations 8, $\mathrm{d}\epsilon_x + \mathrm{d}\epsilon_y + \mathrm{d}\epsilon_z = 0$, indicating constant volume. Substitution of $\sigma_x = X$, $\sigma_y = \sigma_z = 0$ into Equation 8 gives the

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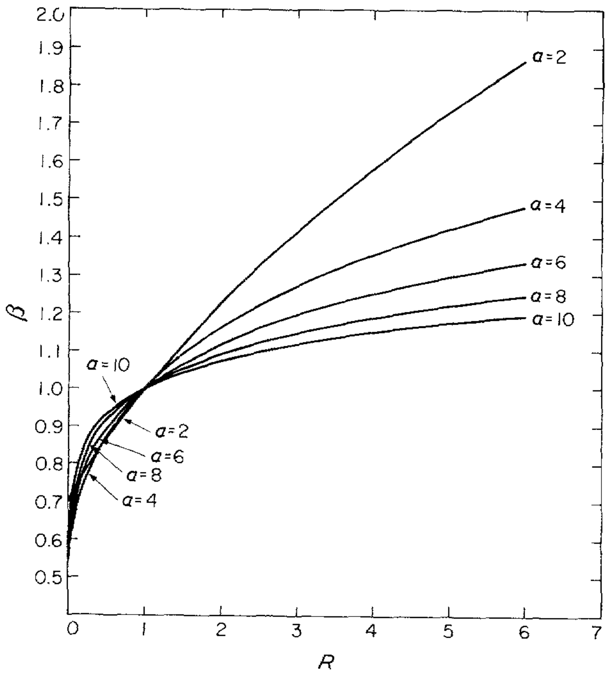


*Figure 1* The calculated relation of the $R$ value and $\beta$ for various $a$ values.

resulting strains,

$$\begin{aligned} d\epsilon_x &= d\lambda(-G+H)X^{a-1} \\ d\epsilon_y &= -d\lambda H X^{a-1} \\ d\epsilon_z &= d\lambda G X^{a-1} \end{aligned} \quad (9)$$

Since the strain ratio for the $x$-direction tension test is defined as $R = R_0 = |d\epsilon_y/d\epsilon_z|$

$$R = \frac{H}{G} \quad (10)$$

Similarly, defining $p = R_{90}$ as the strain ratio in a $y$-direction tension test, $p = d\epsilon_x/d\epsilon_z$ with $\sigma_y = Y$ and $\sigma_x = \sigma_z = 0$, Equations 8 result in

$$P = \frac{H}{F} \quad (11)$$

Equations 10 and 11 allow one to predict the values of the $z$-direction yield stress, $Z$, by conducting $x$- and $y$-direction tension tests and measuring $R$ and $P$ as well as $X$ and $Y$. From Equation 6

$$\frac{Z^a}{X^a} = \frac{G+H}{F+G} = \frac{\frac{1}{R}+1}{1/R+1/P}$$

or

$$Z = Y\left[\frac{R(1+P)}{P+R}\right]^{1/a} \quad (12)$$

Similarly,

$$Z = Y\left[\frac{R(1+P)}{P+R}\right]^{1/a}$$

Substituting $1 = (G+H)X^a$ from Equation 6 and dividing by $G$ Equation 5 becomes

$$\frac{F}{G}|\sigma_y-\sigma_z|^a + |\sigma_z-\sigma_x|^a + \frac{H}{G}|\sigma_x-\sigma_y|^a = \left(1+\frac{H}{G}\right)X^a$$

Substituting $R = H/G$ and $R/P = F/G$ and multiplying by $P$,

$$R|\sigma_y-\sigma_z|^a + P|\sigma_z-\sigma_x|^a + RP|\sigma_x-\sigma_y|^a = P(R+1)X^a \quad (13)$$

Similarly the flow rules, Equations 8, reduce to

$$\begin{aligned} d\epsilon_x : d\epsilon_y : d\epsilon_z = &-|\sigma_z-\sigma_x|^{a-1} + R|\sigma_x-\sigma_y|^{a-1} : \\ &\frac{R}{P}|\sigma_y-\sigma_z|^{a-1} - R|\sigma_x-\sigma_y|^{a-1} : \\ &-\frac{R}{P}|\sigma_y-\sigma_z|^{a-1} + |\sigma_z-\sigma_x|^{a-1} \end{aligned} \quad (14)$$

If the material has rotational symmetry about the $z$-axis (planar isotropy), $F = G$, $L = M$, and $R = P$. In this case, substitution of $P = R$ in Equations 13 and 14 results in

$$|\sigma_y-\sigma_z|^a + |\sigma_z-\sigma_x|^a + R|\sigma_x-\sigma_y|^a = (R+1)X^a \quad (15)$$

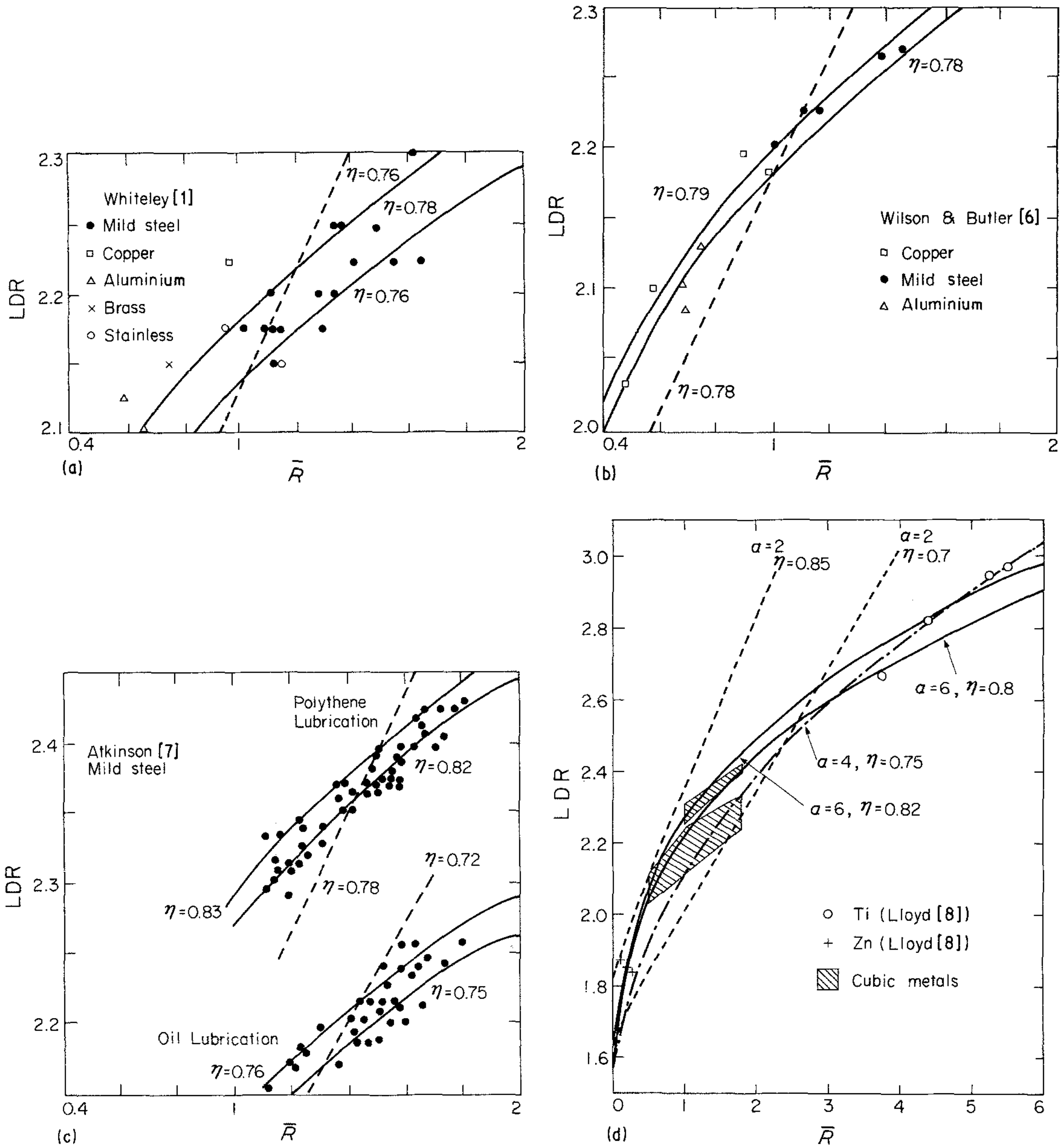


*Figure 2* Dependence of limiting drawing ratio on average strain ratio $\bar{R}$. The experimental points are from Whiteley [1], Wilson and Butler [6], Atkinson [7] and Lloyd [8]. The solid curves and the dashed curves in (a) to (c) are for $a = 8$ and 2, respectively.

and

$$d\epsilon_x : d\epsilon_y : d\epsilon_z = -|\sigma_z - \sigma_x|^{a-1} + R|\sigma_x - \sigma_y|^{a-1} : |\sigma_y - \sigma_z|^{a-1} - R|\sigma_x - \sigma_y|^{a-1} : -|\sigma_y - \sigma_z|^{a-1} + |\sigma_z - \sigma_x|^{a-1} \quad (16)$$

The flow strengths, $\sigma_{\text{flange}(\epsilon_z=0)}$ and $\sigma_{\text{wall}(\epsilon_y=0,\sigma_z=0)}$, may be expressed as $|\sigma_x - \sigma_y|_{\epsilon_z=0}$ and $\sigma_{x(\epsilon_y=0,\sigma_z=0)}$, respectively. In the flange, where $d\epsilon_z = 0$, Equation 16 predict that $\sigma_z = (\sigma_x + \sigma_y)/2$. Substituting this into the yield criterion, Equation 15, results in

$$2|\sigma_y - \sigma_x|^a + 2^a R|\sigma_x - \sigma_y|^a = 2^a(R+1)X^a$$

or

$$\sigma_{\text{flange}} = |\sigma_x - \sigma_y|_{\epsilon_z=0} = |\sigma_y - \sigma_x|_{\epsilon_z=0} = 2X\left[\frac{1+R}{2+2^a R}\right]^{1/a} \quad (17)$$

In the wall, where $d\epsilon_y = 0$ and $\sigma_z = 0$, Equations 17 predict that

$$\sigma_y = \frac{R^{1/(a-1)}}{1 + R^{1/(a-1)}}\sigma_x \tag{18}$$

Substituting Equation 18 into Equation 15 results in

$$\sigma_{\text{wall}} = \sigma_x(\epsilon_{y=0}, \sigma_{z=0})$$
$$= \frac{X(R+1)^{1/a}}{\left[\dfrac{R^{a/(a-1)}}{\left(1 + \dfrac{1}{R^a - 1}\right)^a} + 1 + R\left(1 - \dfrac{R^{1/(a-1)}}{1 + R^{1/(a-1)}}\right)^a\right]^{1/a}} \tag{19}$$

Substituting Equations 17 and 19 into the definition of $\beta$, Equation 2, we obtain

$$\beta = \frac{(2 + 2^a R)^{1/a}}{2\left[\dfrac{R^{1/(a-1)}}{\left(1 + \dfrac{1}{R^{a-1}}\right)^a} + 1 + R\left(1 - \dfrac{R^{1/(a-1)}}{1 + R^{1/a-1)}}\right)^a\right]^{1/a}} \tag{20}$$

Setting $a = 2$, reduces to $[(R + 1)/2]^{1/2}$, which is equivalent to Equation 3. Substitution of Equation 20 into Equation 1 with the efficiency, $\eta$, gives us the relation between the LDR and $R$-value.

Fig. 1 shows the calculated relation of the $R$ value and $\beta$ for various $a$ values. The $R$ value dependence of $\beta$ decreases with increasing $a$ at the $R$ values above unity. Hosford [4] suggested the $a \approx 6$ for bcc metals and $a \approx 8$ to 10 for fcc metals in the yield locus calculation. Several calculated results of the $R$–LDR relation are compared with experimental data in Fig. 2. The $R$–$\beta$ relations calculated with $a = 8$ for cubic metals and $a = 4$ to 6 for hexagonal metals agree very well with the experimental results.

## References


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