Foundations of Physics, Vol. 8, Nos. 7/8, 1978

Algebraic Constraints on Hidden Variables[1]

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Received September 9, 1977

In the contemporary discussion of hidden variable interpretations of quantum mechanics, much attention has been paid to the "no hidden variable" proof contained in an important paper of Kochen and Specker. It is a little noticed fact that Bell published a proof of the same result the preceding year, in his well-known 1966 article, where it is modestly described as a corollary to Gleason's theorem. We want to bring out the great simplicity of Bell's formulation of this result and to show how it can be extended in certain respects.

1. As a basis for comparison of Bell's[(2)] and Kochen and Specker's[(5)] algebraic no-hidden-variable proofs, we begin by outlining Kochen and Specker's strategy, omitting much of the technical detail. Kochen and Specker begin with two assumptions:

(i) Corresponding to each quantum state of a system, there is an underlying phase space Ω. Corresponding to each quantum mechanical observable A (self-adjoint operator on the Hilbert space of the system), there is a measurable function $f_A : \Omega \to \mathbb{R}$ that assigns, for each point in the phase space Ω, a real number, which is to be thought of as the value of the observable for the point in the phase space.

(ii) For each observable A and each Borel function g we have $g(f_A) = f_{g(A)}$, i.e., for each $\omega \in \Omega$, $g[f_A(\omega)] = f_{g(A)}(\omega)$.

Kochen and Specker then proceed by introducing the notions of a partial algebra of observables and a partial Boolean algebra of the projectors on the Hilbert space. Assumptions (i) and (ii) imply that there is an imbedding of the partial algebra of observables into a commutative algebra and an

[1] Work on this paper was partially supported by National Science Foundation Grants SOC 76-82113 and SOC 76-10659.

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imbedding of the partial Boolean algebra of projectors into a Boolean algebra. This last in turn implies that there are homomorphisms from the partial Boolean algebra of projectors to the two-element Boolean algebra Z_2; indeed there is a homomorphism for each pair of distinct observables which separate the pair. Finally, they show that for each three-dimensional Hilbert space, there is a finite partial subalgebra of projectors that cannot be homomorphically mapped to Z_2, thereby showing that (i) and (ii) lead to a contradiction.

We want to describe the last step in the proof in a little more detail. Let P_i, P_j,... range over projectors onto one-dimensional subspaces i, j,... of a three-dimensional Hilbert space H_3. (We shall sometimes use these subscripts to refer to vectors spanning the subspaces.) A homomorphism from the partial Boolean algebra of these projectors to Z_2 assigns to each projector the value 1 or 0 (the maximal and minimal elements of Z_2); and if i, j, and k are pairwise orthogonal, then exactly one of P_i, P_j, P_k gets mapped to 1 and the other two get mapped to 0. This can be pictured conveniently as a mapping of the points on the unit sphere in H_3 to 0 and 1 such that for each triple of orthogonal points on the sphere, one gets mapped to 1 and the other two get mapped to 0. Kochen and Specker use a straightforward geometrical argument to show that under these conditions, the angle subtended by points that get distinct values must be at least $\cos^{-1}1/2$. In other words, points that subtend a smaller angle must both receive the value 0 or both the value 1. The proof is then essentially done; since at least one point gets the value 1, all must get the value 1. But of each orthogonal triple, two are supposed to get the value 0. Furthermore, since all points within a cone of fixed small angle get assigned the same value, it is straightforward to show that there is a finite number of orthogonal triples for which the required assignment of 0's and 1's is impossible. Kochen and Specker's continuation of the argument (Ref. 5, Lemma 2, pp. 68–69) merely counts the number of points needed, as we will illustrate below in the context of Bell's work.

One is struck by the prominent role of the partial Boolean algebra of projectors in Kochen and Specker's work. Judging by the frequent references in discussions of this work, it would appear that a formulation in terms of the partial Boolean algebra of projectors is generally thought to be the correct, or even the only way to understand this material. Using Bell's presentation, we shall show shortly that this idea is mistaken. Indeed, we shall simplify the presentation of this material in two stages. In the first stage we can follow out an idea suggested by Kochen and Specker themselves and notice that each $\omega \in \Omega$ defines what we shall call a *valuation function* v; that is, a function that assigns an exact value (= real number) to each observable. Namely, for each observable A, let

$$v(A) = f_A(\omega)$$

Assuming that the Hilbert space of the system is at least three-dimensional, it is easy to see that (i) and (ii) imply the existence of a three-dimensional subspace such that for each triple i, j, k of its orthogonal one-dimensional subspaces, v assignes the value 1 to exactly one of P_i, P_j, and P_k, and 0 to the other two. Since the implementation of Kochen and Specker's argument requires no more than that there be one such subspace and v, all that needs to be assumed to arrive at a contradiction is that;

(i′) There is a valuation function v from observables to $\mathbb{R}$, on a Hilbert space of at least three dimensions; and

(ii′) For each Borel function g, $v[g(A)] = g[v(A)]$, for each observable A.

Before proceeding to a still simpler formulation, we want first to undermine what we believe is a widespread motivation for the emphasis on partial Boolean algebras of projectors in the discussion of hidden variables, namely the assumption that projectors can be construed as "propositions." The thought seems to be that the partial Boolean algebra of projectors will then in some way capture the "logic" of these propositions. To show the difficulty with this line of thought, we shall first establish the following:

Proposition 1. The functional relation condition (ii′) is satisfied by any valuation v that satisfies:

(iii) $v(P_a) = 1$ iff $v(A) \in \mathcal{O}\!\ell$, for all observables A, where P_a is the projector onto the $\mathcal{O}\!\ell$-eigenspace of A (for $\mathcal{O}\!\ell$ a set of real numbers).

Proof. Where χ_b is the characteristic function of b (for b a single real number or a set of reals) notice that for any observable B, $\chi_b(B)$ is the projector onto the b-eigenspace of B. Repeated use of (iii) yields the following chain of equivalences:

$$v[g(A)] = \lambda \quad \text{iff}$$
$$v(\chi_\lambda[g(A)]) = 1 \quad \text{iff}$$
$$v(\chi_{g^{-1}(\lambda)}(A)) = 1, \text{ since } \chi_\lambda[g(A)] = \chi_{g^{-1}(\lambda)}(A), \quad \text{iff}$$
$$v(A) \in g^{-1}(\lambda) \quad \text{iff}$$
$$g[v(A)] = \lambda$$

i.e., for any observable A and Borel function g,

$$v[g(A)] = g[v(A)]$$

Suppose, now, that each observable of the system has a precise value; i.e., suppose there is a valuation function v that gives the ("true") value that each observable has [in satisfaction of (i′)]. To say that projectors can be interpreted as propositions is to say that the projector P_a [as in (iii)] represents the proposition that the value of A is in $\mathcal{a}$. This requires that

$$P_a \text{ is true iff } v(A) \in \mathcal{a}$$

(where P_a is used ambiguously for the projector or for the proposition it represents). Since a is the $\{1\}$-eigenspace of P_a, this condition implies that

$$P_a \text{ is true iff } v(P_a) = 1$$

But these last two conditions imply that (iii) holds for this valuation v. It follows from Proposition 1 that v satisfies (ii′), which in turn leads to a contradiction, as explained above.

The upshot is that once we have assumed (i′), that is, once we have assumed that each observable gets a precise value, the further requirement that projectors be interpreted as propositions is already contradictory. Thus, in order not to beg the question against hidden variables (at least insofar as the existence of a valuation function is contained in a hidden variable interpretation), we must regard projectors neutrally; i.e., we must regard reference to them and to the partial Boolean algebras they form as no more than a technical convenience in certain proofs or constructions. In particular, one must not suppose that these algebraic structures have any special bearing on the logic of quantum mechanical propositions. In fact, the appeal to partial algebras of any sort turns out to be completely superfluous in connection with the no-hidden-variables results.

2. In order to show this, we shall proceed with the second stage in our simplification of those results and turn to the work of Bell.

Bell starts from the assumption that there is a function v defined on the projectors P_i on the one-dimensional subspaces of the at least three-dimensional Hilbert space H such that:

(iv) $v(P_i) = 0$ or 1 for each i; and

(v) $\sum_{i \in B} v(P_i) = 1$ for each orthonormal basis B of H.

Bell's result is that these conditions lead to a contradiction. Of course, the contradiction is an immediate consequence of Gleason's theorem. It was Bell's intention, however, to exhibit this contradiction without recourse to harmonic analysis and the other technicalia of Gleason's proof. To this end Bell uses (iv) and (v) in an elementary geometrical argument, very similar to the one employed by Kochen and Specker, to show that if $v(P_i) \neq v(P_j)$, then $\| i - j \| \geqslant 1/2$; i.e., that the angle between i and j is at least $2 \sin^{-1}(1/4)$. Bell then observes that (iv) and (v) imply that $v(P_i) = 1$ for at least one i. By the geometrical argument it follows that $v(P_j) = 1$ for all j, since one could interpolate between i and any j a sequence of vectors pairwise separated by an angle no larger than $2 \sin^{-1}(1/4)$. Of course, (iv) and (v) also imply that for some j, $v(P_j) = 0$. Thus the conjunction of (iv) and (v) leads to a contradiction.

We know of only two authors who have noticed any connection between Bell's proof and that of Kochen and Specker. Bellinfante (Ref. 1, Appendix C, Part I) notes that Bell's argument establishes Kochen and Specker's main result. Bub (Ref. 3, pp. 69–71) writes that Bell's argument can be adapted to establish a result weaker than that of Kochen and Specker, and a few words on Bub's remarks will help to make clear the closeness of these two arguments. First of all, Bub correctly remarks that Bell presents his argument in terms of expectation values for dispersion-free states. But this is literally just another name for a valuation function. Second, Bub claims that Kochen and Specker do, while Bell does not, prove the impossibility of a valuation function satisfying (iv) and (v) which breaks down on a finite set of subspaces in H_3. This is just mistaken. Both proofs work by showing that all projectors corresponding to subspaces in any cone of a fixed angle must receive the same value, and given the fixed angle of the cones, the number of projectors needed in both proofs is finite.

Since, as Kochen and Specker recognize (see Ref. 5, *Remark*, p. 70), the primary advantage of their complex argument over a simple application of Gleason's theorem lies in this finitization, an expansion of the preceding remark for Bell's work seems in order. It is most convenient to recast Bell's (iv) and (v) for a three-dimensional subspace of H, and indeed to confine attention to a unit sphere. In this setting Bell's geometrical argument establishes the following result.

Corresponding to each pair of points S, T on the sphere there are six other points on the sphere such that if all eight points are assigned 0's or 1's and if S gets the value 1, then so does T, provided the angle between S and T is no larger than $2 \sin^{-1}(1/4)$. The problem of finitization is to show that there is some finite set of points on the sphere to which it is impossible to assign 0's and 1's in satisfaction of Bell's (iv) and (v). Using the result just cited, such a finite set can be constructed as follows. Start with an orthogonal triple, i, j, k. Interpolate three points between i and j, so that the angular distance between any two neighboring points is less than $2 \sin^{-1}(1/4)$. Similarly, interpolate three such points between j and k. Now for each pair of neighboring points add the six points required by the result of Bell's geometrical argument. All these points taken together constitute the desired finite set. For, suppose that there were an assignment of 0's and 1's to these points, satisfying Bell's (iv) and (v). In a three-dimensional space (iv) and (v) imply that one of i, j, and k gets the value 1, and the other two get the value 0. Without loss of generality, suppose that i gets the value 1 and j the value 0. Bell's geometrical argument now applies successively to show that each point interpolated between i and j, and finally j itself, gets the value 1. But j had the value 0, which is the desired contradiction.

A final comment on the connection between the proofs of Bell and Kochen and Specker concerns the differences in their assumptions (not discussed by Bub). Assumptions (iv) and (v) follow so immediately from (ii′) that they deserve to be called special cases of Kochen and Specker's much more general condition: (iv) follows from (ii′) by taking g to be the squaring function; (v) follows from (ii′) because the sum of commuting observables can be written as a sum of functions of one operator and thus as a function of one operator.

3. In closing, we would like to provide two extensions of Bell's result. Bell's conditions (iv) and (v) are extended from projectors to general observables:

(iv′) $v(A)$ is an eigenvalue of A (*spectrum rule*); and

(v′) $v(\sum A_i) = \sum v(A_i)$, for commuting A_i (*sum rule*).

(Bell's proof uses these assumptions restricted to projectors onto one-dimensional subspaces of H.) If H is infinite dimensional, then of course the sum in (iv′) may be an infinite sum. We want to show that in the presence of the spectrum rule (iv′), the full functional relation rule (ii′) follows merely from the finite counterpart of (v′) alone, i.e., from:

(vi) $v(A + B) = v(A) + v(B)$, for commuting A and B (*finite sum rule*). Throughout we consider only observables with a discrete spectrum and assume the existence of the valuation function v.

Proposition 2. The spectrum and finite sum rules [i.e., (iv′) and (vi)] imply the functional relation rule (ii′).

We shall first establish the following result.

Lemma. The spectrum and finite sum rules imply

(iii′) $v(\chi_{\mathcal{O}}(A)) = 1$ iff $v(A) \in \mathcal{O}$

Proof. Since the spectrum rule implies that 0 and 1 are the only values possible for a projector, we can suppose first that $v(\chi_{\mathcal{O}}(A)) = 0$. Let a be any point of $\mathcal{O}$. Then since $\chi_{\mathcal{O}}(A) = \chi_{\mathcal{O}-\{a\}}(A) + \chi_a(A)$, the spectrum and finite sum rules yield that $v(\chi_a(A)) = 0$. Using that sum rule again, it follows that $v(A) = v(A) + v(\chi_a(A)) = v(A + \chi_a(A))$. Since $a \in \mathcal{O}$ is not an eigenvalue of $(A + \chi_a(A))$, the spectrum rule implies that $v(A) \neq a$. So, if $v[\chi_{\mathcal{O}}(A)] = 0$, then $v(A) \notin \mathcal{Cl}$. Suppose, second, that $v[\chi_{\mathcal{Cl}}(A)] = 1$. Then, where I is the identity, the spectrum and finite sum rules imply that $1 = v(I) = v[\chi_{\mathcal{Cl}}(A) + I - \chi_{\mathcal{Cl}}(A)] = v[\chi_{\mathcal{Cl}}(A)] + v[I - \chi_{\mathcal{Cl}}(A)]$. Hence $v[I - \chi_{\mathcal{Cl}}(A)] = 0$. But $(I - \chi_{\mathcal{Cl}}(A)) = \chi_{\bar{\mathcal{Cl}}}(A)$. Repeating the argument of the first part, it follows that $v(A) \notin \bar{\mathcal{Cl}}$; i.e., that $v(A) \in \mathcal{Cl}$.

Proposition 2 now follows immediately from the preceding Lemma and Proposition 1 (of Section 1). As remarked above, the functional relation rule (ii′) implies the infinite sum rule. Hence Proposition 2 has as a consequence:

Proposition 3. In the presence of the spectrum rule, the finite sum rule implies the infinite sum rule.

Since Bell's result shows that the infinite sum rule is inconsistent with the spectrum rule, Proposition 3 thereby establishes:

Proposition 4. There is no valuation function satisfying the spectrum and finite sum rules. (The reader is referred to Ref. 4, where these questions were first raised, and to Ref. 6, where the sum rule is discussed from an experimental point of view.)

In certain respects these propositions clearly strengthen Bell's results. Moreover, if the Hilbert space is infinite dimensional, Bell's proof actually uses the infinite sum rule. But by Proposition 3, the proof need only use the finite sum rule. In other respects, however, appeal to these propositions weakens the result, because the sum rule used there must be assumed to hold for observables other than projectors, while Bell's proof requires a sum rule only assumed to hold for projectors. Proposition 2 has further interest, nevertheless, since we need it to establish a second result: Just as there can be no valuation function satisfying the spectrum rule and sum rules (v) or (vi), there can be no valuation function satisfying the spectrum rule and the following *product rule*:

(vii) $v(AB) = v(A)\, v(B)$, for all commuting A and B.

Proposition 5. There is no valuation function satisfying the spectrum rule (iv) and the product rule (vii).

Proof. Suppose there were some valuation function v satisfying the spectrum and product rules. Define a function v' from observables to real numbers as follows:

$$v'(A) = \log_2[v(2^A)]$$

Since $v(2^A)$ is assumed to be in the spectrum of 2^A, which is nonnegative, v' is well defined and always $v'(A)$ will be in the spectrum of A. Moreover, a straightforward calculation shows that the product rule for v yields

$v'(A + B) = v'(A) + v'(B)$ for commuting observables A and B. Thus v' is a valuation function satisfying the spectrum and finite sum rules. By Proposition 5, there is no such function. Hence there is no v satisfying the spectrum and product rules.

ACKNOWLEDGMENTS

We thank Sharon Fine for help with some calculations and Dana Fine for help with the notation.

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