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When Can a Classical Electron Accelerate Without Radiating?

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A classical point electron radiates when it accelerates. However, there are classical electron models with extended charge distributions which can accelerate and/or deform without radiating. Can a model be contrived that will undergo radiationless motion while accelerating (on the average) over a distance large compared to its size? The answer is no: we prove that the "center" of the electron is always closer than the electron "diameter" to a fictitious point undergoing constant-velocity motion, if the electron's motion is radiationless.

1. INTRODUCTION

A point charge must radiate if it accelerates, but the same is not true of an extended charge distribution. A uniformly charged spherical shell of radius b can move in *arbitrary* periodic motion of period $2b/c$ without radiating,[1,2] and there are other examples as well.[3],2 Although the individual pieces of charge comprising such models accelerate and emit radiation fields, their superposed radiation fields cancel in all directions.

In the known examples of radiationless motion, the "center" of the electron model oscillates about constant-velocity motion over a distance only of the order of the size of the electron. This leads one to ask whether this must always be the case, or whether the charge distribution of an electron model can so deform while it moves that the "center" of the electron can accelerate over larger distances without the electron as a whole radiating. For example, might the classical nuclear atom be stable against radiative

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[2] Goedecke[3] has discussed rigid, nonradiating charge distributions. Examples of nonrigid, nonradiating charge distributions can be constructed by superposing Schott's spherical shells, each shell of different radius vibrating with its own period.

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decay when the orbiting electrons are extended, deformable charge distributions?

Alas, no: we prove here that the motion of the known examples is typical of the general case. For an electron model composed of charge of one sign, for which $\mathbf{j} = \rho \mathbf{v}$ ($|\mathbf{v}|/c < 1$), we show that for radiationless motion, the "center" of the electron (suitably defined) strays no further than $\sqrt{2} \times$ the electron "radius" (suitably defined) from a fictitious point undergoing constant-velocity motion.

In Section 2, beginning briefly with an argument found elsewhere,[3,4] we develop a condition for a charge distribution not to radiate in the direction of the unit vector $\hat{\mathbf{x}}$. This condition takes the form of a dynamical equation that a world-line lying wholly within the electron's world-tube must satisfy. Then we show in Section 3 (for two space dimensions) and Section 4 (for three space dimensions) that if there is to be no radiation in any direction, the electron's world-tube in Minkowski space, if enlarged in radius by a factor of $\sqrt{2}$, is required to contain a *straight* timelike world-line.

2. CONDITION FOR NO RADIATION

The asymptotic vector potential $A^\mu(x)$ in the Lorentz gauge, due to the current $j^\mu(x)$, is

$$A^\mu(x) \underset{r\to\infty}{\longrightarrow} r^{-1} \int d^4x'\, j^\mu(x')\, \delta(r - t - \tilde{x} \cdot x') \tag{1}$$

where $\tilde{x} \equiv (1, \mathbf{x}/r)$, the scalar product $a \cdot b = -a^0 b^0 + \mathbf{a} \cdot \mathbf{b}$, $c \equiv 1$, and in this section, Greek indices[3] run from 0 to 3, Latin indices from 1 to 3.[5] It follows that asymptotically

$$\partial_\mu A^\nu(x) \to r^{-1} \tilde{x}_\mu L^\nu(x) \tag{2a}$$

$$F^{\mu\nu} \to r^{-1}(\tilde{x}^\mu L^\nu - \tilde{x}^\nu L^\mu) \tag{2b}$$

$$T^{\mu\nu} \to (4\pi r^2)^{-1}\, \tilde{x}^\mu \tilde{x}^\nu L \cdot L \tag{2c}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, $T^{\mu\nu}$ is the energy–momentum–stress density tensor, and L^μ is defined as

$$L^\mu(x) \equiv \int d^4x'\, j^\mu(x')\, \delta'(r - t - \tilde{x} \cdot x') \tag{3}$$

[3] The whole discussion presented here refers to one reference frame: a manifestly Lorentz-invariant discussion is more complicated, and introduces no essential advantages. Four-vectors in this paper such as x^μ, Z^μ are not form-invariant under Lorentz transformations.

If there is to be no radiation in the $\hat{\mathbf{x}}$ direction, it follows from Eq. (2c) that $L \cdot L = 0$. The Lorentz gauge condition $\partial_\mu A^\mu = 0$ implies that $\tilde{x} \cdot L = 0$, according to Eq. (2a). Since $\tilde{x} \cdot \tilde{x} = 0$ (by definition), and since two orthogonal lightlike vectors are either parallel or one of them vanishes, a necessary (and easily seen to be sufficient) condition for no radiation in the $\hat{\mathbf{x}}$ direction is

$$L^\mu(x) = \tilde{x}^\mu f(r - t, \hat{\mathbf{x}}) \tag{4}$$

where f is an unspecified function of its arguments (which may vanish). Up to this point we have followed a previously published argument.[4]

Upon setting $\mu = 0$ in (4), we identity f with L^0. The nontrivial 3-vector part of (4) may be written, using (2b), as

$$rF^{0i} \to L^i - \hat{x}^i L^0 = 0 \tag{5}$$

In other words, the condition for radiationless motion is that *the asymptotic radiation electric field vanishes*. $\mathbf{L}$ is the contribution of the vector potential to the asymptotic electric field, while $\hat{\mathbf{x}}L^0$ is the contribution of the scalar potential.

We shall now show that this condition for radiationless motion can be written in a different form, as an equation of motion for a world-line z^μ that lies wholly within the world-tube traced out by the electron's charge distribution. For this purpose, it is convenient to introduce the Hertz potential[5] $\boldsymbol{\pi}(\mathbf{x}, t)$ and the polarization vector $\mathbf{p}(\mathbf{x}, t)$. The polarization vector is related to the charge and current density by

$$\rho = -\nabla \cdot \mathbf{p} \tag{6a}$$

$$\mathbf{j} = \partial \mathbf{p}/\partial t \tag{6b}$$

while the asymptotic Hertz potential is related to the polarization vector by

$$\boldsymbol{\pi}(\mathbf{x}, t) \to r^{-1} \int d^4x' \, \mathbf{p}(x') \, \delta(r - t - \tilde{x} \cdot x') \tag{7}$$

As is well known, the fields are easily expressed in terms of the Hertz potential:

$$\phi = -\nabla \cdot \boldsymbol{\pi} \tag{8a}$$

$$\mathbf{A} = (\partial/\partial t)\boldsymbol{\pi} \tag{8b}$$

$$\mathbf{B} = (\partial/\partial t)\nabla \times \boldsymbol{\pi}, \tag{8c}$$

$$\mathbf{E} = \nabla \times (\nabla \times \boldsymbol{\pi}) \tag{8d}$$

We may express the asymptotic electric field in terms of π using Eq. (8d). Alternatively, we can write $\mathbf{L}$ using (3) and (6b) as

$$\mathbf{L} = -\frac{\partial}{\partial t}\int d^4x' \left[\frac{\partial}{\partial t'}\mathbf{p}(x')\right]\delta(r - t - \tilde{x}\cdot x') = -\frac{\partial^2}{\partial t^2} r\pi \tag{9}$$

the last step following by integration by parts with respect to t'. Since $L^0 = \hat{\mathbf{x}}\cdot\mathbf{L}$, the vanishing of the asymptotic electric field (5) may be written

$$rF^{i0} \to -\sum_{k=1}^{3}(\delta^{ik} - \hat{x}^i\hat{x}^k)\frac{\partial^2}{\partial t^2} r\pi^k = 0 \tag{10}$$

Having expressed the field in terms of the potential, we proceed to express the potential in terms of the sources ρ, $\mathbf{j}$. Using Eqs. (6), the following identity is readily proven:

$$\begin{aligned}\frac{\partial}{\partial x^{i'}} x^{j'}p^i(x')\,\delta(r - t - \tilde{x}\cdot x') + \frac{\partial}{\partial t'} x^{j'}\hat{\mathbf{x}}\cdot\mathbf{p}(x')\,\delta(r - t - \tilde{x}\cdot x')\\ = \{p^j(x') - x^{j'}[\rho(x') - \hat{\mathbf{x}}\cdot\mathbf{j}(x')]\}\,\delta(r - t - \tilde{x}\cdot x')\end{aligned} \tag{11}$$

Upon multiplying Eq. (11) by d^4x' and integrating, using Gauss' law and Eq. (7), we find

$$r\pi = \int d^4x'\,\mathbf{x}'[\rho(x') - \hat{\mathbf{x}}\cdot\mathbf{j}(x')]\,\delta(r - t - \tilde{x}\cdot x') \tag{12}$$

We want to interpret the integrand of Eq. (12) in a particular way. Let us define an *effective charge density* (for fixed $\hat{\mathbf{x}}$)

$$\rho_{\text{eff}}(\hat{\mathbf{x}}, x') \equiv \rho(x') - \hat{\mathbf{x}}\cdot\mathbf{j}(x') \tag{13}$$

In what follows, we shall restrict consideration to models which are composed of charge of one sign—for definiteness, let us say $\rho \geqslant 0$. We shall also restrict consideration to models for which $|\mathbf{j}| \leqslant \rho$, which includes the important case $\mathbf{j} = \rho\mathbf{v}$, so

$$\rho_{\text{eff}}(\hat{\mathbf{x}}, x') \geqslant 0 \tag{14}$$

There is a charge conservation law associated with ρ_{eff}. To find it most conveniently, let us rewrite Eq. (5) as

$$rF^{0i} \to \frac{\partial}{\partial t}\int d^4x'\,\{\hat{x}^i\rho - j^i\}\,\delta(r - t - \tilde{x}\cdot x') \tag{15}$$

using the definition of L^μ in (3). This asymptotic electric field is transverse, as one might expect: this can be shown by taking the scalar product of the alternative expression (10) with $\hat{\mathbf{x}}$. Thus the scalar product of (15) with $\hat{\mathbf{x}}$ yields identically zero:

$$\frac{\partial}{\partial t}\int d^4x'\ \rho_{\text{eff}}(\hat{\mathbf{x}}, x')\ \delta(r - t - \tilde{x}\cdot x') = 0 \tag{16}$$

Equation (16) says that the total effective charge, located where the plane $r - t + t' - \hat{\mathbf{x}}\cdot\mathbf{x}'$ intersects the electron's world-tube, is conserved as t changes: $\partial Q_{\text{eff}}/\partial t = 0$. This is, to be sure, an unusual charge conservation law, since the charge is not evaluated at a constant time, nor is t the time coordinate of any element of the charge: t is simply a parameter which determines where the plane (which is the asymptotic approximation to the light-cone surface whose apex is at r, t) slices the world-tube.

Now comes the main point of this section. We are going to look at $r\boldsymbol{\pi}/Q_{\text{eff}}$ as the spatial coordinates of a world-line (parametrized by t, for fixed $\hat{\mathbf{x}}$), rather than as the asymptotic (normalized) Hertz potential. We define

$$\mathbf{z}(t - r, \hat{\mathbf{x}}) \equiv \int d^4x'\ \mathbf{x}'\rho_{\text{eff}}(\hat{\mathbf{x}}, x')\ \delta(r - t - \tilde{x}\cdot x')\Big/\int d^4x'\ \rho_{\text{eff}}\delta \tag{17}$$

If there is no radiation in the $\hat{\mathbf{x}}$ direction, $\mathbf{z}$ obeys the equation of motion

$$\sum_{k=1}^{3}(\delta^{ik} - \hat{x}^i\hat{x}^k)\frac{\partial^2}{\partial t^2}z^k = 0 \tag{18}$$

which follows from Eqs. (10) and (16). To complement Eq. (17) we define

$$z^0(t - r, \hat{\mathbf{x}}) \equiv \int d^4x'\ t'\rho_{\text{eff}}(\hat{\mathbf{x}}, x')\ \delta(r - t - \tilde{x}\cdot x')\Big/\int d^4x'\ \rho_{\text{eff}}\delta \tag{19}$$

Using the identity

$$\begin{aligned} 0 &= \int d^4x'\ (r - t - \tilde{x}\cdot x')\ \rho_{\text{eff}}(\hat{\mathbf{x}}, x')\ \delta(r - t - \tilde{x}\cdot x') \\ &= Q_{\text{eff}}(r - t + z^0 - \hat{\mathbf{x}}\cdot\mathbf{z}) \end{aligned} \tag{20}$$

we establish that

$$z^0 = \hat{\mathbf{x}}\cdot\mathbf{z} + t - r \tag{21}$$

so the motion of z^0 (its t dependence) can be found from that of $\mathbf{z}$.

To summarize: if there is to be no radiation in the $\hat{\mathbf{x}}$ direction, the "four-vector" z^μ (see footnote 3) must trace out a world-line in Minkowski space which obeys the dynamical equations (18) and (21). Moreover, according to the definition (17), (19) of z^μ, and because ρ_{eff} is nonnegative and $Q_{\rm eff}$ is conserved, z^μ *is the "center of charge" of* $\rho_{\rm eff}$ along the hyperplanes $r - t - \tilde{x} \cdot x'$.

We shall need to be able to discuss the "center" of the electron and its "radius." We may assume that in each constant-time (not constant-t) hyperplane there is a sphere which completely encloses the electron's charge and current distribution. The world-tube of the electron is hereafter defined as the volume in Minkowski space occupied by these spheres. For simplicity, we can take all spheres to have the same radius b, and require the center of the spheres to trace out a continuous timelike world-line.

Now, because of the convex nature of the electron's world-tube, which totally encloses the effective charge density (13), and because z^μ is a "center of charge," $z^\mu(t - r, \hat{\mathbf{x}})$ *traces out a world-line that lies wholly within the electron's world-tube*, for any fixed $\hat{\mathbf{x}}$. This fact, together with the dynamical equations (18) and (21), is all we shall need in what follows.

3. ARGUMENT IN TWO SPACE DIMENSIONS

If Eq. (18) was simply

$$\partial^2 \mathbf{z} / \partial t^2 = 0 \tag{22}$$

it would follow that

$$\mathbf{z}(t - r, \hat{\mathbf{x}}) = \mathbf{b}(\hat{\mathbf{x}})(t - r) + \mathbf{c}(\hat{\mathbf{x}}) \tag{23}$$

which, together with Eq. (21), implies that z^μ is a straight world-line. Since z^μ must lie inside the electron's world-tube, we would have proven that the center of the electron's world-tube lies no farther than the radius b from a straight world-line, if there is to be no radiation in merely one direction $\hat{\mathbf{x}}$.

Unfortunately, Eq. (18) is more complicated than Eq. (22). Its solution is

$$\mathbf{z}(t - r, \hat{\mathbf{x}}) = \hat{\mathbf{x}} a(t - r, \hat{\mathbf{x}}) + \mathbf{b}(\hat{\mathbf{x}})(t - r) + \mathbf{c}(\hat{\mathbf{x}}) \tag{24}$$

where a is an *arbitrary* function of its arguments, and $\mathbf{b}$ and $\mathbf{c}$ are arbitrary vectors (which can be taken orthogonal to $\hat{\mathbf{x}}$ with no loss of generality). Then, according to Eq. (21),

$$z^0 = a + (t - r) \tag{25}$$

Equations (24) and (25) say that all we can be sure of is that the world-line z^μ lies in a 2-surface (plane) obtained by letting the parameters a and $t - r$ take on all possible values. In order to fully draw the consequences of this, we shall have to consider various directions $\hat{\mathbf{x}}$, and go through a fairly intricate geometrical argument. We therefore believe it is appropriate to present the argument in two space dimensions (i.e., in three-dimensional space time) first, where it may be illustrated and more easily visualized.

Consider application of Eq. (24) to two different directions $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$, in each of which there is no radiation. Suppose we are able to prove that we can choose the two 2-planes containing the world-lines $z_1{}^\mu \equiv z^\mu(t - r, \hat{\mathbf{x}}_1)$ and $z_2{}^\mu \equiv z^\mu(t - r, \hat{\mathbf{x}}_2)$ to be orthogonal (see Fig. 1). Then we can show, at any time T, that the *intersection of the two planes* (a straight timelike line) is no farther than $\sqrt{2}\, b$ from the center of the electron, by the following argument.

Slice the world-tube by a $z^0 = T = \text{const}$ plane. This plane cuts the two 2-planes in two perpendicular lines (see Fig. 2). The point $\mathbf{z}_1$ lies some-

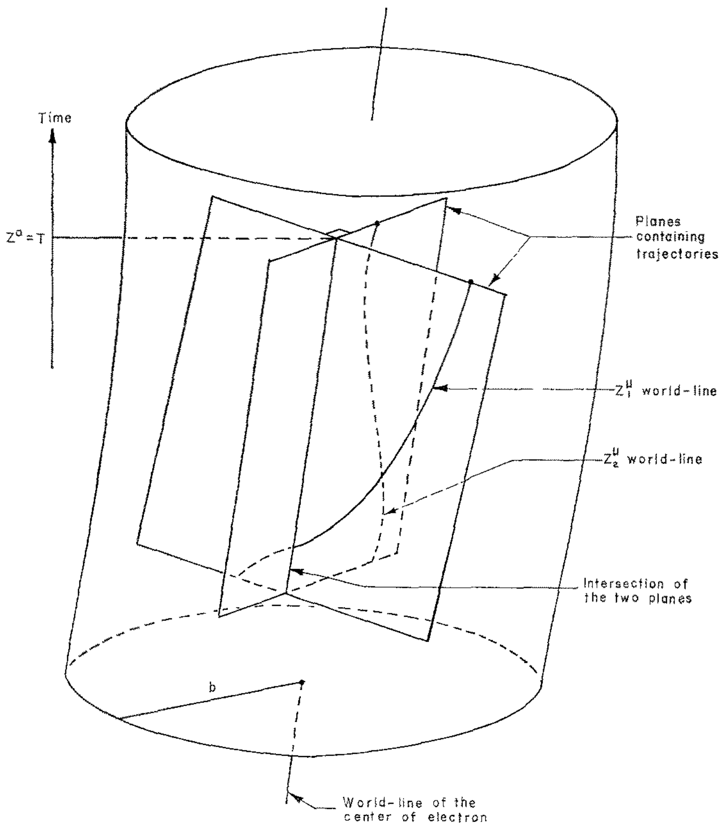

Fig. 1. Illustration of the electron's world-tube (radius b), and two orthogonal planes in which the world-lines $z_1{}^\mu$, $z_2{}^\mu$ lie. We prove that at any time T, the intersection of the two planes (a straight, timelike line) is no farther than $\sqrt{2}\, b$ from the center of the electron.

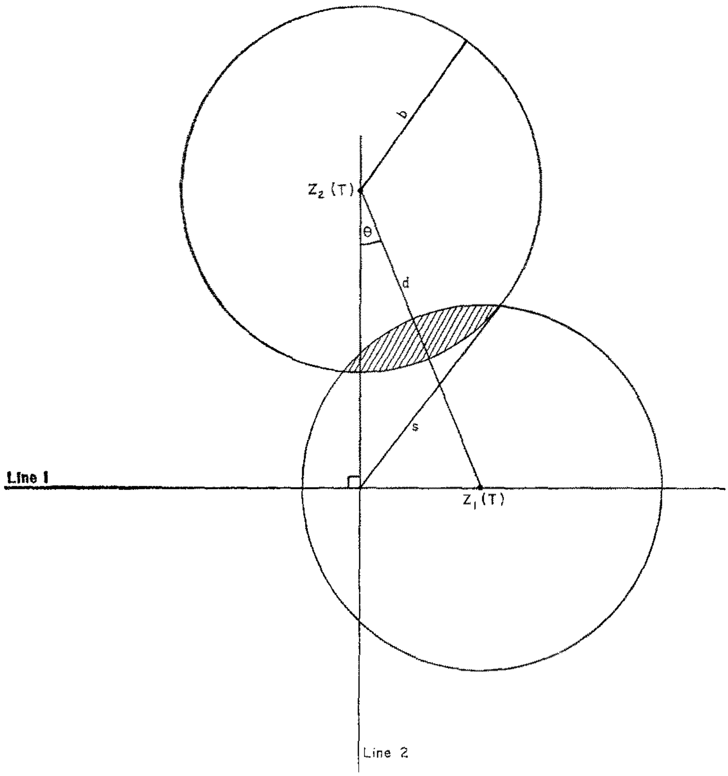

Fig. 2. A $z^0 = T$ plane. The intersections of the world-lines z_1, z_2 with this plane lie on lines 1 and 2 respectively, and are a distance $d < 2b$ apart. The center of the electron lies somewhere in the region of intersection of the two circles. The distance s between the intersection of lines 1 and 2 and the farthest possible location of the center of the electron satisfies $s \leqslant \sqrt{2}\, b$.

where on line 1, and the point $\mathbf{z}_2$ lies somewhere on line 2, but the distance d between these points must be less than $2b$. This is because both points lie inside the electron world-tube, which cuts the $z^0 = T$ plane in a circle of diameter $2b$. Indeed, a circle of radius b surrounding $\mathbf{z}_1$ or $\mathbf{z}_2$ must contain the center of the electron, so the electron's center lies in the region of intersection of these two circles.

Denote the distance from the intersection of lines 1 and 2 to the furthest point where the electron's center could lie, by s. It is a matter of elementary trigonometry to show that

$$s^2 = b^2 + d[b^2 - (d/2)^2]^{1/2} \sin 2\theta \tag{26}$$

where θ is the angle between the line connecting $\mathbf{z}_1$ and $\mathbf{z}_2$ and either line 1 or line 2. The distance s is maximum for $\theta = \pi/4$ and $d = \sqrt{2}\, b$, from which we find that

$$\max[s] = \sqrt{2}\, b \tag{27}$$

Thus the electron's center can lie no farther than $\sqrt{2}\, b$ from the intersection of lines 1 and 2. But this intersection is a point on the straight timelike line which is the intersection of the two orthogonal planes. We may regard this timelike line *as the constant-velocity motion of a fictitious point*, and express our conclusion by saying that the electron's center is always closer than $\sqrt{2}\, b$ to a point that undergoes constant-velocity motion. This concludes the argument.

Therefore, in order to complete the proof, it only remains for us to show that directions $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ can be chosen so that the two 2-planes are orthogonal.

We begin by eliminating a from Eqs. (24) and (25), and replacing the parameter $t - r$ by τ to obtain

$$\mathbf{z}(\tau, \hat{\mathbf{x}}) = \tau[\mathbf{b}(\hat{\mathbf{x}}) - \hat{\mathbf{x}}] + \hat{\mathbf{x}} z_0 + \mathbf{c}(\hat{\mathbf{x}}) \tag{28}$$

As τ is permitted to vary for fixed $z^0 = T$, Eq. (28) traces out the straight line in the $z^0 = T$ plane, somewhere along which the actual world-line point z^μ must lie. Consider two such straight lines associated with two arbitrary directions $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$:

$$\mathbf{z}_i = \tau_i(\mathbf{b}_i - \hat{\mathbf{x}}_i) + \hat{\mathbf{x}}_i T + \mathbf{c}_i\,, \qquad i = 1, 2 \tag{29}$$

The intersection of these two lines is to be found by solving $\mathbf{z}_1 = \mathbf{z}_2$,

$$(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)T + \mathbf{c}_1 - \mathbf{c}_2 = \tau_1(\hat{\mathbf{x}}_1 - \mathbf{b}_1) - \tau_2(\hat{\mathbf{x}}_2 - \mathbf{b}_2) \tag{30}$$

for τ_1 and τ_2, and substituting either value back into Eq. (29). The solution is facilitated by defining the projection operator $\mathbf{P}(\mathbf{v}_1, \mathbf{v}_2)$ on $\mathbf{v}_1$ in the skew coordinate system formed from $\mathbf{v}_1$, $\mathbf{v}_2$:

$$\mathbf{P}(\mathbf{v}_1, \mathbf{v}_2) \equiv |\mathbf{v}_1 \times \mathbf{v}_2|^{-2}\,(v_2{}^2\mathbf{v}_1 - \mathbf{v}_1 \cdot \mathbf{v}_2\mathbf{v}_2) \tag{31}$$

$\mathbf{P}_{12} \equiv \mathbf{P}(\mathbf{v}_1, \mathbf{v}_2)$ has the properties

$$\mathbf{P}_{12} \cdot \mathbf{v}_1 = 1, \qquad \mathbf{P}_{12} \cdot \mathbf{v}_2 = 0 \tag{32}$$

If we set $\mathbf{v}_1 \equiv \hat{\mathbf{x}}_1 - \mathbf{b}_1$, $\mathbf{v}_2 \equiv \hat{\mathbf{x}}_2 - \mathbf{b}_2$, and operate on Eq. (30) with $\mathbf{P}_{12}$, we obtain

$$\tau_1 = \mathbf{P}_{12} \cdot [(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)T + \mathbf{c}_1 - \mathbf{c}_2] \tag{33}$$

which, when inserted into Eq. (29) gives us the intersection point

$$\mathbf{Z} = [\hat{\mathbf{x}}_1 - \mathbf{v}_1\mathbf{P}_{12}\cdot(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)]T + \mathbf{c}_1 - \mathbf{v}_1\mathbf{P}_{12}\cdot(\mathbf{c}_1 - \mathbf{c}_2) \tag{34}$$

We see from Eq. (34) that *the intersection point* $Z^\mu \equiv (T, \mathbf{Z}(T))$ *traces out a straight line* whose tangent vector is $V^\mu \equiv \{1, \mathbf{V}\}$, where

$$\mathbf{V} \equiv d\mathbf{Z}/dT = \hat{\mathbf{x}}_1 - \mathbf{v}_1\lambda_1 \tag{35}$$

[we have written $\lambda_1 = \mathbf{P}_{12}\cdot(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)$]. Now, we can prove that $|\mathbf{V}| < 1$, and $\mathbf{V}$ *must be independent of the directions* $\hat{\mathbf{x}}_1$ *and* $\hat{\mathbf{x}}_2$ that were used to derive Eq. (35)!

For any nonzero angle between the two intersecting lines, the center of the electron must remain within a determined (though possibly large) distance from the intersection point $\mathbf{Z}$ (the argument is similar to that already gone through when the angle is $\pi/2$, as in Fig. 2). Since the center of the electron moves on a timelike trajectory, Z^μ must be a timelike trajectory, or else the distance between the two trajectories will get arbitrarily large as $T \to \infty$. Therefore, V^μ is a timelike vector, i.e.,

$$|\mathbf{V}| < 1 \tag{36}$$

Moreover, if the tangent V^μ calculated with vectors $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ is different from the tangent calculated with vectors $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_4$ or $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$, etc., the corresponding world-lines of intersection Z^μ will separate by an arbitrarily large distance as $T \to \infty$. But this is impossible, as each $\mathbf{Z}$ must remain within a finite distance of the center of the electron. Therefore the slope $(1, \mathbf{V})$ is independent of all directions.

We note that λ_1 in Eq. (35) cannot vanish, else $|\mathbf{V}| = 1$, which contradicts (36). [We also note in passing that λ_1 must be independent of $\hat{\mathbf{x}}_2$, although we shall not need this result.]

Finally, we are in a position to prove that the two intersecting lines $\mathbf{z}_1$, $\mathbf{z}_2$ can be made perpendicular by appropriate choice of $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$. We see from Eq. (29) that the line $\mathbf{z}_i$ is parallel to the vector $\mathbf{v}_i = \mathbf{b}_i - \hat{\mathbf{x}}_i$. Accordingly, the condition $\mathbf{z}_1 \cdot \mathbf{z}_2 = 0$ is equivalent to [using Eq. (35)]

$$0 = \mathbf{v}_1 \cdot \mathbf{v}_2 = \lambda_1^{-1}\lambda_2^{-1}(\hat{\mathbf{x}}_1 - \mathbf{V})\cdot(\hat{\mathbf{x}}_2 - \mathbf{V}) \tag{37}$$

If $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$ can be chosen to satisfy Eq. (37), the intersecting lines can be made orthogonal. For simplicity, let us choose $\hat{\mathbf{x}}_1$ parallel to $\mathbf{V}$. (If $\mathbf{V}$ vanishes, the result (38) still obtains.) Then the condition (37) implies

$$\pm|\mathbf{V}| = \hat{\mathbf{x}}_1 \cdot \hat{\mathbf{x}}_2 \tag{38}$$

A direction $\hat{\mathbf{x}}_2$ can always be chosen to satisfy Eq. (38) because of the inequality (36), and our proof is complete.

4. ARGUMENT IN THREE SPACE DIMENSIONS

In three space dimensions, the following complication is added: at any time T we are still sure that two trajectories $\mathbf{z}_1$, $\mathbf{z}_2$ must lie along two straight lines, but the two straight lines do not have to intersect (as they must in two space dimensions): they are generally skew.

Nonetheless, the argument is similar to that given in the previous section. Suppose: (1) we can choose the two 2-planes containing the world-lines $z_1{}^\mu$ and $z_2{}^\mu$ so that the intersection of the two planes with the $z^0 = T$ hyperplane are two (generally skew) *perpendicular* lines (see Fig. 3). Consider the *point of closest approach* $\mathbf{Z}$ on, say, line 1 to line 2, and further suppose: (2) that we can show that $Z^\mu \equiv (T, \mathbf{Z}(T))$ is a straight, timelike line. Then we can show that $\mathbf{Z}$ lies no further than $\sqrt{2}\, b$ from the center of the electron, by the argument given in the next two paragraphs, and our result is proved.

First, suppose that the two orthogonal straight lines 1 and 2 do happen to intersect. Then Fig. 2 again illustrates the situation, *except* that the center

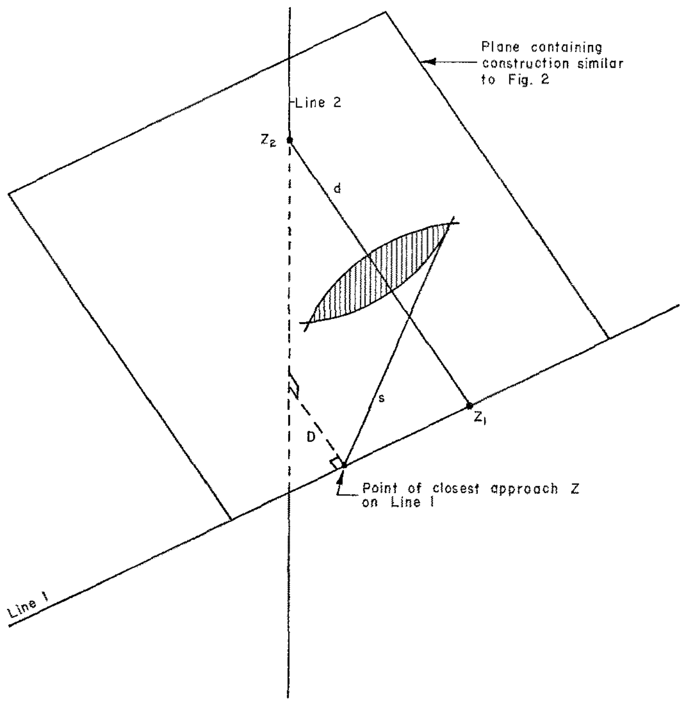

Fig. 3. A $z^0 = T$ hyperplane. The intersections of the world-lines z_1, z_2 with this hyperplane lie on lines 1 and 2, respectively, and are a distance $d < 2b$ apart. Lines 1 and 2 are orthogonal skewlines, separated by a distance D.

of the electron is known to lie in a *spherical* volume of radius b surrounding the trajectory point $\mathbf{z}_1(T)$ [or $\mathbf{z}_2(T)$]. Therefore, the center of the electron must lie in the volume that is the intersection of the two spheres surrounding $\mathbf{z}_1$ and $\mathbf{z}_2$. However, the point in this volume that is farthest from $\mathbf{Z}$, the intersection point of lines 1 and 2, does in fact lie in the plane of lines 1 and 2: the maximum distance s between these two points has been shown to equal $\sqrt{2}\,b$ in the previous section.

Now suppose that the two orthogonal straight lines 1 and 2 do not intersect. As is illustrated in Fig. 3, one can draw a plane containing the point of closest approach $\mathbf{Z}$ on line 1 to line 2, and the two world-line points $\mathbf{z}_1$ and $\mathbf{z}_2$. Then the construction of the maximum possible distance s between $\mathbf{Z}$ and the center of the electron takes place in this plane, and is identical to that discussed in the previous paragraph and illustrated in Fig. 2. Therefore $\mathbf{Z}$ lies within a distance $\sqrt{2}\,b$ of the center of the electron. This concludes the argument.

Therefore, in order to complete the proof, we must calculate Z^μ for two arbitrary directions $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, show that $Z^\mu(T)$ is a straight, timelike line, and show that we can choose $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$ so that the straight lines on which the respective world-line points $\mathbf{z}_1$, $\mathbf{z}_2$ lie are perpendicular.

We begin by finding the points of closest approach on the two straight lines (29), by minimizing the square of the distance vector $|\mathbf{z}_1 - \mathbf{z}_2|^2$ with respect to the parameters τ_1, τ_2:

$$(\partial/\partial\tau_i)\,|(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)T - \tau_1\mathbf{v}_1 + \tau_2\mathbf{v}_2 + \mathbf{c}_1 - \mathbf{c}_2|^2 = 0, \qquad i = 1, 2 \tag{39}$$

The resulting equations

$$\begin{aligned} \mathbf{v}_1 \cdot [(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)\,T + \mathbf{c}_1 - \mathbf{c}_2] &= \tau_1 v_1^2 - \tau_2 \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot [(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)\,T + \mathbf{c}_1 - \mathbf{c}_2] &= \tau_1 \mathbf{v}_1 \cdot \mathbf{v}_2 - \tau_2 v_2^2 \end{aligned} \tag{40}$$

are readily solved for τ_1, τ_2 if $|\mathbf{v}_1 \times \mathbf{v}_2| \neq 0$ (which we shall assume and justify below):

$$\tau_1 = \mathbf{P}_{12} \cdot [(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)\,T + \mathbf{c}_1 - \mathbf{c}_2] \tag{41a}$$

$$\tau_2 = \mathbf{P}_{21} \cdot [(\hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1)\,T + \mathbf{c}_2 - \mathbf{c}_1] \tag{41b}$$

Note that Eq. (41a) is identical to Eq. (33). Upon substituting (41a) into Eq. (29), we obtain Eq. (34) for $\mathbf{Z}$: in three dimensions, it is seen to be the expression for the point on line 1 closest to line 2.

From here, the proof proceeds as in Section 3. We note that $\mathbf{Z}$ depends linearly on T, so $Z^\mu \equiv (T, \mathbf{Z}(T))$ is a straight line. The arguments that Z^μ is timelike, that its slope $(1, \mathbf{V})$ [with $\mathbf{V}$ given by Eq. (35)] is universal, that $|\mathbf{V}| < 1$, and that $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$ can be chosen to satisfy Eq. (38) so that the two straight lines are orthogonal, all go through as before. As a final point, we note [by Eq. (37)] that since $\mathbf{v}_1$, $\mathbf{v}_2$ are the tangents to the orthogonal straight lines, then $|\mathbf{v}_1 \times \mathbf{v}_2| \neq 0$, which we needed to derive Eqs. (41) from Eqs. (40). Our proof is complete.

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