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Quantum Mechanics of Relativistic Spinless Particles

John R. Fanchi[1] and R. Eugene Collins[1]

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A relativistic one-particle, quantum theory for spin-zero particles is constructed upon $L^2(\mathbf{x}, ct)$, *resulting in a positive definite spacetime probability density. A generalized Schrödinger equation having a Hermitian Hamiltonian* H *on* $L^2(\mathbf{x}, ct)$ *for an arbitrary four-vector potential is derived. In this formalism the rest mass is an observable and a scalar particle is described by a wave packet that is a superposition of mass states. The requirements of macroscopic causality are shown to be satisfied by the most probable trajectory of a free tardyon and a nontrivial framework for charged* and *neutral particles is provided. The Klein paradox is resolved and a link to the free particle field operators of quantum field theory is established. A charged particle interacting with a static magnetic field is discussed as an example of the formalism.*

1. INTRODUCTION

At present a consistent relativistic one-particle quantum theory for spin-zero particles has been developed only for free particles[2] and it is generally believed that interactions, such as the electromagnetic interaction, are properly interpreted only within the context of many-body theories.[3] It is shown in this paper, however, that a consistent one-particle theory of relativistic spinless particles (RSP) in the presence of an arbitrary four-vector potential can be constructed using a formalism developed from ideas first introduced by Fock(3) and Stückelberg(4) over thirty years ago. More recently these ideas have been discussed by Nambu,(5) Schwinger,(6) Feynman,(7) and Cooke.(8) A logical framework for the ideas presented in Refs. 3–8 is developed in this paper. Furthermore, a number of physical

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[1] University of Houston, Houston, Texas.
[2] A statement to this effect is made explicitly in Ref. 1.
[3] This contention is prevalent in, for example, the widely used text by Bjorken and Drell.(2)

and mathematical results that have not been discussed previously are presented in detail.

The theory presented here has a number of distinguishing features. The four most important are the following: a relativistically covariant parameter τ; a positive-definite probability density based on the Born interpretation of the wave function; a scalar product defined in $L^2(\mathbf{x}, ct)$; and generators representing an 11-parameter continuous group.

In Section 2 the foundation of the theory proposed here—henceforth referred to as the four-space formulation (FSF)—will be laid. It parallels Collins'[9,10] treatment of NRQM[4] and places probabilistic concepts at the basis of relativistic quantum mechanics. Assumptions will be made which restrict the subsequent investigation to spinless particles undergoing only electromagnetic interactions. These restrictions make it possible to simplify the presentation while conveying the essential concepts of the FSF. It will be observed that the FSF can be extended to particles of nonzero spin and nonelectromagnetic interactions, but the details will not be developed in this paper. The primary goal of Section 2 is the derivation of the equation

$$i\hbar \frac{\partial \psi}{\partial \tau}(\mathbf{x}, ct, \tau) = H\psi(\mathbf{x}, ct, \tau) \tag{1}$$

with $\psi(\mathbf{x}, ct, \tau)$ an element of $L^2(\mathbf{x}, ct)$, H a Hermitian operator in this space, and τ an invariant parameter labeling the variation of dynamical variables in a manner similar to that described in Refs. 3–8. Note that Eq. (1) is postulated by Refs. 3–8 as the quantum mechanical extension of relativistic Hamilton–Jacobi mechanics, whereas here Eq. (1) is *derived* from probabilistic considerations. Also, a special case of Eq. (1) is the Klein–Gordon equation[11–13] as will be shown below.

The similarity of Eq. (1) to the Schrödinger equation guides the development of the FSF in Section 3. A classical correspondence for the relativistic theory is constructed in the Heisenberg representation by identifying H as a Hamiltonian and then proceeding as in NRQM. Next, the concept of superposition of mass states is introduced. This concept, analogous to the nonrelativistic concept of superposition of energy states, is examined further in Section 4.

Unlike Sections 2 and 3, which are concerned primarily with the mathematical foundation of the FSF, Sections 4 and 5 focus on the physical notions of the FSF.

The free particle is discussed at length in Section 4. This is so because the free particle can be used to vividly illustrate the concepts of macroscopic

[4] The techniques of nonrelativistic quantum mechanics (NRQM) are discussed in many well-known texts, e.g., Ref. 14.

causality and superposition of mass states. In order to appreciate this discussion, however, one first obtains, as a natural consequence of the FSF, the Stückelberg–Feynman[15,16] interpretation. It is then learned, as a result of combining the Stückelberg–Feynman interpretation with the concept of charge conjugation, that a particle and its antiparticle[5] propagate in opposite spatial *and* temporal directions for a given potential configuration. This interpretation, first suggested by Feshbach and Villars[18] for charged particles, is shown to be valid for *both* charged and neutral particles. The macroscopic causality principle is derived next and, lastly, the free particle example is completed.

The problem of RSP scattering from a step potential is examined in Section 5. It is shown that the reflection coefficient never exceeds unity, unlike conventional treatments,[19,20] which, for potentials of sufficient magnitude, yield reflection coefficients that are never less than unity (a result generally known as the Klein paradox). The reason for this difference is discussed, and it is observed that this example provides a means of experimentally testing the validity of the FSF.

By the end of Section 5 the FSF has been developed into a usable theory. The way in which the FSF relates to existing relativistic theories is examined in Sections 6 and 7 by considering two distinguishing mathematical features of the FSF: the group properties, and the scalar product.

It is observed that Eq. (1) for a free particle is invariant with respect to linear transformations having the form

$$x' \to \Lambda x + a \tag{2a}$$

$$\tau' \to \tau + \Delta\tau \tag{2b}$$

where Eq. (2a) represents an inhomogeneous Lorentz transformation[6,7] and Eq. (2b) corresponds to translations along the τ axis. It is shown in Section 6 that $\{a, \Delta\tau, \Lambda\}$ are elements of an 11-parameter continuous group, here called the FSF group. Then it is shown that the Poincaré group[24,25] is a subgroup of the FSF group corresponding to $\Delta\tau = 0$. Finally, it is shown that, in addition to the ten generators of the Poincaré group, there exists a Hermitian operator [H of Eq. (1)] which generates infinitesimal translations along the τ axis. Thus, all 11 generators of the FSF group are determined.

[5] The distinction between particle and antiparticle is clearly stated by Weinberg on p. 1321 of Ref. 17.

[6] The first in-depth investigation of the infinite-dimensional representations of the Lorentz group was made by Majorana.[21] Majorana's paper is reviewed by Fredkin.[22]

[7] The irreducible representations of the proper inhomogeneous Lorentz group have been determined by Wigner and Bargmann.[23]

Expectation values are examined in Section 7. It is observed that the expectation value defined within the FSF is a special case of the general scalar product defined by Wigner (Ref. 23a). Then it is shown that the expectation values defined within the FSF reduce, in the appropriate limit, to the expected nonrelativistic values. A link to quantum field theory[17] is then established by requiring that only a single mass state contributes to the wave function. Finally, microscopic causality is imposed, with the expected consequences that spinless particles must be bosons and have antiparticles.

The fundamentals of the FSF and the relationship of the FSF to existing theories is established by the end of Section 7. The physical significance of the FSF and its experimental interpretation is illustrated and clarified by analyzing a simple physical system in Section 8. In particular, an ensemble of charged particles moving in a magnetic field is described using the FSF.

The notation used below is that of Bjorken and Drell,[2] and the Einstein summation convention is imposed unless otherwise specified.

2. FOUR-SPACE FORMULATION

In order to treat space and time symmetrically the probability density ρ should represent a joint distribution in the space *and* time coordinates, though ρ may be conditioned by some invariant parameters. In particular, it is postulated that ρ is conditioned by an invariant parameter τ. Then the conditional probability density is written as $\rho(\mathbf{x}, ct \mid \tau)$, and the corresponding normalization condition is

$$\int_D \rho(\mathbf{x}, ct \mid \tau)\, d^4x = 1; \qquad d^4x = dx^0\, dx^1\, dx^2\, dx^3 \tag{3}$$

The quantity D is the domain of definition on which ρ may be nonzero, and $\rho(\mathbf{x}, ct \mid \tau)\, dx^0\, d^3x$ is the probability that the particle is at the world-point $(\mathbf{x}, ct)$ when the parameter has the value τ. Since a particle can occupy any world-point in spacetime, it follows that D must extend over *all* of space and time. This point and the physical interpretation of τ are discussed more fully in Ref. 26.

Preservation of the norm in four-space is assured by requiring that $\rho(\mathbf{x}, ct \mid \tau)$ vanishes as $\mid x^\mu \mid \to \infty$ and obeys the equation

$$\frac{\partial}{\partial \tau} \rho(\mathbf{x}, ct \mid \tau) + \frac{\partial}{\partial x^\mu} [\rho(\mathbf{x}, ct \mid \tau)\, V^\mu] = 0 \tag{4}$$

where V^μ is not yet defined. In this equation it is necessary that ρ and $\{V^\mu\}$ be single-valued and differentiable, and it is also *assumed* that the only nonzero elements of the metric are

$$g_{00} = 1 = -g_{11} = -g_{22} = -g_{33} \tag{5}$$

Rather than simply assuming Eq. (4), one could have required that $\rho\, d^4x$ be invariant with respect to a five-dimensional velocity field having as components $\{V^\mu\}$ and $V^4 = 1$, where a fifth coordinate x^4 is defined to be τ. The techniques of a procedure such as this are discussed by Kiehn,[27] and it can be shown that Eq. (4) is one of two conditions for the invariance of $\rho\, d^4x$ with respect to propagation down the trajectories of $(\{V^\mu\}, V^4 = 1)$. The other condition is either $\partial V^\mu/\partial\tau$ is zero for all values of μ or $d\tau$ is zero. Either approach, simply assuming Eq. (4) is valid or else requiring $\rho\, d^4x$ to be invariant with respect to $(\{V^\mu\}, V^4 = 1)$, can be used here.

The physical meaning of the velocity field $\{V^\mu\}$ is determined by examining the expectation value of the spacetime position vector of the particle,

$$\langle x^\mu \rangle = \int_D x^\mu \rho\, d^4x \tag{6}$$

Differentiating $\langle x^\mu \rangle$ with respect to τ, substituting Eq. (4) for $\partial\rho/\partial\tau$, and applying the divergence theorem with the boundary condition that ρ vanishes as $| x^\mu | \to \infty$, yields

$$d\langle x^\mu \rangle/d\tau = \int_D V^\mu \rho\, d^4x \tag{7}$$

In other words, the expectation value of V^μ is the derivative, with respect to τ, of the expectation value of the four-position vector. This fact, along with Eq. (4), motivates the characterization of the quantities τ and $d\langle x^\mu \rangle/d\tau$ as statistical analogs of the classical proper time and proper velocity, respectively. This characterization will be further justified below.

A Hilbert space formalism follows from this probabilistic description by first observing that, since ρ must be nonnegative and differentiable, all derivatives of ρ must vanish when ρ vanishes, otherwise ρ would be somewhere negative. The Born representation is an acceptable mathematical form for assuring this constraint; thus write

$$\rho(\mathbf{x}, ct \mid \tau) = \psi^*(\mathbf{x}, ct, \tau)\, \psi(\mathbf{x}, ct, \tau) \geqslant 0 \tag{8}$$

where ψ and ψ^* are Lorentz-invariant scalars. By assuming that ρ is positive definite, one avoids many of the difficulties associated with existing one-particle theories.[8] The function ρ satisfies the requirements of integrability,

[8] A critique of existing one-particle theories has been given by Fanchi.[28]

continuity, and differentiability if ψ is both Lebesgue square-integrable and differentiable. The quantity ψ has the form

$$\psi(\mathbf{x}, ct, \tau) = \rho(\mathbf{x}, ct \mid \tau)^{1/2} \exp[i\phi(\mathbf{x}, ct, \tau)] \tag{9}$$

where ϕ is a real scalar function as yet undetermined.

The four-vector $\{V^\mu\}$ of Eq. (4) can always be written as

$$V^\mu = \frac{\hbar}{\overline{m}} \frac{\partial \phi}{\partial x_\mu} + \left(V^\mu - \frac{\hbar}{\overline{m}} \frac{\partial \phi}{\partial x_\mu}\right) \tag{10}$$

where $\hbar/\overline{m}$ is an unspecified constant with units such that ϕ is dimensionless. One can then define

$$\epsilon A^\mu \equiv V^\mu - \frac{\hbar}{\overline{m}} \frac{\partial \phi}{\partial x_\mu} \tag{11}$$

in terms of which Eq. (11) becomes

$$V^\mu = \frac{\hbar}{\overline{m}} \frac{\partial \phi}{\partial x_\mu} + \epsilon A^\mu \tag{12}$$

Here ϵ is a constant setting the scale and units of A^μ. The quantity A^μ has the same harmonic and rotational parts as does V^μ, although their solenoidal parts may differ. Equation (12) expresses V^μ in terms of two quantities which depend on the phase of ψ. A relationship between A^μ, later to be identified as the four-vector potential of the electromagnetic field, and ϕ has been suggested before.[29,30] The consequence of Eq. (12) is the following.

The value of the density ρ is unchanged by the transformation

$$\psi' = \psi \exp[-i(\epsilon\overline{m}/\hbar)\Lambda] \tag{13}$$

where Λ is a real scalar function of $(\mathbf{x}, ct, \tau)$. This implies that ψ is specified only to within a gauge transformation of the first kind. The gauge transformation in Eq. (12) corresponds to the phase change

$$\phi' = \phi - (\epsilon\overline{m}/\hbar)\Lambda \tag{14}$$

and must be accompanied by the gauge transformation of the second kind

$$A'^\mu = A^\mu + \partial\Lambda/\partial x_\mu \tag{15}$$

so that Eq. (4) remains invariant with respect to the above transformations. The final result is that V^μ and ρ are unchanged by these gauge transformations, and one concludes that both A^μ and ψ are specified only to within a gauge transformation.

The function ϕ has a topological significance which will not be examined here (see Ref. 30 for a brief discussion of the nonrelativistic analog).

Using Eqs. (8) and (12) in Eq. (4) yields

$$\frac{\partial}{\partial\tau}(\psi^*\psi)+\frac{\partial}{\partial x^\mu}\left[\psi^*\left(\frac{\hbar}{\bar{m}}\frac{\partial\phi}{\partial x_\mu}+\epsilon A^\mu\right)\psi\right]=0 \tag{16}$$

Also from Eq. (9) there results

$$\frac{\partial\phi}{\partial x_\mu}=\frac{-i}{2\rho}\left(\psi^*\frac{\partial\psi}{\partial x_\mu}-\psi\frac{\partial\psi^*}{\partial x_\mu}\right) \tag{17}$$

which, when substituted into Eq. (15), yields

$$\frac{\partial}{\partial\tau}(\psi^*\psi)+\frac{\partial}{\partial x^\mu}\left[\frac{-i\hbar}{2\bar{m}}\left(\psi^*\frac{\partial\psi}{\partial x_\mu}-\psi\frac{\partial\psi^*}{\partial x_\mu}\right)+\epsilon A^\mu\psi^*\psi\right]=0 \tag{18}$$

Now this can be rearranged to read

$$\psi^*F=F^*\psi=(\psi^*F)^* \tag{19}$$

where the quantity F is

$$F=i\hbar\frac{\partial\psi}{\partial\tau}+\frac{\hbar^2}{2\bar{m}}\frac{\partial^2\psi}{\partial x_\mu\partial x^\mu}-\frac{i\hbar}{2\bar{m}c}\left(\frac{\partial A^\mu\psi}{\partial x^\mu}+A^\mu\frac{\partial\psi}{\partial x^\mu}\right) \tag{20}$$

Thus the product ψ^*F must be *real*. This is assured, for arbitrary ψ, if F has the form $U\psi$ with U a real scalar. The simplest possible form for U which keeps Eq. (20) gauge- and Lorentz-invariant is $e^2A^\mu A_\mu/2\bar{m}c^2$, where ϵ is now written as

$$\epsilon=-e/\bar{m}c \tag{21}$$

with e and c unspecified constants. Setting F equal to this $U\psi$ yields

$$i\hbar\frac{\partial\psi}{\partial\tau}=\frac{1}{2\bar{m}}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_\mu}-\frac{e}{c}A^\mu\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x^\mu}-\frac{e}{c}A_\mu\right)\psi\equiv p^\mu p_\mu\frac{\psi}{2\bar{m}} \tag{22}$$

with $p^\mu p_\mu$ a scalar operator.

Equation (22) is a generalized form of the KG equation if A^μ is identified as the four-vector potential, e as the electric charge of the particle, $\bar{m}$ as a constant with mass units, $\hbar$ as Planck's constant divided by 2π, and c as the speed of light.

Note that a multicomponent wave function could have been used in the procedure for deriving Eq. (22). Furthermore, the form chosen for U is not unique. These facts suggest points of departure for the generalization of the FSF to include nonelectromagnetic interactions and particles with nonzero spin. The details of a generalization along these lines will be developed elsewhere.

Equation (22) has the form of the Schrödinger equation except that Eq. (22) is defined on a four-space with a Lorentz metric. Since it does have the form of the Schrödinger equation, many of the procedures and results of Schrödinger wave mechanics can be paralleled by the four-space formulation, although care must be exercised in working with the metric.

The meaning of the operator p^μ in the formalism here is made clear by observing, in the manner of Collins[9] for the nonrelativistic case, that if one inserts the above representations for V^μ and ρ in terms of ψ and A^μ into Eq. (7) for $d\langle x^\mu\rangle/d\tau$ there results

$$\bar{m}\frac{d\langle x^\mu\rangle}{d\tau} = \int \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} - \frac{e}{c}A^\mu\right)\psi\, d^4x = \langle p^\mu\rangle \tag{23}$$

This defines the expectation value of a relativistic "proper" momentum. From this definition of the momentum operator p^μ follows the familiar commutation rules for canonically conjugate coordinates and momenta, and from these follow the corresponding uncertainty relationships; however, the energy–time relationship is now on the same mathematical basis as the momentum–spatial coordinate relationships.

Direct generalization then yields the definition

$$\langle \Omega\rangle = \int \psi^* \Omega \psi\, d^4x \tag{24}$$

for the expectation value of any observable associated with the particle, that is, the expectation value for any function of the x^μ or any derivative of such an expectation value, as p^μ above. Also note that Eq. (24) is the definition of a scalar product on $L^2(\mathbf{x}, ct)$, and is a special case of the general scalar product—a Stieltjes integral in four-momentum space—defined by Wigner in Ref. 23a, pp. 185. Expectation values will be examined further in Section 7.

Recalling that ρ and $\{V^\mu\}$ have been assumed single-valued and continuous, two particularly familiar boundary conditions are obtained by assuming that ψ and $\{A^\mu\}$ are also single-valued and continuous. Integrating Eq. (4) over a "pillbox" in spacetime, letting its length normal to the boundary go to zero, and employing the divergence theorem as in electrostatic theory[31] yields

$$\begin{aligned}\frac{-i\hbar}{2\bar{m}}\left(\psi_{\mathrm{I}}^*\frac{\partial\psi_{\mathrm{I}}}{\partial n} - \psi_{\mathrm{I}}\frac{\partial\psi_{\mathrm{I}}^*}{\partial n}\right) - \frac{eA_{\mathrm{I}}^\mu}{\bar{m}c}n_\mu\rho_{\mathrm{I}} \\ = \frac{-i\hbar}{2\bar{m}}\left(\psi_{\mathrm{II}}^*\frac{\partial\psi_{\mathrm{II}}}{\partial n} - \psi_{\mathrm{II}}\frac{\partial\psi_{\mathrm{II}}^*}{\partial n}\right) - \frac{eA_{\mathrm{II}}^\mu}{\bar{m}c}n_\mu\rho_{\mathrm{II}}\end{aligned} \tag{25}$$

where $\partial/\partial n$ represents the normal derivative, n_μ is the unit normal vector, and all quantities are evaluated at the boundary between regions I and II.

Since ψ is single-valued

$$\psi_{\mathrm{I}} = \psi_{\mathrm{II}} \tag{26}$$

on the boundary. Equations (25) and (26), with the fact that $\{A^\mu\}$ is single-valued, imply the other boundary condition, namely

$$\partial\psi_{\mathrm{I}}/\partial n = \partial\psi_{\mathrm{II}}/\partial n \tag{27}$$

Equations (26) and (27) state that ψ and its normal derivative are continuous at the boundary.

In addition, observe that ψ must vanish at any boundary where the four-vector potential has an infinite discontinuity. If this were not true, then the probability flux $\{\rho V^\mu\}$ would be unbounded across the boundary.

3. GENERALIZED SCHRÖDINGER EQUATION

It is straightforward to prove that p^μ is Hermitian and, consequently, that $p^\mu p_\mu$ is also Hermitian. One can use this fact to write Eq. (22) as

$$i\hbar\, \partial\psi/\partial\tau = H\psi; \qquad H \equiv (1/2\overline{m})\, p^\mu p_\mu \tag{28}$$

where H is a Hermitian operator. Equation (28) is essentially a generalized Schrödinger equation. Since H is Hermitian, there exists a set of wave functions which constitute a basis of eigenvectors for $L^2(\mathbf{x}, ct)$ obeying the orthonormality condition

$$\int \psi_{q'}^* \psi_q \, d^4x = \delta_{q'q} \tag{29}$$

where ψ_q is a solution of the equation

$$q\psi_q = p^\mu p_\mu \psi_q \tag{30}$$

The form of Eq. (30) suggests that q represents the square of an invariant momentum. Therefore let us define the magnitude of the expectation value of $p^\mu p_\mu$ as $m_0{}^2c^2$. A similar identification of a (mass)^2 operator has been made by Feynman *et al.*[32] and also by Cooke.[8] This definition will be elaborated upon shortly. First observe that Eqs. (28)–(30) are valid whether potentials are present or not. Such orthonormality relationships cannot be consistently defined within conventional theories except for the free particle case. Furthermore, H is Hermitian regardless of the strength of the potentials. This is a claim that conventional theories, such as the two-component theory of Feshbach and Villars,[18] cannot make because the KG equation with potentials is not, in general, diagonalizable into separate positive-and negative-energy parts.[9]

Given Eq. (28), a formalism is readily obtained which parallels non-relativistic Schrödinger theory. For example, there now exists a classical correspondence for the relativistic theory[28] because of the Hamiltonian form of Eq. (28). However, despite the mathematical similarities, the FSF contains concepts which do not exist in the nonrelativistic formalism. One such concept is the superposition of mass states.

The general solution of Eq. (28) is a superposition of the eigenfunctions, i.e.,[10]

$$\psi(\mathbf{x}, ct, \tau) = \sum_q A(q)\, \psi_q(\mathbf{x}, ct) \exp(iq\tau/2\overline{m}\hbar) \tag{31}$$

with $A(q)$ denoting the expansion coefficients. Since both positive and negative values of q are admissible, it is possible for the expectation value of $p^\mu p_\mu$ to be negative; the FSF thus includes both tardyons and tachyons. For tardyons,

$$\langle p^\mu p_\mu \rangle = m_0{}^2 c^2 > 0 \tag{32}$$

which is the special-relativistic light-cone constraint in energy–momentum space. Furthermore, the general solution, Eq. (31), can now be thought of as a superposition of mass states, a concept that is mathematically similar to the nonrelativistic concept of superposition of energy states. This interpretation will be discussed in more detail below for the free particle and also in Section 8.

4. CHARGED AND NEUTRAL PARTICLES: THE FREE PARTICLE AND CHARGE CONJUGATION

The free particle is defined by setting the four-vector $\{A^\mu\}$ to zero everywhere, or by setting the charge e to zero. Equation (22) becomes, with the metric now explicit,

$$-i\hbar \frac{\partial \psi}{\partial \tau} = \frac{\hbar^2}{2\overline{m}} \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi \tag{33}$$

[9] This is because the Hamiltonian of the two-component formalism, obtained by transforming the second-order *KG* equation into two first-order equations[18] and then applying the Foldy–Wouthuysen[33] transformation, will not converge or else will not be Hermitian except for weak, slowly varying potentials.[2]

[10] If the eigenvalues are continuous, or part of the eigenvalue range is continuous and the rest discrete, then the sum in Eq. (31) is to be replaced by an integral, or an integral and a sum defined over the appropriate range of eigenvalues, respectively.

with the general solution

$$\psi(\mathbf{x}, ct, \tau) = \int_{-\infty}^{\infty} \int \left\{ \frac{A(\mathbf{k}, \omega)}{(2\pi)^4} \exp\left[\frac{i\hbar^2}{2\overline{m}\hbar} \left(\mathbf{k}\cdot\mathbf{k} - \frac{\omega^2}{c^2} \right) \tau + i(\omega t - \mathbf{k}\cdot\mathbf{x}) \right] \right\} d^3k \, \frac{d\omega}{c} \tag{34}$$

Here x^0 is expressed as ct. The notion of the direction of particle propagation in space and time can be thought of as follows.

Using Eq. (34) in Eq. (7) yields

$$d\langle x^\mu \rangle / d\tau = (\hbar/m)\langle k^\mu \rangle = \langle V^\mu \rangle \tag{35}$$

where the invariant interval $d\tau$ is taken to be positive. The space components of the quantity $\{V^\mu\}$ correspond to the space components of the phase velocity, and V^0 corresponds to the temporal component of the phase velocity, which is interpreted as follows. If, in fact, a particle is progressing into the future when $d\langle ct \rangle$ is positive, and regressing into the past when $d\langle ct \rangle$ is negative, then Eq. (35) says that a positive-frequency wave propagates forward in time and a negative-frequency wave propagates backward in time. This result parallels the Feynman–Stückelberg interpretation.[15,16] Furthermore, in the nonrelativistic limit when $m_0^2 c^2 \gg \hbar^2 \langle \mathbf{k}\cdot\mathbf{k} \rangle$, one has

$$|\langle V^0 \rangle| \to c \tag{36}$$

as, indeed, it must.

The Feynman–Stückelberg interpretation in this format provides an acceptable physical interpretation of positive- and negative-energy solutions *within* the framework of a one-particle theory; thus it is *not* necessary, because of the existence of negative-energy solutions, to reinterpret the KG equation or the generalized Schrödinger equation, Eq. (28), as field equations which can be understood only within the framework of many-body theories.

The Feynman–Stückelberg interpretation can be combined with the concept of charge conjugation to obtain a familiar result. The complex conjugate of Eq. (30) is

$$\left(\frac{\hbar}{i} \frac{\partial}{\partial x^\mu} + \frac{e}{c} A_\mu \right) \left(\frac{\hbar}{i} \frac{\partial}{\partial x_\mu} + \frac{e}{c} A^\mu \right) \psi_q{}^* = q \psi_q{}^* \tag{37}$$

where the quantity $\psi_q{}^*$ is the solution of the KG equation with the charge e replaced by $-e$. Thus the probability density remains unchanged with respect to charge conjugation. What does change?

If one writes ψ as $Re^{i\Phi}$, then ρV^μ becomes

$$j^\mu \equiv \rho V^\mu = \left(\frac{\hbar}{\bar{m}} \frac{\partial \Phi}{\partial x_\mu} - \frac{eA^\mu}{\bar{m}c}\right) R^2 \tag{38}$$

where j^μ is the probability flux. Replacing e by $-e$ and interchanging ψ_q with $\psi_q{}^*$ will change the sign of V^μ and $\langle V^\mu \rangle$. This says that a charged particle and its oppositely charged antiparticle propagate in opposite spatial *and* temporal directions for a given potential configuration. Feshbach and Villars[18] arrived at this interpretation by defining a "negative probability density," but the FSF avoids this difficulty. Furthermore, theses remarks are applicable to the free particle solutions, which, in turn, can be used to describe neutral particles. Thus a nontrivial interpretation of j^0/c can be provided for both charged and neutral particles by the FSF (see Ref. 28 for details).

Returning now to the free particle, let us observe that the rest mass of a free tardyon is given by Eq. (32), i.e.,

$$m_0{}^2c^2 = \langle p^\mu p_\mu \rangle = \hbar^2[\langle \omega^2 \rangle / c^2 - \langle \mathbf{k} \cdot \mathbf{k} \rangle] \tag{39}$$

which is obtained by substituting Eq. (34) into Eq. (24) with Ω being $p^\mu p_\mu$. Thus one obtains the familiar relationship of relativity theory between energy $\hbar\omega$, momentum $\hbar\mathbf{k}$, and the observed rest mass in terms of expectation values. Equations (35) and (39) can be used to generate another significant result.

Integrating Eq. (35) for a small increment $\delta\tau$ in τ, one finds

$$\delta\langle \mathbf{x} \rangle = (\hbar/\bar{m})\langle \mathbf{k} \rangle\, \delta\tau \qquad \text{and} \qquad \delta\langle ct \rangle = (\hbar/\bar{m})\langle \omega/c \rangle\, \delta\tau \tag{40}$$

where the space and time components have been separated and explicitly written. Forming the inner product of the four-vector $\{\delta(x^\mu)\}$ yields

$$\begin{aligned} \delta\langle x^\mu \rangle\, \delta\langle x_\mu \rangle &= (\delta\langle ct \rangle)^2 - \delta\langle \mathbf{x} \rangle \cdot \delta\langle \mathbf{x} \rangle \\ &= (\hbar^2/\bar{m}^2)(\langle \omega/c \rangle^2 - \langle \mathbf{k} \rangle \cdot \langle \mathbf{k} \rangle)\, \delta\tau^2 \end{aligned} \tag{41}$$

Using Eq. (39) in Eq. (41) gives

$$(\delta\langle ct \rangle)^2 - \delta\langle \mathbf{x} \rangle \cdot \delta\langle \mathbf{x} \rangle = \frac{c^2 m_0{}^2 \delta\tau^2}{\bar{m}^2} - \left(\frac{\Delta^2\omega}{c^2} - \Delta^2\mathbf{k}\right) \frac{\hbar^2\delta\tau^2}{\bar{m}^2} \tag{42}$$

where $\Delta^2\omega$ is the dispersion, $\langle \omega^2 \rangle - \langle \omega \rangle^2$, in ω and $\Delta^2\mathbf{k}$ is similarly defined. One thus obtains, in the limit of negligible dispersions, the timelike constraint of special relativity expressed in terms of average spatial and temporal displacements. Thus the FSF obeys a "macroscopic causality" principle,[24] i.e., the tardyon's *most probable trajectory* remains within the light cone. Furthermore, this analysis is another example of how the properties of τ are analogous to those of proper time.

It appears here that the as yet unspecified constant $\bar{m}$ should be identified as m_0, the expectation value of the mass. If this is done, then in the non-quantum limit of zero dispersions Eq. (42) reduces to the classical definition of proper-time interval, but in terms of expectation values.

Here, then, one sees that the FSF can be used to derive results that are familiar from special relativity. Furthermore, Eq. (39) represents the fact the rest mass has been elevated in the FSF from simply a specified constant to an observable. An examination of Eq. (34) shows that ψ is a superposition of states, each of which corresponds to a particular combination of $\mathbf{k}$ and ω. Thus one obtains the reasonable result that measurements of the mass of the relativistic particle are not sharply defined, but have a distribution that depends on the probabilistic weight of each pure state, namely $|A(\mathbf{k}, \omega)|^2$. As an example let us consider the minimum wave packet representation of the free particle.

The expansion coefficients for the minimum wave packet are

$$A(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \left\{ [\exp(-ik^\mu x_\mu)] \prod_{\mu=0}^{3} N_\mu \Phi_\mu \right\} d^4x \tag{43}$$

where $\{N_\mu\}$ are normalization constants and $\{\Phi_\mu\}$ represent the initial wave components at $\tau = 0$ for the minimum wave packet; these are given by

$$\Phi_\mu = [2\pi(\Delta x^\mu)^2]^{1/2} \exp\{-[(x^\mu - \langle x_0{}^\mu\rangle)^2/4(\Delta x^\mu)^2] - i\langle k_{\mu 0}\rangle x^\mu\} \tag{44}$$

Here Δx^μ is the uncertainty in x^μ, and $\langle x_0{}^\mu\rangle$ and $\langle k_0{}^\mu\rangle$ represent the average position and momentum of the particle at the initial instant $\tau = 0$. The minimum wave packet is then constructed by substituting Eq. (43) into Eq. (34) and evaluating the integral. The result is

$$\begin{aligned}\psi_{\min}(\mathbf{x}, ct, \tau) = \prod_{\mu=0}^{3} \frac{N_\mu}{(2\pi)^{1/2}} &\left[\frac{\pi}{(\Delta x^\mu)^2 + i\hbar\tau g_{\mu\mu}/2\bar{m}}\right]^{1/2} \\ &\times \exp\left\{ i\langle k_\mu\rangle x^\mu - \frac{i\hbar\tau}{2\bar{m}} \langle k_0{}^\mu\rangle\langle k_{\mu 0}\rangle \right. \\ &\left. - \frac{(x^\mu - \langle x_0{}^\mu\rangle - \hbar\langle k_0{}^\mu\rangle\, \tau/\bar{m})^2}{4[(\Delta x^\mu)^2 + i\hbar\tau g_{\mu\mu}/2\bar{m}]} \right\}\end{aligned} \tag{45}$$

where $g_{\mu\mu}$ is given by

$$g_{00} = 1 = -g_{11} = -g_{22} = -g_{33} \tag{46}$$

Before proceeding to the physically interesting points regarding $\psi_{\min}$, let us first emphasize two important mathematical points. First, the repeated indices in Eqs. (44) and (45) do *not* imply summation. Second, the appearance of $g_{\mu\mu}$ in Eq. (45) is necessary in order to ensure that $\psi_{\min}$ is a solution of Eq. (22) for the free particle case. With these remarks out of the way, let us now consider four physically interesting points.

First, the average $\langle \mathbf{x} \rangle$ and $\langle ct \rangle$ are $\langle \mathbf{x}_0 \rangle + (\langle \hbar \mathbf{k}_0 \rangle / \bar{m})\tau$ and $c\langle t_0 \rangle + (\hbar \langle \omega_0 \rangle / \bar{m}c)\tau$, respectively. These are expected results.

Second, by taking the absolute square of $A(\mathbf{k}, \omega)$ one obtains the probability distribution in the momentum–energy representation. The result is a Gaussian distribution, which indicates that the wave packet is formed as a superposition of states that separately correspond to a particular mass, i.e., a particular combination of $\mathbf{k}$ and ω. This is also expected.

Third, the absolute square of $\psi_{\min}$ gives the probability distribution —Gaussian—in space *and* time. From this joint distribution a marginal probability of observing the particle somewhere in space during a specified interval of time, i.e., $\rho(\mathbf{x}, ct \mid \tau)$ is integrated over all space to yield $\bar{\rho}(ct \mid \tau)$. Whenever the marginal probability distribution is zero the particle cannot be observed *anywhere* in space, i.e., the particle effectively does not exist when $\bar{\rho}$ is zero. This capability, not present in conventional theories, is necessary for describing unstable particles while simultaneously retaining probability conservation.

Finally it is of interest to note that the wave packet solution $\psi_{\min}$ is not Lorentz-invariant. This is a result of the choice of the minimum wave packet as the initial value. Since $\prod_{\mu=0}^{3} \Psi_\mu$ is not Lorentz-invariant, it is clear that the form of the initial value changes from one Lorentz frame to another. The consequence is that $\psi_{\min}$ has a form that depends on the particular Lorentz frame. Although $\psi_{\min}$ is not a relativistically proper wave packet, it does illustrate the concept of superposition of mass states.

5. SCATTERING FROM A STEP POTENTIAL

It is important to observe that the solutions of the usual KG equation have *not* been changed by going to a four-space formalism although the *use* of those solutions has changed. In addition, the definition of spatial probability flux as ρV^j $(j = 1, 2, 3)$ remains unchanged. If one recalls that scattering calculations require only the continuity of spatial probability flux, and do *not* require the use of normalized solutions, then it is evident that the usual KG theory can be used to compute scattering cross sections under certain circumstances. These have been discussed by Cooke.[8]

In general one must represent an incident free particle by a spacetime wave packet which is a superposition of mass states, just as in NRQM an incident free particle is represented by a wave packet which is a superposition of energy states. The formal development of such a theory has promise as a fertile area for future research. For the purposes of this paper, however, it will be sufficient to consider only the simple example of a RSP scattering from a step potential.

An interesting feature of existing theories is the prediction that a RSP incident on a step potential of sufficient strength will be reflected by the barrier with a reflection coefficient that exceeds unity. This result is a direct consequence of identifying ej^0/c as the charge density.[19,20] Even though total charge is conserved, the prediction that more particles will be reflected than were incident is a surprising consequence which has not been experimentally justified. This situation does not arise in the FSF.

The four-vector potential for this problem is

$$A^0(x) = \begin{cases} 0, & x < 0 \\ \alpha, & x > 0 \end{cases} \tag{47}$$

$$\mathbf{A}(\mathbf{x}, ct) = 0$$

Here region I refers to values of $x < 0$, and region II refers to values of $x \geqslant 0$. It is assumed throughout this calculation that a spinless tardyon is incident from $x = -\infty$ with energy ω_1 and momentum k_1, where ω_1 and k_1 are both real and positive. The boundary conditions to be satisfied are that the wave function and its first derivative with respect to x must be continuous across the boundary at $x = 0$.

The solutions of the equations

$$2\overline{m}i\hbar \frac{\partial \psi_{\mathrm{I}}}{\partial \tau} = q_{\mathrm{I}}\psi_{\mathrm{I}} = -\hbar^2 \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\psi_{\mathrm{I}}\right) \tag{48}$$

and

$$2\overline{m}i\hbar \frac{\partial \psi_{\mathrm{II}}}{\partial \tau} = q_{\mathrm{II}}\psi_{\mathrm{II}} = -\hbar^2 \frac{\partial^2}{\partial x^2}\psi_{\mathrm{II}} + \left(\frac{\hbar}{ic}\frac{\partial}{\partial t} - \frac{e\alpha}{c}\right)^2 \psi_{\mathrm{II}} \tag{49}$$

are the wave functions for regions I and II, respectively.

In region I the solution ψ_{I} is

$$\psi_{\mathrm{I}} = [a_1 \exp(-ik_1x) + b_1 \exp(ik_1x)] \exp(i\omega_1 t) \exp(-iq_{\mathrm{I}}\tau/2\overline{m}\hbar) \tag{50}$$

where a_1 and b_1 are the coefficients of the incident and reflected plane wave solutions, respectively. These identifications are established in the appendix. The eigenvalue relation in region I is

$$q_{\mathrm{I}} \equiv m^2c^2 = \hbar^2[(\omega_1{}^2/c^2) - k_1{}^2] \geqslant 0 \tag{51}$$

where the inequality is valid for incident tardyons and the real quantity m is defined by Eq. (51). Since ω_1 and k_1 are assumed known, so also is the observed rest mass m of the particle.

A description of the situation in region II is a little more involved than that given above for region I. The solution of Eq. (49) is

$$\psi = [a_2 \exp(-ik_2x) + b_2 \exp(ik_2x)] \exp(i\omega_2 t) \exp(-iq\tau/2\bar{m}\hbar) \tag{52}$$

To assure that the wave function and its first x-derivative are continuous at $x = 0$ for all values of t and τ, one must have

$$\omega_1 = \omega_2 \tag{53}$$

and

$$q_{\mathrm{I}} = q_{\mathrm{II}} \tag{54}$$

respectively. The eigenvalue relation for region II can now be written as

$$\hbar^2 k_2^2 = [(\hbar\omega_1/c) - (e\alpha/c)]^2 - m^2c^2 \tag{55}$$

where k_2 may be real or imaginary. The four possible cases, the appropriate spatial solutions, and the reflection and transmission coefficients for each case, found in the same manner as that of NRQM,[14] are listed in Table I. The quantities ω_1, k_1, k_2, m^2c^2, and $e\alpha$ are all taken to be real and positive in Table I. The coefficient b_2 is zero in cases 1 and 3 because no particles are incident on the barrier from $x = +\infty$. For cases 2 and 4 the quantity k_2 of Eq. (52) is positive imaginary, thus the coefficient of the term $\exp(|k_2| x)$ must be zero because the wave function must be finite for all values of x. Are the results of Table I physically realistic?

In Table I it is evident that the reflection coefficient for each case does not exceed unity and that the sum of the reflection and transmission coefficients for each case is unity. These are desired results. All of the results for cases 1 and 4 are easy to accept since these particular cases correspond to the familiar results of NRQM. The details of cases 2 and 3 deserve further attention.

It is readily shown that case 2 is physically realistic. First observe that $\hbar\omega_1 > mc^2$ because of Eq. (51). The eigenvalue condition for case 2 can be written as

$$|e\alpha| > \hbar\omega_1 - mc^2 > 0 \tag{56}$$

Recalling that the kinetic energy of the particle in region I is $\hbar\omega_1 - mc^2$, it is clear that Eq. (56) asserts that the barrier height exceeds the kinetic energy of the incident particle and, hence, that reflection at the barrier is expected. Thus the results of case 2 are realistic.

Table I. Scattering from a Step Potential

Case	Conditions[a]	Spatial solution	Reflection coefficient R	Transmission coefficient T	$R + T = 1$?
1	$\hbar\omega_1 > e\alpha$ $[(\hbar\omega_1/c) - (e\alpha/c)]^2 \geqslant m^2c^2$	$a_2e^{-ik_2x}$	$[(k_1 - k_2)/(k_1 + k_2)]^2$	$4k_1k_2/(k_1 + k_2)^2$	Yes
2	$\hbar\omega_1 > e\alpha$ $[(\hbar\omega_1/c) - (e\alpha/c)]^2 < m^2c^2$	$b_2e^{-k_2x}$	1	0	Yes
3	$\hbar\omega_1 < e\alpha$ $[(\hbar\omega_1/c) - (e\alpha/c)]^2 \geqslant m^2c^2$	$a_2e^{-ik_2x}$	$[(k_1 - k_2)/(k_1 + k_2)]^2$	$4k_1k_2/(k_1 + k_2)^2$	Yes
4	$\hbar\omega_1 < e\alpha$ $[(\hbar\omega_1/c) - (e\alpha/c)]^2 < m^2c^2$	$b_2e^{-k_2x}$	1	0	Yes

[a] All of the quantities ω_1, $e\alpha$, m^2c^2, k_1, and k_2 are real and positive here.

Now consider case 3. The smallest value of $| e\alpha |$ for which the conditions of case 3 are satisfied is $2mc^2$. This is a very large value—on the order of 280 MeV for a charged pion—and is not yet experimentally accessible; consequently one cannot say with certainty whether or not case 3 is physically realistic. It is interesting to observe that the particles transmitted into region II are propagating backward in time, and the fraction of reflected particles is $[(k_1 - k_2)/(k_1 + k_2)]^2$. For this same case existing theories predict that antiparticles propagating forward in time are transmitted into region II, and that the fraction of reflected particles is $[(k_1 + k_2)/(k_1 - k_2)]^2$! An experiment capable of measuring the fraction of particles reflected from an appropriate step potential could ascertain which, if either, result is correct. Such an experiment could serve as a test of the FSF *and* of existing theories.

6. SYMMETRY CONSIDERATIONS

It is straightforward to show that Eq. (28) for a free particle is invariant with respect to the linear transformation

$$Y_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu \tag{57a}$$

$$\tau' = \tau + \Delta\tau \tag{57b}$$

where the quantities $\{\Lambda_\mu{}^\nu\}$ represent a homogeneous Lorentz tranformation[23] and the quantities $\{a_\mu, \Delta\tau\}$ represent translations along the $\{x_\mu, \tau\}$ axes, respectively. Equations (57) can be represented as $\{a, \Lambda, \Delta\tau\}$ and will be referred to as the FSF transformation.

Let us define a set F containing, as elements, all FSF transformations. Clearly the set F contains an identity element, namely the transformation

$$Y_\mu = x_\mu , \qquad \tau' = \tau \tag{58}$$

The product of two FSF transformations $\{a_1, \Lambda_1, \Delta\tau_1\}$ and $\{a_2, \Lambda_2, \Delta\tau_2\}$ is given by

$$\{a_1, \Lambda_1, \Delta\tau_1\}\{a_2, \Lambda_2, \Delta\tau_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2, \Delta\tau_1 + \Delta\tau_2\} \tag{59}$$

Using Eq. (59), one can readily show that the elements of F are associative. Furthermore, every element of F, every FSF transformation, has an inverse. Therefore the set F, together with the binary operation represented by Eq. (59), forms a group, which will be referred to below as the FSF group. If one lets $\Delta\tau \to 0$, then one obtains the Poincaré group[24,25] as a subgroup of the FSF group.

The FSF group has one more continuous parameter ($\Delta\tau$) then the Poincaré group; thus the FSF group is an 11-parameter continuous group. The corresponding 11 generators are the ten generators of the Poincaré group plus one additional generator that generates infinitesimal displacements of state vectors along the τ axis. This generator is obviously the Hamiltonian H of Eq. (28).

Although all of the mathematical subtleties of the FSF group have not yet been discussed, the above remarks do place into perspective the relationship between the FSF group and the more familiar Poincaré group.

7. EXPECTATION VALUES

The expectation value defined by Eq. (24), denoted below as $\langle\cdots\rangle_{\text{FSF}}$, also defines the scalar product of two vectors. According to this point of view, $\langle\Omega\rangle_{\text{FSF}}$ is the transition probability from the state $\Omega\psi$ into ψ, or conversely. The scalar product $\langle\cdots\rangle_{\text{FSF}}$ is a special case of the general scalar product defined by Wigner[23] that corresponds to replacing Wigner's $df(p, \zeta)$ by d^4k. The physical significance of this replacement is that all volume elements of four-momentum space are weighted equally. One can obtain the more familiar results of quantum field theory by altering the weight distribution of the volume elements of four-momentum space. This will be shown below. First, however, let us obtain the nonrelativistic limit of $\langle\cdots\rangle_{\text{FSF}}$.

Observe that the joint probability density $\rho(\mathbf{x}, ct \mid \tau)$ can always be written as the product of a maginal, $\bar{\rho}(t \mid \tau)$, and a conditional, $\rho_c(\mathbf{x} \mid ct, \tau)$, probability density:

$$\rho(\mathbf{x}, ct \mid \tau) = \bar{\rho}(t \mid \tau)\, \rho_c(\mathbf{x} \mid ct, \tau) \tag{60}$$

Now $\langle g\rangle_{\text{FSF}}$ is given by

$$\langle g\rangle_{\text{FSF}} = \int \bar{\rho}(t \mid \tau)\Big[\int g(\mathbf{x}, ct)\, \rho_c(\mathbf{x} \mid ct, \tau)\, d^3x\Big]\, d(ct) \tag{61}$$

where $g(\mathbf{x}, ct)$ is, at present, unspecified. Equation (61) reduces to the three-space definition of expectation value only for those cases in which the function g is independent of t or else $\bar{\rho}(t \mid \tau)$ is a delta function in t. This is a mathematical constraint. To completely define the region in which $\langle\cdots\rangle_{\text{FSF}}$ should agree with the nonrelativistic value $\langle\cdots\rangle_{\text{NRQM}}$, one must also impose the physical constraint that $\hbar\omega \sim m_0c^2$, where m_0 is the rest mass of the particle. This same physical requirement must also be imposed on the expectation value defined within conventional theories $\langle\cdots\rangle_{\text{con}}$ before it can be compared with $\langle\cdots\rangle_{\text{NRQM}}$.

A special case which satisfies all of the above conditions is that in which one takes the expectation value of a function of the spatial coordinates $f(\mathbf{x})$ and the time dependence of ψ_q is $[\exp(i\omega t)]/(2T)^{1/2}$, where $-T < t < +T$ in the limit as $T \to \infty$. The resulting expectation values are[11]

$$\langle f\rangle_{\text{con}} = \frac{\int_{-\infty}^{\infty} \psi_q{}^*\psi_q(\hbar\omega - eA^0)\, f(\mathbf{x})\, d^3x}{\int_{-\infty}^{\infty} \psi_q{}^*\psi_q(\hbar\omega - eA^0)\, d^3x} \tag{62}$$

and

$$\langle f\rangle_{\text{FSF}} = \frac{\lim_{T\to\infty} \int_{-T}^{T} \int_{-\infty}^{\infty} f(\mathbf{x})\, \psi_q{}^*\psi_q\, d^3x\, d(ct)}{\lim_{T\to\infty} \int_{-T}^{T} \int_{-\infty}^{\infty} \psi_q{}^*\psi_q\, d^3x\, d(ct)} \tag{63}$$

where box normalization has been imposed in the time domain and ψ_q is defined by Eq. (30). Equation (63) can be written in the equivalent form

$$\langle f\rangle_{\text{FSF}} = \frac{\int_{-\infty}^{\infty} f(\mathbf{x})\, \psi_q{}^*\psi_q\, d^3x}{\int_{-\infty}^{\infty} \psi_q{}^*\psi_q\, d^3x} \tag{64}$$

The important difference between $\langle f\rangle_{\text{con}}$ and $\langle f\rangle_{\text{FSF}}$ is due to the eA^0 term in $\langle f\rangle_{\text{con}}$. Clearly for the free particle $\langle f\rangle_{\text{con}}$ and $\langle f\rangle_{\text{FSF}}$ agree. Thus it is not surprising that relativistic free particles can be consistently described using existing theories.[(1)] In fact, if either A^0 or f is independent of the spatial coordinates, then Eqs. (62) and (64) agree; generally they do not.

It is well known that the KG equation reduces[(14)] to the Schrödinger equation in the nonrelativistic limit, although the reduction violates Lorentz invariance and the gauge is altered. Nevertheless, under these circumstances the KG solutions are just the solutions of the Schrödinger equation. Thus Eq. (64) is then the usual nonrelativistic definition of the expectation value. Under the same circumstances, however, $\langle f\rangle_{\text{con}}$ does not generally reduce to $\langle\cdots\rangle_{\text{NRQM}}$, because of the term of order $eA^0/\hbar\omega$. The magnitude of this discrepancy is calculated in Ref. 28 for the case in which A^0 is an attractive Coulomb potential. There it is shown that $\langle r\rangle_{\text{con}}$ is 20% larger than $\langle r\rangle_{\text{FSF}}$ for the $2P$ state of a π^+–π^- system.

As remarked earlier, the FSF can be linked to conventional quantum field theory. First observe that the general solution of the free particle equation can be written as

$$\begin{aligned}\psi(\mathbf{x}, ct, \tau) = \iint \Big\{ &A(\mathbf{k}, \omega) \exp(i\omega t - i\mathbf{k}\cdot\mathbf{x}) \\ &\times \exp\left[\frac{-i\hbar\tau}{2\bar{m}}\left(\frac{\omega^2}{c^2} - \mathbf{k}\cdot\mathbf{k}\right)\right]\Big\}\, d^3k\, \frac{d\omega}{c}\end{aligned} \tag{65}$$

[11] This definition is used by Feshbach and Villars[(18)] and also by Bjorken and Drell.[(2)] See Ref. 28 for additional details.

where $A(\mathbf{k}, \omega)$ is an arbitrary expansion coefficient. Now, conventional quantum field theory assumes that any given elementary particle has a unique rest mass, say m_0, which can be measured to an arbitrary degree of accuracy. This assumption corresponds, here, to asserting that only a single mass state contributed to the wave function; thus

$$A(\mathbf{k}, \omega) = b(\mathbf{k}, \omega)\, \delta\left(\hbar^2 \frac{\omega^2}{c^2} - \hbar^2 \mathbf{k} \cdot \mathbf{k} - m_0^{\,2} c^2\right) \tag{66}$$

where $b(\mathbf{k}, \omega)$ is arbitrary. This delta function can be rewritten as[34]

$$\delta\left(\hbar^2 \frac{\omega^2}{c^2} - \hbar^2 \mathbf{k} \cdot \mathbf{k} - m_0^{\,2} c^2\right) = \frac{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)}{(2\hbar^2/c^2)\, \omega_0} \tag{67}$$

where

$$\omega_0(\mathbf{k}) = (c/\hbar)(m_0^{\,2} c^2 + \hbar^2 \mathbf{k} \cdot \mathbf{k})^{1/2} > 0 \tag{68}$$

Substituting Eqs. (66) and (67) into Eq. (65) and evaluating the ω integral yields

$$\begin{aligned} \psi(\mathbf{x}, ct, \tau) = e \frac{-i m_0^{\,2} c^2 \tau}{2\bar{m}\hbar} \int [&b(\mathbf{k}, \omega_0) \exp(i\omega_0 t - i\mathbf{k} \cdot \mathbf{x}) \\ &+ b(\mathbf{k}, -\omega_0) \exp(-i\omega_0 t - i\mathbf{k} \cdot \mathbf{x})] \frac{c\, d^3k}{2\hbar^2 \omega_0} \end{aligned} \tag{69}$$

The corresponding result for the antiparticle is

$$\begin{aligned} \psi^*(\mathbf{x}, ct, \tau) = e \frac{i m_0^{\,2} c^2 \tau}{2\bar{m}\hbar} \int \{&b^*(\mathbf{k}, \omega_0) \exp[-i(\omega_0 t - \mathbf{k} \cdot \mathbf{x})] \\ &+ b^*(\mathbf{k}, -\omega_0) \exp[i(\omega_0 t + \mathbf{k} \cdot \mathbf{x})]\} \frac{c\, d^3k}{2\hbar^2 \omega_0} \end{aligned} \tag{70}$$

The τ dependence of ψ and ψ^* is now just a numerical factor which has no effect on the probability density $(\psi^*\psi)$ and can therefore be removed by a gauge transformation or, equivalently, by setting $\tau = 0$ in Eqs. (69) and (70). If one now compares $\psi(\mathbf{x}, ct, 0)$ and $\psi^*(\mathbf{x}, ct, 0)$ to the ψ operators of quantum field theory,[17,35][12] one observes that these functions are *precisely* the same when the coefficients $b(\mathbf{k}, \omega_0)$ and $b^*(\mathbf{k}, +\omega_0)$ are interpreted as particle creation and annihilation operators, respectively, and the coefficients $b^*(\mathbf{k}, -\omega_0)$ and $b(\mathbf{k}, -\omega_0)$ are interpreted as antiparticle creation and annihilation operators, respectively. Thus the field operators of conventional relativistic quantum field theory can be derived from the FSF by assuming that only a single mass state contributes to the wave function and then replacing coefficients by the conventional manner of the second quantization formalism.

[12] Also see Ref. 1, p. 197.

If it is now required that *every* field obeys microscopic causality,[25] i.e., any two field operators either commute or anticommute for a spacelike interval, then, as Pauli[36] has shown, two fields having the form of Eq. (69) must commute for spacelike intervals. Thus one obtains the expected conclusion that spinless particles are bosons. Furthermore, Weinberg[17] has shown that microscopic causality also requires every particle to have an antiparticle. Finally, it should be noted that the usual theory of scattering[17] is applicable under the conditions specified above.

8. APPLICATION OF THE FSF TO A SIMPLE PHYSICAL EXPERIMENT

In the preceding sections a theory has been developed which defines a mass operator and asserts that the state of a particle is a superposition of mass states. Consequently the particle does not necessarily have a sharply defined mass. One may justifiably ask how this feature of the FSF can be reconciled with the existence of, for example, beams of particles having sharply defined masses. The answer to this question, as well as the related question of why mass is quantized, can be demonstrated by using the FSF to describe a particle beam interacting with a magnetic field.

First consider how a particle beam may be obtained from an ensemble of charged particles moving in a static magnetic field with magnitude B. Without loss of generality a frame of reference can be chosen that has an origin about which the particles execute circular motion and the magnetic field is oriented parallel to the z axis. This assumes, of course, that the particles of differing rest mass have the same total energy and execute concentric trajectories, and that the particles are not interacting with one another. Then the rest mass m_0 of a particle with charge e and total classical energy ϵ_c is given by

$$m_0^2 = \epsilon_c^2 - e^2B^2a^2 \tag{71}$$

where a is the radius of gyration. The usual interpretation of Eq. (71) is that the total classical energy is given in terms of the rest mass and the momentum; however, the actual measured quantities are ϵ_c, e, B, and a, from which m_0 is calculated. In other words, the energy and momentum are the independent observables.[8] Now the particle having rest mass m_0 can be observed by placing a detector at a or, alternatively, a beam of particles having rest mass m_0 can be obtained by putting collectors at all radii except a. Thus a beam of particles having a particular mass m_0 can be obtained by eliminating all particles having a mass other than m_0. Quantum mechanically, this is equivalent to *preparing* a pure mass state by eliminating contributions from all other mass states. Once the pure mass state is obtained, then the wave function describing the particle beam–magnetic field system can be renormalized. Let us quantify these results.

The equation to be solved is Eq. (22), where the gauge is chosen such that

$$A_1 = -yB, \qquad A_0 = A_2 = A_3 = 0 \tag{72}$$

Writing $\psi(x, \tau)$ as $[\exp(-iq\tau/2\overline{m}\hbar)]\ \phi(x)$ gives

$$q\phi(x) = p^{\mu}p_{\mu}\phi(x) \tag{73}$$

The solution to Eq. (73) has been derived by Lam[37] and, for $\hbar = c = 1$, it is

$$\phi(x) = \exp[i(k_x x + k_z z - \epsilon t)]\ P(y) \tag{74}$$

where

$$P(\xi) = \exp(\tfrac{1}{2}i\xi^2)\ g_j(i\alpha - i\xi^2); \qquad j = 1, 2 \tag{75}$$

$$\xi^2 = i \mid eB \mid (y + k_x/\mid eB \mid)^2 \tag{76}$$

and

$$\alpha = (m^2 - \epsilon^2)/i \mid eB \mid \tag{77}$$

The g_j are given in terms of confluent hypergeometric functions,[34] namely

$$g_1(b, \tau') = F(\tfrac{1}{4}(1 + b), \tfrac{1}{2}, \tau') \tag{78}$$

and

$$g_2(b, \tau') = \tau^{1/2}F(\tfrac{1}{4}(1 + b) + \tfrac{1}{2}, \tfrac{3}{2}, \tau') \tag{79}$$

The eigenvalue relation is

$$m_0^2 = q = \epsilon^2 - (2n + 1) \mid e \mid B; \qquad n = 0, 1, 2,... \tag{80}$$

where as in the classical case it has been assumed that k_z is zero. For a given energy ϵ, the mass spectrum represented by Eq. (80) is discrete; otherwise it is continuous. This result is due to the field configuration and is consistent with the discussion in Ref. 26. The general solution can be written as

$$\psi(x, \tau) = \sum_l \int A(p, l)[\exp(-iq\tau/2\overline{m}\hbar)]\ \phi_{p,l}(x)\ d\epsilon \tag{81}$$

where p and l represent the energy and momentum eigenvalues, respectively.

As a check of the quantum mechanical result, let us demonstrate the classical correspondence. Employing Eq. (24), one finds

$$2 \mid e \mid B\langle y_n^2\rangle = 2n + 1 \tag{82}$$

which, when substituted into Eq. (80), yields

$$q = \epsilon^2 - 2e^2B^2\langle y_n{}^2\rangle \tag{83}$$

Noting that the classical time-averaged value of y^2, denoted by $\langle y^2\rangle_t$, is

$$\langle y^2\rangle_t = \tfrac{1}{2}a^2 \tag{84}$$

it is seen that the classical and quantum results agree in the classical limit, as they should.

If a particle beam has been prepared experimentally as described at the beginning of this section, then a pure mass state exists and is treated analytically by defining

$$A(p, l) = B(p, l)\,\delta(p - \epsilon)\,\delta_{ln} \tag{85}$$

where $B(p, l)$ is a factor fixed by renormalizing the particle beam–magnetic field system. Thus a beam of particles having a sharply defined mass m_0 can exist because the system has been prepared such that all mass states other than m_0 are eliminated.

The particle beam–magnetic field system illustrates an important aspect of the FSF. In particular, the square of the rest mass [q in Eq. (73)] is an observable that depends on the four-vector potential. This point of view has been suggested by Cooke[8] and more recently by Feynman *et al.*[32] If the electromagnetic potential is perturbed, then the observable rest mass can be altered. Furthermore, since the operator $p^\mu p_\mu$ is Hermitian on $L^2(\mathbf{x}, ct)$ for an arbitrary four-vector potential, it can be inferred that the eigenvalues q are real and that the set of eigenfunctions $\{\phi_{p,l}(x)\}$ of $p^\mu p_\mu$ span the space $L^2(\mathbf{x}, ct)$. Therefore $\{\phi_{p,l}(x)\}$ can be used as a basis set in terms of which a perturbation calculation of the rest mass change can be made.

As an example, suppose that in addition to the experiment described previously an independent experiment is performed which is identical to the first except for the magnitude of the magnetic field. It is important to emphasize that the location of the experimentalist's tools, e.g., the particle detectors or collectors, has also not changed. If the perturbing four-vector potential $\{a_\mu\}$ is

$$a_1 = y\delta_b\,, \qquad a_0 = a_2 = a_3 = 0, \qquad |\delta_B| < |B| \tag{86}$$

then one can ask what rest mass will be observed at the gyration radius a. This question can readily be answered.

Making use of the completeness condition on the wave function, it can be shown that the first-order difference in the squared rest mass is

$$m_1{}^2 - m_0{}^2 = \int (\phi_{p,l}(x))^*\, Q\phi_{p,l}(x)\, d^4x \tag{87}$$

where

$$Q \equiv \left\{\frac{ea^\mu}{i}\frac{\partial}{\partial x_\mu} - \frac{e}{i}\frac{\partial a^\mu}{\partial x^\mu} + 2e^2A^\mu a_\mu\right\} \tag{88}$$

The quantities m_0 and $\phi_{p,l}(x)$ are given by Eqs. (80) and (74), respectively, and m_1 is the rest mass correct to first order of the perturbed system. Performing the necessary integrations yields

$$m_1^2 = m_0^2 + (2n+1)\,|e|\,\delta_B\,, \qquad n = 0, 1,... \tag{89}$$

Solving Eq. (89) for m_0^2 and substituting into Eq. (81) gives

$$m_1^2 = \epsilon^2 - (2n+1)\,|e|\,(B - \delta_B) \tag{90}$$

This result is the same as that obtained by solving Eq. (73) directly for a four-vector potential $\{A_\mu + a_\mu\}$. Perturbation theory has been used here primarily because it illustrates the usefulness of the completeness condition on the wavefunction, and also because perturbation theory is applicable for an *arbitrary* four-vector potential, including those for which the Klein–Gordon equation cannot be solved.

APPENDIX. DIRECTION OF PARTICLE PROPAGATION FOR THE CASE OF SCATTERING FROM A STEP POTENTIAL

Recall that

$$\rho V^\mu = \frac{-i\hbar}{2\bar{m}}\left(\psi^*\frac{\partial\psi}{\partial x_\mu} - \psi\frac{\partial\psi^*}{\partial x_\mu}\right) - \frac{eA^\mu}{\bar{m}c}\psi^*\psi \tag{A1}$$

and

$$\begin{aligned} &\text{region I:} && (A^0, A^1) = (0, 0) \\ &\text{region II:} && (A^0, A^1) = (\alpha, 0) \end{aligned} \tag{A2}$$

For the t and x components one has

$$\rho V^0 = \frac{-i\hbar}{2\bar{m}c}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{eA^0}{\bar{m}c}\psi^*\psi \tag{A3}$$

$$\rho V^1 = \frac{i\hbar}{2\bar{m}}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) \tag{A4}$$

respectively. One can now determine the propagation direction for a given plane wave solution in a specified region. These results are tabulated in Table II.

Table II. Propagation Direction of Plane Wave Solutions[a]

Region		V^0	V^1	Propagation direction
I	$e^{i\omega t}$	$\hbar\omega/\bar{m}c$	—	Forward in t
	$e^{-i\omega t}$	$-\hbar\omega/\bar{m}c$	—	Backward in t
	e^{ikx}	—	$-\hbar k/\bar{m}$	Backward in x
	e^{-ikx}	—	$\hbar k/\bar{m}$	Forward in x
II	$e^{i\omega t}$	$(\hbar\omega - e\alpha)/\bar{m}c$	—	Forward in t if $\hbar\omega > e\alpha$; backward in t if $\hbar\omega < e\alpha$; stationary in t if $\hbar\omega = e\alpha$
	$e^{-i\omega t}$	$(-\hbar\omega - e\alpha)/\bar{m}c$	—	Backward in t
	e^{ikx}	—	$-\hbar k/\bar{m}$	Backward in x
	e^{-ikx}	—	$\hbar k/\bar{m}$	Forward in x

[a] The quantities ω, k, and e are all real and positive.

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