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Flow in Porous Media III: Deformable Media

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Abstract. Stokes flow in a deformable medium is considered in terms of an isotropic, linearly elastic solid matrix. The analysis is restricted to steady forms of the momentum equations and small deformation of the solid phase. Darcy's law can be used to determine the motion of the fluid phase; however, the determination of the Darcy's law permeability tensor represents part of the closure problem in which the position of the fluid-solid interface must be determined.

Key words. Darcy's law, elastic media, boundary conditions, closure.

0. Nomenclature

Roman Letters

$\mathscr{A}_{\beta\sigma}$ interfacial area of the β–σ interface contained within the macroscopic system, m^2
$A_{\beta\sigma}$ interfacial area of the β–σ interface contained within the averaging volume, m^2
$\mathscr{A}_{\sigma e}$ area of entrances and exits for the σ-phase contained within the macroscopic system, m^2
$A^*_{\beta\sigma}$ interfacial area of the β–σ interface contained within a unit cell, m^2
$A^*_{\sigma e}$ area of entrances and exits for the σ-phase contained within a unit cell, m^2
E_σ Young's modulus for the σ-phase, N/m^2
$\mathbf{e}_i$ unit base vectors ($i = 1, 2, 3$)
$\mathbf{g}$ gravity vector, m^2/s
H height of elastic, porous bed, m
$\mathbf{k}$ unit base vector ($= \mathbf{e}_3$)
ℓ_ω characteristic length scale for the ω-phase, m
L characteristic length scale for volume-averaged quantities, m
$\mathbf{n}_{\beta\sigma}$ unit normal vector pointing from the β-phase toward the σ-phase ($\mathbf{n}_{\beta\sigma} = -\mathbf{n}_{\sigma\beta}$)
p_β pressure in the β-phase, N/m^2
P_β $p_\beta - \rho_\beta \mathbf{g} \cdot \mathbf{r}$, N/m^2
r_0 radius of the averaging volume, m
$\mathbf{r}$ position vector, m
t time, s
$\mathbf{T}_\omega$ total stress tensor in the ω-phase, N/m^2

$\mathbf{T}_\sigma^0$ hydrostatic stress tensor for the σ-phase, N/m^2
$\mathbf{u}_\sigma$ displacement vector for the σ-phase, m
$\mathscr{V}$ averaging volume, m^3
V_ω volume of the ω-phase contained within the averaging volume, m^3
$\mathbf{v}_\omega$ velocity vector for the ω-phase, m/s

Greek Letters

ϵ_ω $V_\omega/\mathscr{V}$, volume fraction of the ω-phase
ρ_ω mass density of the ω-phase, kg/m^3
μ_β shear coefficient of viscosity for the β-phase, Nt/m^2
μ_σ first Lamé coefficient for the σ-phase, N/m^2
λ_σ second Lamé coefficient for the σ-phase, N/m^2
κ_β bulk coefficient of viscosity for the β-phase, Nt/m^2
$\boldsymbol{\tau}_\sigma$ $\mathbf{T}_\sigma - \mathbf{T}_\sigma^0$, a deviatoric stress tensor for the σ-phase, N/m^2

1. Introduction

Deformable porous media are of interest to the seismologist (Bourbie, 1984) who is concerned with wave propagation phenomena, and the engineer who seeks a fundamental understanding of the processes that take place during the drying of cellular material (Crapiste *et al.*, 1984). Soil scientists are concerned with the rapid motion that takes place during earthquakes and the quasi-steady motion that occurs during subsidence (Narasimhan and Witherspoon, 1977), while chemical engineers must deal with the compaction that takes place during filtration processes (Tiller and Horng, 1983). All of these systems have in common a structure that can be loosely identified as a mixture of solid and fluid, and they have in common the characteristic that they are deformable. For granular porous media this deformation most likely takes place in terms of a re-orientation of the solid particles, while the solid phase of cellular materials is more likely to undergo an actual swelling or shrinking depending on the moisture content. Processes of interest to seismologists are likely to consist of both of these types of deformation.

In terms of practical problems, there would appear to be none in which the solid matrix could be adequately described in terms of an isotropic, linear elastic material. However, one can easily construct a porous medium of elastic spheres, thus the possibility exists for comparing theory and experiment in the absence of adjustable parameters. For example, the system shown in Figure 1 is easy to construct, and both the pressure profile and the deformation at the top of the bed can be easily measured as a function of flow rate. This is precisely the type of experiment that is done with compressible filter cakes (Tiller and Horng, 1983), and from those studies it is clear that eliminating the effect of the side walls is not a trivial problem.

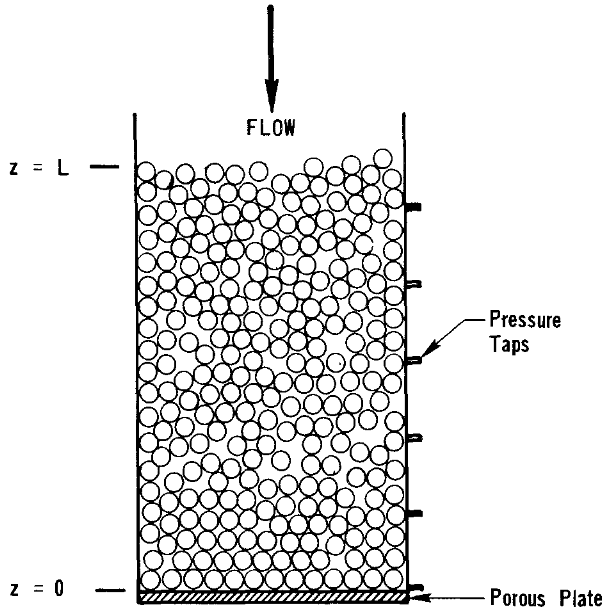

Fig. 1.

While the isotropic, linear elastic solid matrix might not be a predominate feature of practical problems, it does contain the essential feature of all deformable porous media, i.e., the location of the phase interface is a crucial part of the problem. It is this aspect of the problem that has been ignored in all prior work and it is only considered here in terms of small deformation theory. In the first study of deformable porous media, Biot (1941) avoided the closure problem entirely by means of an intuitive homogenization of the governing equations, while later studies have been based on mixture theory and constitutive assumptions (Crochet and Naghdi, 1966). In this work, the closure problem is attacked directly in the manner described in Parts I and II of this paper (Whitaker, 1986a,b), and this allows for a comparison between theory and experiment in terms of parameters describing the geometry. This type of analysis can be extended to nonlinear phenomena, large deformations, and the case in which particle-particle interaction plays a crucial role in the deformation process. However, it is always best to begin a new development with the simplest case, for this provides both a connection with prior work and a well-understood special case with which future studies can be compared.

2. Theory

The general system under consideration is illustrated in Figure 2, and the general problem that we would like to solve can be stated as

$$\frac{\partial \rho_\beta}{\partial t} + \nabla \cdot (\rho_\beta \mathbf{v}_\beta) = 0, \tag{2.1}$$

$$\rho_\beta \frac{\partial \mathbf{v}_\beta}{\partial t} = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla \cdot (\nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta^T) + + (\kappa_\beta - \tfrac{2}{3}\mu_\beta) \nabla (\nabla \cdot \mathbf{v}_\beta) \tag{2.2}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = \mathbf{v}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma} \tag{2.3}$$

$$\text{B.C.2} \quad \mathbf{n}_{\beta\sigma} \cdot \mathbf{T}_\beta = \mathbf{n}_{\beta\sigma} \cdot \mathbf{T}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{2.4}$$

$$\frac{\partial \rho_\sigma}{\partial t} + \nabla \cdot (\rho_\sigma \mathbf{v}_\sigma) = 0, \tag{2.5}$$

$$\rho_\sigma \frac{\partial \mathbf{v}_\sigma}{\partial t} = \rho_\sigma \mathbf{g} + \nabla \cdot \mathbf{T}_\sigma . \tag{2.6}$$

Here we have neglected convective inertial effects with the idea that they will be negligible for most problems of interest. In addition, we have neglected any mechanical effects associated with the particle-particle contact area. These effects, which are primarily electrostatic in nature, are especially important in soils and fine powders and must be taken into account in an extension of the current theory.

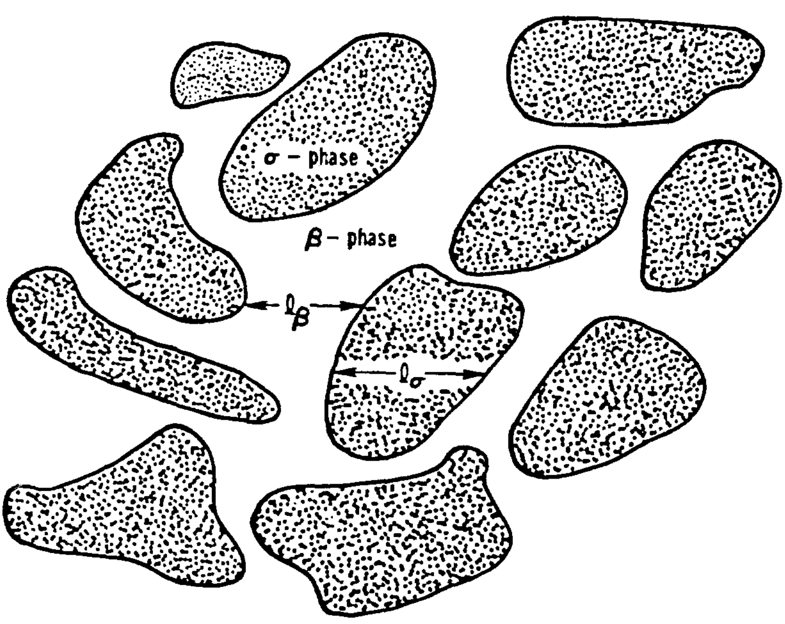

Fig. 2. Two-phase system.

A solution of the problem given by Equations (2.1) through (2.6) would provide information about the propagation of acoustic waves in a saturated, deformable porous media. However, that problem will be part of a subsequent study, and at this point we wish to identify the constraints that must be satisfied in order that the boundary value problem be steady and the flow incompressible. We begin this delineation with the continuity equation for the fluid phase, and express Equation (2.1) as

$$\begin{aligned}\nabla\cdot\mathbf{v}_\beta &= -\left\{\frac{1}{\rho_\beta}\left(\frac{\partial\rho_\beta}{\partial t}\right)+\frac{1}{\rho_\beta}(\mathbf{v}_\beta\cdot\nabla\rho_\beta)\right\}\\ &= \mathbf{O}\left\{\frac{1}{\rho_\beta}\left(\frac{\Delta\rho_\beta}{t}\right),\frac{\mathbf{v}_\beta\Delta\rho_\beta}{\rho_\beta L}\right\}.\end{aligned}\tag{2.7}$$

In proposing these order of magnitude estimates, we have assumed that significant changes in the density occur during the time t and over the distance L. For the wave-propagation problem t would be the inverse of the frequency and L would be the wavelength. The left-hand side of Equation (2.7) consists of three terms of order $\mathbf{v}_\beta/\ell_\beta$ when there is a pressure-driven flow in the porous medium, and for that case Equation (2.7) reduces to

$$\nabla\cdot\mathbf{v}_\beta=0\tag{2.8}$$

when the following two constraints are satisfied

$$\frac{\rho_\beta\mathbf{v}_\beta t}{\Delta\rho_\beta\ell_\beta}\gg 1,\qquad\left(\frac{L}{\ell_\beta}\right)\left(\frac{\rho_\beta}{\Delta\rho_\beta}\right)\gg 1.\tag{2.9}$$

For practical applications one would want to relate $\Delta\rho_\beta$ to the pressure field in order to determine when these constraints are satisfied; however, the form given by Equations (2.9) is sufficient for our purposes.

The fact that the continuity equation takes on the incompressible form when the constraints indicated by Equations (2.9) are imposed does not automatically mean that the compressibility effects in the momentum equation can be discarded. In order that these effects be small in Equation (2.2) we require that

$$\mu_\beta\nabla^2\mathbf{v}_\beta\gg\mu_\beta\nabla(\nabla\cdot\mathbf{v}_\beta)\tag{2.10}$$

since μ_β and κ_β are the same order of magnitude. If we use the estimates

$$\begin{aligned}\nabla^2\mathbf{v}_\beta &= \mathbf{O}(\mathbf{v}_\beta/\ell_\beta^2),\\ \nabla(\nabla\cdot\mathbf{v}_\beta) &= -\nabla\left\{\frac{1}{\rho_\beta}\left(\frac{\partial\rho_\beta}{\partial t}\right)+\frac{1}{\rho_\beta}(\mathbf{v}_\beta\cdot\nabla\rho_\beta)\right\}\\ &= \mathbf{O}\left\{\frac{1}{\rho_\beta}\left(\frac{\Delta\rho_\beta}{Lt}\right),\frac{\mathbf{v}_\beta\Delta\rho_\beta}{\rho_\beta L^2}\right\}\end{aligned}\tag{2.11}$$

we see that Equation (2.10) leads to

$$\frac{\rho_\beta \mathbf{v}_\beta t}{\Delta \rho_\beta \ell_\beta}\left(\frac{L}{\ell_\beta}\right) \gg 1, \qquad \left(\frac{L}{\ell_\beta}\right)^2 \left(\frac{\rho_\beta}{\Delta \rho_\beta}\right) \gg 1. \tag{2.12}$$

These constraints are more easily satisfied than those given by Equations (2.9) and they allow us to simplify Equation (2.2) to

$$\rho_\beta \frac{\partial \mathbf{v}_\beta}{\partial t} = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta . \tag{2.13}$$

In order that the local acceleration terms in Equations (2.2) and (2.6) be negligible, we require

$$\frac{\mu_\beta t}{\rho_\beta \ell_\beta^2} \gg 1, \qquad \frac{E_\sigma t^2}{\rho_\sigma L^2} \gg 1 \tag{2.14}$$

in which E_σ represents Young's modulus for the solid phase. To obtain the first of these we have again assumed that the fluid velocity undergoes significant changes over the small length scale ℓ_β, and we have required that the local acceleration be small compared to the viscous forces. The second of the restrictions given by Equation (2.14) results from the requirement that the local acceleration for the solid phase be small compared to the elastic forces. In estimating the elastic forces we have assumed that significant variations in the *displacement vector field* occur over the macroscopic length scale, L. It is important to note that the no-slip condition for the fluid velocity, $\mathbf{v}_\beta$, gives rise to a characteristic length scale of ℓ_β for the fluid velocity when we are dealing with a process of flow through a porous media. For the displacement vector $\mathbf{u}_\sigma$, there is boundary condition comparable to the no-slip condition, and a little thought will indicate that $\mathbf{u}_\sigma$ undergoes significant variations over the macroscopic length scale. This difference in the characteristic length scales for $\mathbf{v}_\beta$ and $\mathbf{u}_\sigma$ gives rise to a significant difference in the structure of the closure scheme for the solid mechanics problem.

When the constraints given by Equations (2.14) are satisfied (along with those given by Equations (2.9) and (2.12)) the problem under consideration simplifies to

$$4\ \nabla \cdot \mathbf{v}_\beta = 0, \tag{2.15}$$

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta , \tag{2.16}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = \mathbf{v}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{2.17}$$

$$\text{B.C.2} \quad \mathbf{n}_{\beta\sigma} \cdot \mathbf{T}_\beta = \mathbf{n}_{\beta\sigma} \cdot \mathbf{T}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{2.18}$$

$$0 = \rho_\sigma \mathbf{g} + \nabla \cdot \mathbf{T}_\sigma . \tag{2.19}$$

At this point the problem has been reduced to a quasi-steady problem, i.e., no time derivatives appear in the governing differential equations, but the boundary condition given by Equation (2.17) gives rise to a time dependency. We will comment further on this point in subsequent paragraphs; however, at this point we impose the constraint that the intrinsic phase average velocity is large compared to the velocity of the solid phase. The latter can be estimated as $\mathbf{u}_\sigma/t$, and this leads to yet another constraint on the characteristic process time given by

$$\frac{\langle \mathbf{v}_\beta \rangle^\beta t}{\mathbf{u}_\sigma} \gg 1. \tag{2.20}$$

If this constraint is not imposed, the problem is quasi-steady and the volume-averaged equations (which contain the effect of Equation (2.17)) become transient equations. In this analysis, the constraint given by Equation (2.20) will be imposed and the final form of our boundary value problem is given by

$$\nabla \cdot \mathbf{v}_\beta = 0, \tag{2.21}$$

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta, \tag{2.22}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{2.23}$$

$$\text{B.C.2} \quad \mathbf{n}_{\beta\sigma} \cdot \mathbf{T}_\beta = \mathbf{n}_{\beta\sigma} \cdot \mathbf{T}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{2.24}$$

$$0 = \rho_\sigma \mathbf{g} + \nabla \cdot \mathbf{T}_\sigma. \tag{2.25}$$

In this particular problem it is convenient to remove the effect of gravity by use of the definitions

$$\mathbf{T}_\sigma = \mathbf{T}_\sigma^0 + \boldsymbol{\tau}_\sigma, \qquad P_\beta = p_\beta - \rho_\beta \mathbf{g} \cdot \mathbf{r} \tag{2.26}$$

in which $\mathbf{r}$ is the position vector and $\mathbf{T}_\sigma^0$ is the solution for the hydrostatic problem given by

$$\text{B.C.2} \quad -\mathbf{n}_{\beta\sigma} \rho_\beta \mathbf{g} \cdot \mathbf{r} = \mathbf{n}_{\beta\sigma} \cdot \mathbf{T}_\sigma^0, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{2.27}$$

$$0 = \rho_\sigma \mathbf{g} + \nabla \cdot \mathbf{T}_\sigma^0. \tag{2.28}$$

When the fluid phase stress tensor is represented by

$$\mathbf{T}_\beta = -\mathbf{I} p_\beta + \mu_\beta (\nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta^T), \quad \text{at } \mathscr{A}_{\beta\sigma} \tag{2.29}$$

the problem represented by Equations (2.21) through (2.25) can be expressed as

$$\nabla \cdot \mathbf{v}_\beta = 0, \tag{2.30}$$

$$0 = -\nabla P_\beta + \mu_\beta \nabla^2 \mathbf{v}_\beta, \tag{2.31}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{2.32}$$

$$\text{B.C.2} \quad -\mathbf{n}_{\beta\sigma} P_\beta + \mu_\beta \left(\mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta \cdot \mathbf{n}_{\beta\sigma} \right) = \mathbf{n}_{\beta\sigma} \cdot \boldsymbol{\tau}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{2.33}$$

$$0 = \nabla \cdot \boldsymbol{\tau}_\sigma, \tag{2.34}$$

$$\boldsymbol{\tau}_\sigma = \mu_\sigma (\nabla \mathbf{u}_\sigma + \nabla \mathbf{u}^T) + \lambda_\sigma \mathbf{I} \nabla \cdot \mathbf{u}_\sigma, \tag{2.35}$$

Here μ_σ and λ_σ represent the Lamé coefficients, both of which are the same order of magnitude as Young's modulus, E_σ.

It should be clear that, in the steady form, one can solve the flow problem independently, and in order to connect Equations (2.31) and (2.32) with the development given in Part I of this paper (Whitaker, 1986a) a bit of analysis is required. We begin by forming the intrinsic phase average of the second of Equations (2.26) to obtain

$$\langle P_\beta \rangle^\beta = \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} \cdot \langle \mathbf{r} \rangle^\beta . \tag{2.36}$$

The position vector can be expressed as

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\eta} \tag{2.37}$$

in which $\mathbf{r}_0$ locates the centroid of the averaging-volume and $\boldsymbol{\eta}$ locates points in the β-phase contained within the averaging-volume. Use of Equation (2.37) in Equation (2.36) yields

$$\langle P_\beta \rangle^\beta = \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} \cdot \mathbf{r}_0 - \rho_\beta \mathbf{g} \cdot \langle \boldsymbol{\eta} \rangle^\beta \tag{2.38}$$

and from the work of Carbonell and Whitaker ((1984), Section 2) we have the estimate

$$\langle \boldsymbol{\eta} \rangle^\beta = \mathbf{O}(\epsilon_\beta^{-1} r_0^2 \nabla \epsilon_\beta) \tag{2.39}$$

in which r_0 is the radius of the averaging-volume. Use of this result in Equation (2.38) leads to

$$\langle P_\beta \rangle^\beta = \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} \cdot \mathbf{r}_0 + \mathbf{O}[\rho_\beta \mathbf{g} r_0 (r_0/L)] \tag{2.40}$$

where one should keep in mind that, in this case, the length L tends to infinity for a homogeneous system. In systems of practical importance, a pressure change on the order of $\rho_\beta \mathbf{g} r_0$ will be unimportant and the factor of (r_0/L) in Equation (2.40) allows us to use

$$\langle P_\beta \rangle^\beta = \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} \cdot \mathbf{r}, \tag{2.41}$$

Here we have dropped the subscript on the position vector locating the centroid of the averaging volume, since both $\langle P_\beta \rangle^\beta$ and $\langle p_\beta \rangle^\beta$ are evaluated at the centroid and no confusion should result from replacing $\mathbf{r}_0$ with $\mathbf{r}$ in Equation (2.41).

In addition to a relation connecting $\langle P_\beta \rangle^\beta$ and $\langle p_\beta \rangle^\beta$, we need to relate $\tilde{P}_\beta$ to $\tilde{p}_\beta$ so that the solution of the closure problem in Part I can be used in this problem. From the second of Equations (2.26), and the decompositions

$$P_\beta = \langle P_\beta \rangle^\beta + \tilde{P}_\beta , \qquad p_\beta = \langle p_\beta \rangle^\beta + \tilde{p}_\beta \tag{2.42}$$

we can use Equation (2.40) to obtain

$$\tilde{P}_\beta = \tilde{p}_\beta + \mathbf{O}[\rho_\beta \mathbf{g} r_0 (r_0/L)] \tag{2.43}$$

In Part I of this paper, the estimate of $\tilde{p}_\beta$ was given by

$$\tilde{p}_\beta = \mathbf{O}(\mu_\beta \langle \mathbf{v}_\beta \rangle^\beta / \ell_\beta) \tag{2.44}$$

and we can use this to express Equation (2.43) as

$$\tilde{P}_\beta = \tilde{p}_\beta \tag{2.45}$$

whenever the following constraint is satisfied

$$\left[\frac{\mu_\beta \langle \mathbf{v}_\beta \rangle^\beta}{\rho_\beta \mathbf{g} r_0 \ell_\beta}\right]\left[\frac{L}{r_0}\right] \gg 1. \tag{2.46}$$

On the basis of Equations (2.41) and (2.45), we can use the development of Part I to express the solution to Equations (2.30) through (2.32) as

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle = 0, \tag{2.47}$$

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}}{\mu_\beta} \cdot \nabla \langle P_\beta \rangle^\beta, \tag{2.48}$$

$$\tilde{\mathbf{v}}_\beta = \epsilon_\beta^{-1} \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle, \tag{2.49}$$

$$\tilde{P}_\beta = \epsilon_\beta^{-1} \mu_\beta \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle. \tag{2.50}$$

Here $\mathbf{b}$ and $\mathbf{B}$ depend only on the geometry of the system and can be determined by the boundary value problem outlined in Part I, *provided* the position of the β–σ interface is known for some unit-cell representation of the porous medium. In this respect, the fluid motion is coupled to the solid deformation since the Darcy's law permeability tensor will depend on the deformation of the solid phase.

It is of some interest to note that if the constraint indicated by Equation (2.20) is *not* imposed, the problem is quasi-steady and a few lines of analysis will indicate that the volume-averaged form of the continuity equation is given by

$$\frac{\partial \epsilon_\beta}{\partial t} + \nabla \cdot \langle \mathbf{v}_\beta \rangle = 0. \tag{2.51}$$

This is the form used by Biot ((1941), Equation 4.3) and it is quite correct for the quasi-steady problem. In a later paper, Biot ((1955), Equation 2.11) also incorporated the boundary condition given by Equation (2.17) into Darcy's law to obtain a result resembling

$$\langle \mathbf{v}_\beta \rangle^\beta = -\frac{\epsilon_\beta^{-1}\mathbf{K}}{\mu_\beta} \cdot \nabla \langle P_\beta \rangle^\beta + \langle \mathbf{v}_\sigma \rangle^\sigma. \tag{2.52}$$

This form of Darcy's law is derivable using the method outlined in Part I of this paper, provided the following constraint is satisfied

$$\tilde{\mathbf{v}}_\sigma \ll \langle \mathbf{v}_\sigma \rangle^\sigma. \tag{2.53}$$

Under these circumstances Equation (3.3) of Part I is replaced by

$$\text{B.C.1} \quad \tilde{\mathbf{v}}_\beta = \langle \mathbf{v}_\sigma \rangle^\sigma - \langle \mathbf{v}_\beta \rangle^\beta, \quad \text{on } \mathscr{A}_{\beta\sigma} \tag{2.54}$$

and the analysis can be repeated to obtain

$$\langle \mathbf{v}_\beta \rangle^\beta = -\frac{\mathbf{C}^{-1}}{\mu_\beta} \cdot [\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}] + \langle \mathbf{v}_\sigma \rangle^\sigma \tag{2.55}$$

instead of Equation (3.36) in Part I. Some justification for the restriction given by Equation (2.53) is provided in Section 4. In this development we will impose the constraints given by Equations (2.9), (2.12), (2.14) and (2.20) so that the analysis is restricted to steady, incompressible flow in the presence of time-independent deformation.

3. Volume-Averaging for the Solid Phase

Having dispensed with the analysis of the fluid phase in terms of the analysis given in Part I of this paper, we turn our attention to the solid phase. The equations under consideration are given by

$$\nabla \cdot \boldsymbol{\tau}_\sigma = 0, \tag{3.1}$$

$$\boldsymbol{\tau}_\sigma = \mu_\sigma(\nabla \mathbf{u}_\sigma + \nabla \mathbf{u}_\sigma^T) + \lambda_\sigma \mathbf{I} \nabla \cdot \mathbf{u}_\sigma \tag{3.2}$$

and we would like to develop the volume-averaged form of these equations, in addition to the displacement equation which is given by

$$\mu_\sigma \nabla^2 \mathbf{u}_\sigma + (\mu_\sigma + \lambda_\sigma) \nabla (\nabla \cdot \mathbf{u}_\sigma) = 0. \tag{3.3}$$

Here we have assumed that the Lamé coefficients can be treated as constants. The averaging procedure for Equation (3.3) follows that given in Section 2 of Part I, and we simply note here that the phase average form of Equation (3.3) is given by

$$\begin{aligned} &\mu_\sigma \left\{ \nabla \cdot \left[\nabla \langle \mathbf{u}_\sigma \rangle + \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \mathbf{u}_\sigma \, \mathrm{d}A \right] + \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \cdot \nabla \mathbf{u}_\sigma \, \mathrm{d}A \right\} + \\ &+ (\mu_\sigma + \lambda_\sigma) \left\{ \nabla \left[\nabla \cdot \langle \mathbf{u}_\sigma \rangle + \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \cdot \mathbf{u}_\sigma \, \mathrm{d}A \right] + \right. \\ &\left. + \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \nabla \cdot \mathbf{u}_\sigma \, \mathrm{d}A \right\} = 0. \end{aligned} \tag{3.4}$$

While it is convenient to work with the *phase average* of the fluid velocity, we require the *intrinsic phase average* of the displacement vector field. Thus we make use of

$$\langle \mathbf{u}_\sigma \rangle = \epsilon_\sigma \langle \mathbf{u}_\sigma \rangle^\sigma, \qquad \mathbf{u}_\sigma = \langle \mathbf{u}_\sigma \rangle^\sigma + \tilde{\mathbf{u}}_\sigma \tag{3.5}$$

and follow the development in Section 2 of Part I in order to express Equation

(3.4) as

$$
\begin{aligned}
&\mu_\sigma\left\{\nabla\cdot\left[\epsilon_\sigma\nabla\langle\mathbf{u}_\sigma\rangle^\sigma+\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A\right]+\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\nabla\mathbf{u}_\sigma\,\mathrm{d}A\right\}+\\
&\quad+(\mu_\sigma+\lambda_\sigma)\left\{\nabla\left[\epsilon_\sigma\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma+\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A\right]+\right.\\
&\quad\left.+\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\nabla\cdot\mathbf{u}_\sigma\,\mathrm{d}A\right\}=0.
\end{aligned}
\tag{3.6}
$$

Here we have used the approximation

$$
\frac{1}{\mathscr{V}}\;{}_{A_{\sigma\beta}}\;\mathbf{n}_{\sigma\beta}\langle\mathbf{u}_\sigma\rangle^\sigma\,\mathrm{d}A=\frac{1}{\mathscr{V}}\;{}_{A_{\sigma\beta}}\;\mathbf{n}_{\sigma\beta}\,\mathrm{d}A\;\;\langle\mathbf{u}_\sigma\rangle^\sigma
\tag{3.7}
$$

on the basis of the length scale constraint given by Equation (2.19) in Part I. We continue this line of analysis to obtain

$$
\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\nabla\mathbf{u}_\sigma\,\mathrm{d}A=\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\nabla\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A-\nabla\epsilon_\sigma\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma
\tag{3.8}
$$

so that Equation (3.6) can be simplified to

$$
\begin{aligned}
&\mu_\sigma\left\{\epsilon_\sigma\nabla^2\langle\mathbf{u}_\sigma\rangle^\sigma+\nabla\cdot\left[\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A\right]+\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\nabla\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A\right\}+\\
&\quad+(\mu_\sigma+\lambda_\sigma)\left\{\epsilon_\sigma\nabla\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma+\nabla\left[\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A\right]+\right.\\
&\quad\left.+\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\nabla\cdot\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A\right\}=0.
\end{aligned}
\tag{3.9}
$$

This completes the spatial smoothing of Equation (3.3); however, we also need similar forms for Equations (3.1) and (3.2). The averaging procedure for Equation (3.2) is identical to that presented above and leads to

$$
\begin{aligned}
\langle\boldsymbol{\tau}_\sigma\rangle^\sigma=\;&\mu_\sigma\left\{\nabla\langle\mathbf{u}_\sigma\rangle^\sigma+(\nabla\langle\mathbf{u}_\sigma\rangle^\sigma)^T+\frac{\epsilon_\sigma^{-1}}{\mathscr{V}}\int_{A_{\sigma\beta}}(\mathbf{n}_{\sigma\beta}\tilde{\mathbf{u}}_\sigma+\tilde{\mathbf{u}}_\sigma\mathbf{n}_{\sigma\beta})\,\mathrm{d}A\right\}+\\
&+\lambda_\sigma\left\{\mathbf{I}\left[\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma+\frac{\epsilon_\sigma^{-1}}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\tilde{\mathbf{u}}_\sigma\,\mathrm{d}A\right]\right\}.
\end{aligned}
\tag{3.10}
$$

The averaging procedure for Equation (3.1) begins with the *phase average* form given by

$$
\nabla\cdot\langle\boldsymbol{\tau}_\sigma\rangle+\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}}\mathbf{n}_{\sigma\beta}\cdot\boldsymbol{\tau}_\sigma\,\mathrm{d}A=0
\tag{3.11}
$$

and one makes use of

$$\langle \boldsymbol{\tau}_\sigma \rangle = \epsilon_\sigma \langle \boldsymbol{\tau}_\sigma \rangle^\sigma, \qquad \boldsymbol{\tau}_\sigma = \langle \boldsymbol{\tau}_\sigma \rangle^\sigma + \tilde{\boldsymbol{\tau}}_\sigma \tag{3.12}$$

to obtain

$$\nabla \cdot \langle \boldsymbol{\tau}_\sigma \rangle^\sigma + \frac{\epsilon_\sigma^{-1}}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \cdot \tilde{\boldsymbol{\tau}}_\sigma \, \mathrm{d}A = 0. \tag{3.13}$$

Here we have used Equation (2.20) of Part I. From Equations (3.2) and (3.10) we find that the spatial deviation of $\boldsymbol{\tau}_\sigma$ is given by

$$\begin{aligned} \tilde{\boldsymbol{\tau}}_\sigma &= \mu_\sigma(\nabla \tilde{\mathbf{u}}_\sigma + \nabla \tilde{\mathbf{u}}_\sigma^T) + \lambda_\sigma \mathbf{I} \nabla \cdot \tilde{\mathbf{u}}_\sigma - \\ &- \mu_\sigma \left\{ \frac{\epsilon_\sigma^{-1}}{\mathscr{V}} \int_{A_{\sigma\beta}} (\mathbf{n}_{\sigma\beta} \tilde{\mathbf{u}}_\sigma + \tilde{\mathbf{u}}_\sigma \mathbf{n}_{\sigma\beta}) \, \mathrm{d}A \right\} - \lambda_\sigma \mathbf{I} \left\{ \frac{\epsilon_\sigma^{-1}}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \cdot \tilde{\mathbf{u}}_\sigma \, \mathrm{d}A \right\}. \end{aligned} \tag{3.14}$$

When this result is used in Equation (3.13) we treat the area integrals and ϵ_σ as constants with respect to integration over $A_{\sigma\beta}$ in order to obtain

$$\begin{aligned} &\nabla \cdot \langle \boldsymbol{\tau}_\sigma \rangle^\sigma + \mu_\sigma \left\{ \frac{\epsilon_\sigma^{-1}}{\mathscr{V}} \int_{A_{\sigma\beta}} (\mathbf{n}_{\sigma\beta} \cdot \nabla \tilde{\mathbf{u}}_\sigma + \nabla \tilde{\mathbf{u}}_\sigma \cdot n_{\sigma\beta}) \, \mathrm{d}A \right\} + \\ &\quad + \lambda_\sigma \left\{ \frac{\epsilon_\sigma^{-1}}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \nabla \cdot \tilde{\mathbf{u}}_\sigma \, \mathrm{d}A \right\} + \\ &\quad + \mu_\sigma \epsilon_\sigma^{-2} \nabla \epsilon_\sigma \cdot \left\{ \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} (\mathbf{n}_{\sigma\beta} \tilde{\mathbf{u}}_\sigma + \tilde{\mathbf{u}}_\sigma \mathbf{n}_{\sigma\beta}) \, \mathrm{d}A \right\} + \\ &\quad + \lambda_\sigma \epsilon_\sigma^{-2} \nabla \epsilon_\sigma \left\{ \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} \cdot \tilde{\mathbf{u}}_\sigma \, \mathrm{d}A \right\} = 0. \end{aligned} \tag{3.15}$$

In order to use Equations (3.9), (3.10) and (3.15) to determine the volume-averaged stress and displacement, we need a representation for $\tilde{\mathbf{u}}_\sigma$ in terms of the dependent variables. This will be provided by the closure scheme which must, in addition, provide a means for locating the β–σ interface so that the Darcy's law permeability tensor can be calculated using the closure scheme for the β-phase momentum transfer problem.

4. Closure

The governing differential equation for $\tilde{\mathbf{u}}_\sigma$ is obtained using the approach outlined by Crapiste *et al.* (1985) and illustrated in Section 3 of Part I of this paper. The result is given by

$$\mu_\sigma \nabla^2 \tilde{\mathbf{u}}_\sigma + (\mu_\sigma + \lambda_\sigma) \nabla (\nabla \cdot \tilde{\mathbf{u}}_\sigma) = \frac{1}{V_\sigma} \int_{V_\sigma} [\mu_\sigma \nabla^2 \tilde{\mathbf{u}}_\sigma + (\mu_\sigma + \lambda_\sigma) \nabla (\nabla \cdot \tilde{\mathbf{u}}_\sigma)] \, \mathrm{d}V \tag{4.1}$$

and we need only develop the boundary condition at the β–σ interface to complete the formulation of the closure scheme. Use of Equation (2.35) in Equation (2.33) along with the obvious decompositions for P_β, $\mathbf{v}_\beta$ and $\mathbf{u}_\sigma$ leads to

$$\begin{aligned}
&-\mathbf{n}_{\beta\sigma}\langle P_\beta\rangle^\beta - \mathbf{n}_{\beta\sigma}\tilde{P}_\beta + \mu_\beta\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\langle\mathbf{v}_\beta\rangle^\beta + \nabla\langle\mathbf{v}_\beta\rangle^\beta\cdot\mathbf{n}_{\beta\sigma}\Big) + \\
&\quad + \mu_\beta\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{v}}_\beta + \nabla\tilde{\mathbf{v}}_\beta\cdot\mathbf{n}_{\beta\sigma}\Big) \\
&\quad = \mu_\sigma\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + \nabla\langle\mathbf{u}_\sigma\rangle^\sigma\cdot\mathbf{n}_{\beta\sigma}\Big) + \\
&\quad + \mu_\sigma\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{u}}_\sigma + \nabla\tilde{\mathbf{u}}_\sigma\cdot\mathbf{n}_{\beta\sigma}\Big) + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma + \\
&\quad + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\tilde{\mathbf{u}}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma}.
\end{aligned} \tag{4.2}$$

The no-slip condition given by Equation (2.32), along with the length scales associated with $\tilde{\mathbf{v}}_\beta$ and $\langle\mathbf{v}_\beta\rangle^\beta$, allow for the simplification

$$\nabla\tilde{\mathbf{v}}_\beta \gg \nabla\langle\mathbf{v}_\beta\rangle^\beta \tag{4.3}$$

and Equation (4.2) takes the form

$$\begin{aligned}
&-\mathbf{n}_{\beta\sigma}\langle P_\beta\rangle^\beta - \mathbf{n}_{\beta\sigma}\tilde{P}_\beta + \mu_\beta\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{v}}_\beta + \nabla\tilde{\mathbf{v}}_\beta\cdot\mathbf{n}_{\beta\sigma}\Big) \\
&\quad = \mu_\sigma\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + \nabla\langle\mathbf{u}_\sigma\rangle^\sigma\cdot\mathbf{n}_{\beta\sigma}\Big) + \mu_\sigma\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{u}}_\sigma + \nabla\tilde{\mathbf{u}}_\sigma\cdot\mathbf{n}_{\beta\sigma}\Big) + \\
&\quad + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\tilde{\mathbf{u}}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma},
\end{aligned} \tag{4.4}$$

It is important to note that a similar simplification with respect to the displacement vector field *does not exist* since there is no constraint for $\mathbf{u}_\sigma$ that is similar to that given for the velocity by Equation (2.32). However, there are some comments that we can make concerning the relative magnitudes of $\langle\mathbf{u}_\sigma\rangle^\sigma$ and $\tilde{\mathbf{u}}_\sigma$, and this is as good a place as any to make them.

If we think of a porous system made up of solid spheres and air undergoing the compression process illustrated in Figure 3, one can argue that the stress at the β–σ interface is negligible, i.e.,

$$\mathbf{n}_{\beta\sigma}\cdot\mathbf{T}_\beta = 0, \quad \text{at } \mathscr{A}_{\beta\sigma}. \tag{4.5}$$

Under these circumstances, the boundary condition given by Equation (4.2) can be expressed as

$$\begin{aligned}
&\mu_\sigma\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{u}}_\sigma + \nabla\tilde{\mathbf{u}}_\sigma\cdot\mathbf{n}_{\beta\sigma}\Big) + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\tilde{\mathbf{u}}_\sigma \\
&\quad = -\Big[\mu_\sigma\Big(\mathbf{n}_{\beta\sigma}\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + \nabla\langle\mathbf{u}_\sigma\rangle^\sigma\cdot\mathbf{n}_{\beta\sigma}\Big) + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma\Big], \quad \mathscr{A}_{\beta\sigma}.
\end{aligned} \tag{4.6}$$

This leads to the order of magnitude estimate given by

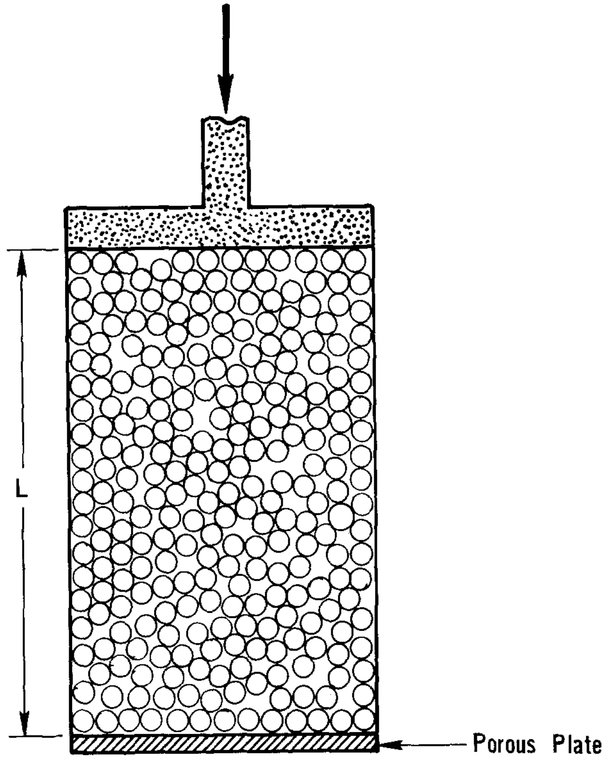

Fig. 3. Compression for an elastic porous medium.

$$\nabla \tilde{\mathbf{u}}_\sigma = \mathbf{O}(\nabla \langle \mathbf{u}_\sigma \rangle^\sigma) \tag{4.7}$$

and we conclude that the spatial deviation of the displacement vector is related to the intrinsic phase average value by

$$\tilde{\mathbf{u}}_\sigma = \mathbf{O}\left[\left(\frac{\ell_\sigma}{L}\right)\langle \mathbf{u}_\sigma \rangle^\sigma\right]. \tag{4.8}$$

This result is analogous to the situation that occurs during diffusion in porous media (Ryan *et al.*, 1981), for in that case there is no boundary condition comparable to the no-slip condition and one arrives at a condition for the concentration field that is comparable to Equation (4.8). As one might expect, a similar situation is encountered in the analysis of heat conduction in multiphase systems (Nozad *et al.*, 1985).

Since Equation (4.8) requires that

$$\tilde{\mathbf{u}}_\sigma \ll \langle \mathbf{u}_\sigma \rangle^\sigma \tag{4.9}$$

there will be certain simplifications available to us; however, Equation (4.9) does not mean that terms involving $\tilde{\mathbf{u}}_\sigma$ can be discarded carelessly. In writing an inequality such as that given by Equation (4.9) for a vector quantity, the intent is that the *largest component* of $\langle \mathbf{u}_\sigma \rangle^\sigma$ is much, much greater than *any component* of $\tilde{\mathbf{u}}_\sigma$. For the case of pure extension this means that

$$\tilde{u}_{\sigma z} \ll \langle u_\sigma \rangle_z^\sigma, \qquad \tilde{u}_{\sigma r} \ll \langle u_\sigma \rangle_z^\sigma \tag{4.10}$$

but it does not mean that $\tilde{u}_{\sigma r}$ is small compared to $\langle u_\sigma \rangle_r^\sigma$ since the latter is zero. It is of some interest to note that if the time scales for $\tilde{\mathbf{u}}_\sigma$ and $\langle \mathbf{u}_\sigma \rangle^\sigma$ are similar, Equation (4.9) immediately translates into Equation (2.53) thus supporting the modified form of Darcy's law given by Equation (2.55).

We now return to the boundary condition given by Equation (4.4) and follow a line of analysis that parallels our study of the stress condition for two-phase flow in a rigid porous medium. Following Equation (3.12) of Part II (Whitaker, 1986b), we take the *area-average* of the normal component of Equation (4.4) and consider the volume-averaged quantities to be constant with respect to integration over $A_{\beta\sigma}$. This leads to

$$\begin{aligned} -\langle P_\beta \rangle^\beta &+ \langle -\tilde{P}_\beta + 2\mu_\beta \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta \cdot \mathbf{n}_{\beta\sigma} \rangle \\ &= 2\mu_\sigma \mathbf{G} : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \lambda_\sigma \nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma + \\ &\quad + \langle 2\mu_\sigma \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{u}}_\sigma \cdot \mathbf{n}_{\beta\sigma} \rangle_{\beta\sigma} + \langle \lambda_\sigma \nabla \cdot \tilde{\mathbf{u}}_\sigma \rangle_{\beta\sigma} . \end{aligned} \tag{4.11}$$

Here the symmetric tensor $\mathbf{G}$ depends only on the *geometry* of the β–σ interface and is given explicitly by

$$\mathbf{G} = \frac{1}{A_{\beta\sigma}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \mathbf{n}_{\beta\sigma} \, \mathrm{d}A. \tag{4.12}$$

At this point we put forth a *plausible approximation* represented by the following inequality

$$\langle \nabla \tilde{\mathbf{u}}_\sigma \rangle_{\beta\sigma} \ll \nabla \langle \mathbf{u}_\sigma \rangle^\sigma. \tag{4.13}$$

The thought here is that the area-average of a deviation will be much, much less than the order of magnitude of the deviation, i.e.,

$$\langle \nabla \tilde{\mathbf{u}}_\sigma \rangle_{\beta\sigma} \ll \nabla \tilde{\mathbf{u}}_\sigma \tag{4.14}$$

and when this result is used with Equation (4.7) we obtain Equation (4.13). This allows us to express Equation (4.11) as

$$\begin{aligned} -\langle P_\beta \rangle^\beta &= 2\mu_\sigma \mathbf{G} : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \lambda_\sigma \nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma + \\ &\quad + \langle \tilde{P}_\beta - 2\mu_\beta \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta \cdot \mathbf{n}_{\beta\sigma} \rangle_{\beta\sigma} . \end{aligned} \tag{4.15}$$

Use of this result in Equation (4.4) allows us to eliminate the volume-averaged pressure from the interfacial stress condition, and Equation (4.4) takes the form

$$\begin{aligned} -\mathbf{n}_{\beta\sigma} \tilde{P}_\beta &+ \mu_\beta (\mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta + \nabla \tilde{\mathbf{v}}_\beta \cdot \mathbf{n}_{\beta\sigma}) - \mathbf{n}_{\beta\sigma} \langle -\tilde{P}_\beta + 2\mu_\beta \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta \cdot \mathbf{n}_{\beta\sigma} \rangle_{\beta\sigma} \\ &= \mu_\sigma (\mathbf{n}_{\beta\sigma} \cdot \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \cdot \mathbf{n}_{\beta\sigma}) - 2\mu_\sigma \mathbf{G} : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \mathbf{n}_{\beta\sigma} + \\ &\quad + \mu_\sigma (\mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{u}}_\sigma + \nabla \tilde{\mathbf{u}}_\sigma \cdot \mathbf{n}_{\beta\sigma}) + \lambda_\sigma \mathbf{n}_{\beta\sigma} \nabla \cdot \tilde{\mathbf{u}}_\sigma \end{aligned} \tag{4.16}$$

The point of view illustrated by Equation (4.13) can again be used to extract the obvious simplification from Equation (4.16) which is given by

$$\begin{aligned}
&-\mathbf{n}_{\beta\sigma}\tilde{P}_\beta + \mu_\beta(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{v}}_\beta + \nabla\tilde{\mathbf{v}}_\beta\cdot\mathbf{n}_{\beta\sigma}) + 2\mu_\sigma\mathbf{G}:\nabla\langle\mathbf{u}_\sigma\rangle^\sigma\mathbf{n}_{\beta\sigma} - \\
&\quad - \mu_\sigma(\mathbf{n}_{\beta\sigma}\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + \nabla\langle\mathbf{u}_\sigma\rangle^\sigma\cdot\mathbf{n}_{\beta\sigma}) \\
&\quad = \mu_\sigma(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{u}}_\sigma + \nabla\tilde{\mathbf{u}}_\sigma\cdot\mathbf{n}_{\beta\sigma}) + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\tilde{\mathbf{u}}_\sigma, \quad \text{at } \mathscr{A}_{\beta\sigma}.
\end{aligned} \tag{4.17}$$

The pressure and velocity deviations can now be expressed in terms of the phase average velocity by means of Equations (2.43) and (2.44), and this leads to

$$\begin{aligned}
&\mu_\sigma\left(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{u}}_\sigma + \nabla\tilde{\mathbf{u}}_\sigma\cdot\mathbf{n}_{\beta\sigma}\right) + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\tilde{\mathbf{u}}_\sigma \\
&\quad = \mu_\beta\mathbf{H}_\beta\cdot\langle\mathbf{v}_\beta\rangle + 2\mu_\sigma\mathbf{G}:\nabla\langle\mathbf{u}_\sigma\rangle^\sigma\mathbf{n}_{\beta\sigma} \\
&\quad\quad - \mu_\sigma\left(\mathbf{n}_{\beta\sigma}\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + \nabla\langle\mathbf{u}_\sigma\rangle^\sigma\cdot\mathbf{n}_{\beta\sigma}\right), \quad \text{at } \mathscr{A}_{\beta\sigma}.
\end{aligned} \tag{4.18}$$

Here the tensor $\mathbf{H}_\beta$ is associated with the *hydrodynamic* effects and is defined by

$$\mathbf{H}_\beta = \epsilon_\beta^{-1}\left(-\mathbf{n}_{\beta\sigma}\mathbf{b} + \mathbf{n}_{\beta\sigma}\cdot\nabla\mathbf{B} + \nabla\mathbf{B}^T\cdot\mathbf{n}_{\beta\sigma}\right). \tag{4.19}$$

It should be kept in mind that $\mathbf{H}_\beta$ is available from the solution to the closure problem described by Equations (3.13) through (3.17) in Part I.

At this point we note that the governing differential equation for $\tilde{\mathbf{u}}_\sigma$ (given by Equation (4.1)) is homogeneous, while the nonhomogeneities in the boundary condition given by Equation (4.18) can be represented in terms of $\langle\mathbf{v}_\beta\rangle$ and $\nabla\langle\mathbf{u}_\sigma\rangle^\sigma$. Without going into the details that were covered in Parts I and II, we simply note that $\tilde{\mathbf{u}}_\sigma$ can be expressed as

$$\tilde{\mathbf{u}}_\sigma = \mathbf{C}\cdot\langle\mathbf{v}_\beta\rangle + \mathscr{D}:\nabla\langle\mathbf{u}_\sigma\rangle^\sigma \tag{4.20}$$

in which $\mathbf{C}$ is a second-order tensor relating $\tilde{\mathbf{u}}_\sigma$ to the phase average velocity and $\mathscr{D}$ is a third-order tensor relating $\tilde{\mathbf{u}}_\sigma$ to the gradient of the intrinsic phase average displacement vector field.

Since the closure problem will be solved in terms of a unit cell, the boundary value problem for the second-order tensor $\mathbf{C}$ is given by

$$\begin{aligned}
&\mu_\sigma\nabla^2\mathbf{C} + (\mu_\sigma + \lambda_\sigma)\nabla(\nabla\cdot\mathbf{C}) \\
&\quad = \frac{1}{V_\sigma}\int_{V_\sigma}[\mu_\sigma\nabla^2\mathbf{C} + (\mu_\sigma + \lambda_\sigma)\nabla(\nabla\cdot\mathbf{C})]\,\mathrm{d}V,
\end{aligned} \tag{4.21}$$

B.C.1 $\quad \mu_\sigma\left(\mathbf{n}_{\beta\sigma}\cdot\nabla\mathbf{C} + \nabla\mathbf{C}^T\cdot\mathbf{n}_{\beta\sigma}\right) + \lambda_\sigma\mathbf{n}_{\beta\sigma}\nabla\cdot\mathbf{C} = \mu_\beta\mathbf{H}_\beta, \quad \text{at } A_{\beta\sigma}, \quad (4.22)$

B.C.2 $\quad \mathbf{C}(\mathbf{r} + \ell_i) = \mathbf{C}(\mathbf{r}), \quad i = 1, 2, 3, \quad (4.23)$

$$\langle\mathbf{C}\rangle^\sigma = 0. \tag{4.24}$$

The governing equation for the third-order tensor $\mathscr{D}$ is identical in form to Equation (4.21) and is given by

$$\mu_\sigma \nabla^2 \mathscr{D} + (\mu_\sigma + \lambda_\sigma) \nabla (\nabla \cdot \mathscr{D}) = \frac{1}{V_\sigma} \int_{V_\sigma} \left[\mu_\sigma \nabla^2 \mathscr{D} + (\mu_\sigma + \lambda_\sigma) \nabla (\nabla \cdot \mathscr{D}) \right] \mathrm{d}V \tag{4.25}$$

however, the boundary condition at the β–σ interface can not be conveniently expressed in dyadic notation. For clarity, we use Cartesian tensor index notation to obtain

$$\text{B.C.1} \quad \mu_\sigma \left[n_i \left(\frac{\partial D_{jk\ell}}{\partial x_i} \right) + \left(\frac{\partial D_{ik\ell}}{\partial x_j} \right) n_i \right] + \lambda_\sigma n_j \left(\frac{\partial D_{ik\ell}}{\partial x_i} \right)$$
$$= -\mu_\sigma (n_k \delta_{j\ell} + \delta_{jk} n_\ell) + 2\mu_\sigma n_j G_{\ell k}, \quad \text{at } A_{\beta\sigma}. \tag{4.26}$$

Here we have used the representations

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}, \qquad \mathbf{n}_{\beta\sigma} = \mathbf{e}_i n_i, \qquad \mathscr{D} = \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k D_{ijk}$$
$$\mathbf{I} = \mathbf{e}_i \mathbf{e}_j \delta_{ij}, \qquad \mathbf{G} = \mathbf{e}_i \mathbf{e}_j G_{ij} \tag{4.27}$$

in which the summation convention has been used throughout. The vector $\mathbf{e}_i$ represents the rectangular, Cartesian base vectors, and in terms of the traditional nomenclature we express $\mathbf{e}_i$ as

$$\mathbf{e}_1 = \mathbf{i}, \qquad \mathbf{e}_2 = \mathbf{j}, \qquad \mathbf{e}_3 = \mathbf{k}.$$

The periodicity condition provides the remaining boundary condition

$$\text{B.C.2} \quad \mathscr{D}(\mathbf{r} + \ell_i) = \mathscr{D}(\mathbf{r}), \quad i = 1, 2, 3 \tag{4.28}$$

and we again require that the average be zero.

$$\langle \mathscr{D} \rangle^\sigma = 0. \tag{4.29}$$

While the closure problem for deformable media might seem to be excessively complex, the typical one-dimensional problem would require the determination of only three components of $\mathbf{C}$ and three components of $\mathscr{D}$ in addition to the solution of the hydrodynamic closure problem.

If *shear deformation* of the solid phase can be neglected at the volume-averaged level, Equation (4.18) simplifies to

$$\mu_\sigma \left(\mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{u}}_\sigma + \nabla \tilde{\mathbf{u}}_\sigma \cdot \mathbf{n}_{\beta\sigma} \right) + \lambda_\sigma \mathbf{n}_{\beta\sigma} \nabla \cdot \tilde{\mathbf{u}}_\sigma$$
$$= \mu_\beta \mathbf{H}_\beta \cdot \langle \mathbf{v}_\beta \rangle + \tfrac{2}{3} \mu_\sigma \mathbf{n}_{\beta\sigma} (\nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma)(\mathbf{G} : \mathbf{I} - 1) \tag{4.30}$$

and the third-order tensor $\mathscr{D}$ is reduced to

$$\mathscr{D} = \mathbf{f}\mathbf{I}. \tag{4.31}$$

Under these circumstances, Equation (4.20) simplifies to

$$\tilde{\mathbf{u}}_\sigma = \mathbf{C} \cdot \langle \mathbf{v}_\beta \rangle + \mathbf{f} \left(\nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma \right), \quad \text{negligible macroscopic shear} \tag{4.32}$$

and the boundary value problem for $\mathbf{f}$ is given by

$$\mu_\sigma \nabla^2 \mathbf{f} + (\mu_\sigma + \lambda_\sigma) \nabla (\nabla \cdot \mathbf{f}) = \frac{1}{V_\sigma} \int_{V_\sigma} \Big[\mu_\sigma \nabla^2 \mathbf{f} + (\mu_\sigma + \lambda_\sigma) \nabla (\nabla \cdot \mathbf{f}) \Big] \,\mathrm{d}V, \tag{4.33}$$

$$\text{B.C.1} \quad \mu_\sigma \Big(\mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{f} + \nabla \mathbf{f} \cdot \mathbf{n}_{\beta\sigma} \Big) + \lambda_\sigma \mathbf{n}_{\beta\sigma} (\nabla \cdot \mathbf{f})$$

$$= \tfrac{2}{3} \mu_\sigma \mathbf{n}_{\beta\sigma} (\mathbf{G} : \mathbf{I} - 1), \quad \text{at } A_{\beta\sigma}, \tag{4.34}$$

$$\text{B.C.2} \quad \mathbf{f}(\mathbf{r} + \ell_i) = \mathbf{f}(\mathbf{r}), \quad i = 1, 2, 3, \tag{4.35}$$

$$\langle \mathbf{f} \rangle^\sigma = 0. \tag{4.36}$$

Closure problems of this type have been solved by Ryan *et al.* (1981) in a study of diffusion and reaction in porous media, by Eidsath *et al.* (1983) in an analysis of dispersion in porous media, and most recently by Nozad *et al.* (1985) in an investigation of heat conduction in multiphase systems.

It is of some interest to note that if the solid phase is composed of *isotropic particles*, the symmetric tensor $\mathbf{G}$ takes the form

$$\mathbf{G} = \tfrac{1}{3} \mathbf{I}, \text{ for isotropic particles} \tag{4.37}$$

and the boundary condition given by Equation (4.34) is replaced with

$$\text{B.C.1}' \quad \mu_\sigma \Big(\mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{f} + \nabla \mathbf{f} \cdot \mathbf{n}_{\beta\sigma} \Big) + \lambda_\sigma \mathbf{n}_{\beta\sigma} (\nabla \cdot \mathbf{f}) = 0, \quad \text{at } A_{\beta\sigma}. \tag{4.38}$$

At this point one can follow the type of development given in Appendix B of Part I or the Appendix of Part II (Whitaker, 1986a, b) in order to prove that $\mathbf{f}$ is zero. Under these circumstances the deviation displacement vector $\tilde{\mathbf{u}}_\sigma$ is independent of the deformation $\nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma$, and this indicates that the *geometry* of the β–σ interface plays a crucial role in the closure problem. It is easy to show that Equation (4.37) is true for spheres and cubes, and a little throught will suggest that it is true for any regular polyhedron in 3-space.

We now return to the general problem, as illustrated by Equation (4.20), and make use of that result in Equation (3.9). In doing so, we repeatedly make use of approximations of the type

$$\nabla \tilde{\mathbf{u}}_\sigma = \nabla \mathbf{C} \cdot \langle \mathbf{v}_\beta \rangle + \nabla \mathcal{D} : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \tag{4.39}$$

and the basis of the length scale constraints discussed in Part I of this paper. The closed form of Equation (3.9) is given by

$$\begin{aligned} &\mu_\sigma \Big\{ \epsilon_\sigma \nabla^2 \langle \mathbf{u}_\sigma \rangle^\sigma + \nabla \cdot \Big(H_C^{(3)} \cdot \langle \mathbf{v}_\beta \rangle \Big) + \nabla \cdot \Big(M_C^{(4)} \colon \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \Big) + \\ &\quad + H_A^{(2)} \cdot \langle \mathbf{v}_\beta \rangle + M_A^{(3)} \colon \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \Big\} + \\ &\quad + (\mu_\sigma + \lambda_\sigma) \Big\{ \epsilon_\sigma \nabla (\nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma) + \nabla \Big(H_D^{(1)} \cdot \langle \mathbf{v}_\beta \rangle \Big) + \nabla \Big(M_D^{(2)} \colon \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \Big) + \\ &\quad + H_B^{(2)} \cdot \langle \mathbf{v}_\beta \rangle + M_B^{(3)} \colon \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \Big\} = 0. \end{aligned} \tag{4.40}$$

Here we have used a superscript in parenthesis to indicate the order of the various tensor coefficients that are defined by

$$
\begin{aligned}
H_A^{(2)} &= \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\cdot\nabla\mathbf{C}\,\mathrm{d}A, \qquad H_B^{(2)} = \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\nabla\cdot\mathbf{C}\,\mathrm{d}A,\\
H_C^{(3)} &= \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\mathbf{C}\,\mathrm{d}A, \qquad H_D^{(1)} = \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\cdot\mathbf{C}\,\mathrm{d}A,
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
M_A^{(3)} &= \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\cdot\nabla\mathcal{D}\,\mathrm{d}A, \qquad M_B^{(3)} = \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\nabla\cdot\mathcal{D}\,\mathrm{d}A,\\
M_C^{(4)} &= \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\mathcal{D}\,\mathrm{d}A, \qquad M_D^{(2)} = \frac{1}{\mathcal{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\cdot\mathcal{D}\,\mathrm{d}A.
\end{aligned}
\tag{4.42}
$$

If we require the tensor coefficients in Equation (4.40) to be *isotropic* and *constant*, we obtain

$$
\begin{aligned}
&\mu_\sigma(\epsilon_\sigma + m_2)\nabla^2\langle\mathbf{u}_\sigma\rangle^\sigma +\\
&+\Big[\mu_\sigma(m_1+m_2) + (\mu_\sigma+\lambda_\sigma)(\epsilon_\sigma + M_D)\Big]\nabla\Big(\nabla\cdot\langle u_\sigma\rangle^\sigma\Big) +\\
&+\Big[\mu_\sigma H_A + (\mu_\sigma+\lambda_\sigma)H_B\Big]\langle\mathbf{v}_\beta\rangle = 0
\end{aligned}
\tag{4.43}
$$

in which m_1 and m_2 are the two distinct components of $M_C^{(4)}$, M_D is the single distinct component of $M_D^{(2)}$, and H_A and H_B are the distinct components of $H_A^{(2)}$ and $H_B^{(2)}$, respectively. If the velocity is expressed in terms of Darcy's law, one recovers the precise *form* of Biot's ((1941), Equation 4.1) original work which has recently been examined with great care by Coussy and Bourbie (1984). The coefficients in this work are different than those given by Biot, because he began his analysis with homogenized forms of Equations (3.1) and (3.2) with the idea that the material coefficients would be determined by macroscopic experiments.

The result given by Equation (4.40) represents the closed form of the volume-averaged version of Equation (3.3), and for most problems we will also require the comparable form for the constitutive equation given by Equation (3.2). This is obtained by the use of Equation (4.20) in Equation (3.10) to obtain

$$
\begin{aligned}
\langle\tau_\sigma\rangle^\sigma &= \mu_\sigma\Big\{\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + (\nabla\langle\mathbf{u}_\sigma\rangle^\sigma)^T\Big\} + \lambda_\sigma\mathbf{I}\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma +\\
&+\epsilon_\sigma^{-1}\Big(\mu_\sigma M_E^{(4)} + \lambda_\sigma M_F^{(4)}\Big) : \nabla\langle\mathbf{u}_\sigma\rangle^\sigma +\\
&+\epsilon_\sigma^{-1}\Big(\mu_\sigma H_E^{(3)} + \lambda_\sigma H_F^{(3)}\Big)\cdot\langle\mathbf{v}_\beta\rangle.
\end{aligned}
\tag{4.44}
$$

Here we have used the nomenclature indicated by Equations (4.41) and (4.42) and the tensor coefficients in Equation (4.44) are expressed in Cartesian tensor notation by

$$H_E^{(3)} = \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} n_i C_{jk} (\mathbf{e}_i \mathbf{e}_j \mathbf{e}_k + \mathbf{e}_j \mathbf{e}_i \mathbf{e}_k) \, \mathrm{d}A, \tag{4.45a}$$

$$H_F^{(3)} = \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{I}\mathbf{n}_{\sigma\beta} \cdot \mathbf{C} \, \mathrm{d}A, \tag{4.45b}$$

$$M_E^{(4)} = \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} n_i D_{jk\ell} (\mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_\ell + \mathbf{e}_j \mathbf{e}_i \mathbf{e}_k \mathbf{e}_\ell) \, \mathrm{d}A, \tag{4.46a}$$

$$M_F^{(4)} = \frac{1}{\mathscr{V}} \int_{A_{\sigma\beta}} \mathbf{I}\mathbf{n}_{\sigma\beta} \cdot \mathscr{D} \, \mathrm{d}A. \tag{4.46b}$$

Here we have used the representations

$$\mathbf{n}_{\sigma\beta} = \mathbf{e}_i n_i , \qquad \mathbf{C} = \mathbf{e}_j \mathbf{e}_k C_{jk} , \qquad \mathscr{D} = \mathbf{e}_j \mathbf{e}_k \mathbf{e}_\ell D_{jk\ell} \tag{4.47}$$

and one must be careful to remember that $\mathbf{n}_{\sigma\beta} = -\mathbf{n}_{\beta\sigma}$ so that sign errors are not generated in Equations (4.26), (4.45a) and (4.46a).

If one considers an isotropic system and makes use of Equation (4.15) in Equation (4.44), the *general form* of Biot's ((1941), Equation 2.11) original expression for the stress in terms of the strain and the fluid pressure is recovered.

5. Solution of the Closure Problem

In order to determine the coefficients in the displacement vector equation given by Equation (4.40), or in the stress-strain relation given by Equation (3.10), one need only solve the local boundary value problems given by Equations (4.21) through (4.29) in addition to the closure problem given in Part I of this paper. Examples of the numerical methods that can be used for problems of this type are given by Ryan *et al.* (1981), Eidsath *et al.* (1983), and Nozad *et al.* (1985). Solution of the closure problem removes the coefficients in the volume-averaged equations from the list of *adjustable parameters*; however, the "adjustment" still takes place in terms of the geometry of the unit cell to be used in conjunction with Equations (4.37) and (4.38), it is clear that the solution to the closure problem will be sensitive to the geometry of the β–σ interface within the unit cell.

For the systems of spheres illustrated in Figures 1 and 3, one could choose a unit cell such as that shown in Figure 4. Within the constraints of small deformation theory, $\nabla \mathbf{u}_\sigma \ll 1$, one can solve the closure problem with the geometry specified by the unit cell. This will provide the coefficients in Equation (4.40) in addition to the Darcy's law permeability tensor in Equation (2.48), thus the pressure field can be determined when the flow rate is specified. To determine the volume-averaged deformation, one would make use of Equation (4.40) and

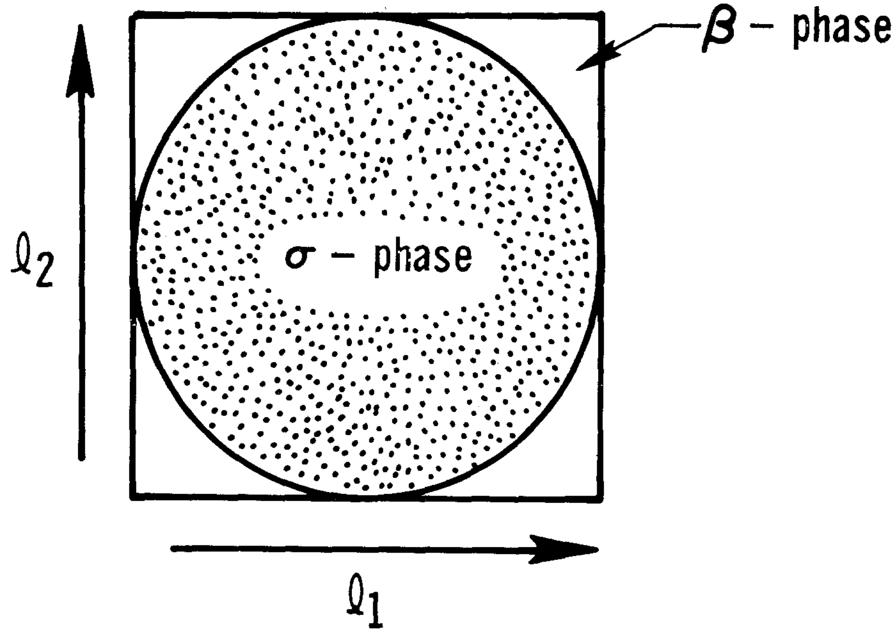

Fig. 4. Unit cell of an elastic porous medium.

the boundary condition;

$$\text{B.C.1} \quad \langle \mathbf{u}_\sigma \rangle^\sigma \cdot \mathbf{k} = 0, \qquad z = 0. \tag{5.1}$$

however, a boundary condition at the top of the bed illustrated in Figure 1 would also be required. On a purely intuitive basis, one would balance the normal stress from Equation (4.44) against the hydrostatic pressure and write

$$\begin{aligned}\text{B.C.2} \quad -\langle P_\beta \rangle^\beta &= \mathbf{k} \cdot \langle \boldsymbol{\tau}_\sigma \rangle^\sigma \cdot \mathbf{k} \\ &= 2\mu_\sigma \mathbf{kk} : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \lambda_\sigma \nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma + \\ &\quad + \epsilon_\sigma^{-1} \mathbf{kk} : \left(\mu_\sigma M_E^{(4)} + \lambda_\sigma M_F^{(4)} \right) : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \\ &\quad + \epsilon_\sigma^{-1} \mathbf{kk} : \left(\mu_\sigma H_E^{(3)} + \lambda_\sigma H_F^{(3)} \right) \cdot \langle \mathbf{v}_\beta \rangle, \quad z = H.\end{aligned} \tag{5.2}$$

This is the type of boundary condition proposed by Biot ((1941), Equation 5.3), and the origin of this result rests with Equation (2.33). A derivation is offered in Appendix A. The pressure can be eliminated from Equation (5.2) by means of Equation (4.15), and this leads to a boundary condition of the form

$$\begin{aligned}&\left[2\mu_\sigma (\mathbf{G} - \mathbf{kk}) - \epsilon_\sigma^{-1} \mathbf{kk} : \left(\mu_\sigma M_E^{(4)} + \lambda_\sigma M_F^{(4)} \right) \right] : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \\ &= \left[\epsilon_\sigma^{-1} \mathbf{kk} : \left(\mu_\sigma H_E^{(3)} + \lambda_\sigma H_F^{(3)} \right) + \right. \\ &\quad \left. + \epsilon_\beta^{-1} \mu_\beta \left\langle \mathbf{b} - 2\mathbf{n}_{\beta\sigma} \mathbf{n} \beta_\sigma : \nabla \mathbf{B} \right\rangle_{\beta\sigma} \right] \cdot \langle \mathbf{v}_\beta \rangle, \quad z = H.\end{aligned} \tag{5.3}$$

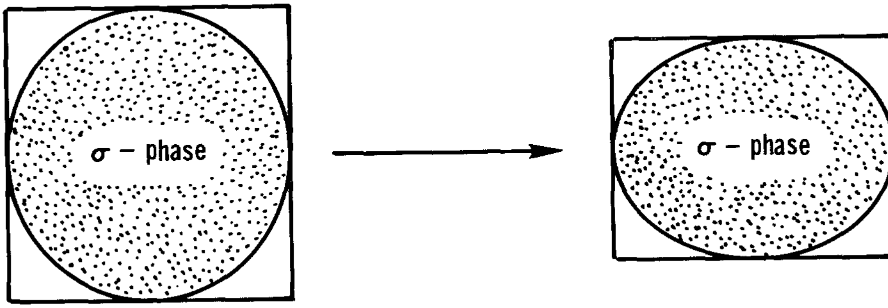

Fig. 5. Deformation of a unit cell.

Here we expect that the very last term, being an average of a deviation, would be negligible; however, it can be estimated as calculated using the unit cell illustrated in Figure 4. After having solved the closure problem to determine the coefficients in Equation (4.40) and the coefficients in the boundary condition given by Equation (5.3), one can solve for $\langle \mathbf{u}_\sigma \rangle^\sigma$ using Equation (4.40) and for $\tilde{\mathbf{u}}_\sigma$ using Equation (4.20). This means that the *point* displacement vector field is available to us through

$$\mathbf{u}_\sigma = \langle \mathbf{u}_\sigma \rangle^\sigma + \tilde{\mathbf{u}}_\sigma = \langle \mathbf{u}_\sigma \rangle^\sigma + \mathscr{D} : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \mathbf{C} \cdot \langle \mathbf{v}_\beta \rangle \tag{5.4}$$

and the original unit cell would be deformed as illustrated in Figure 5 in an exaggerated manner. A prudent procedure at this point would be to resolve the closure problem with the deformed unit cell and recompute the displacement vector field to be certain that a converged solution has been obtained. If the velocity, $\langle \mathbf{v}_\beta \rangle$, is sufficiently small, a single iteration will suffice and a complete closure to the small deformation problem will have been obtained. There are many processes, especially in the area of filtration, in which large deformations are encountered and for those processes the solid mechanics problem must be reformulated in terms of large deformation theory. This aspect of the problem has been discussed by Kubik (1982), but no method of closure is available.

6. Conclusions

The method of volume-averaging has been applied to the process of steady, incompressible flow through an elastic porous medium. For small deformations, a closure scheme is available that provides theoretical values for the Darcy's law permeability tensor in addition to the tensor coefficients that appear in the volume-averaged displacement vector equation.

Appendix: The Volume-Averaged Boundary Condition

In Section 5 we made use of a *plausible* boundary condition at the interface between a deformable porous medium and the surrounding fluid, and in this appendix we would like to provide a derivation of that boundary condition. We begin with the *point* condition given earlier by Equation (2.33)

$$-\mathbf{n}_{\beta\sigma}P_\beta + \mu_\beta\left(\mathbf{n}_{\beta\sigma}\cdot\nabla\mathbf{v}_\beta + \nabla\mathbf{v}_\beta\cdot\mathbf{n}_{\beta\sigma}\right) = \mathbf{n}_{\beta\sigma}\cdot\boldsymbol{\tau}_\sigma, \quad \text{at } \mathscr{A}^*_{\beta\sigma} \tag{A.1}$$

and focus our attention on the region illustrated in Figure 6. There we have placed a *dividing surface* (shown as a solid straight line) in the porous medium near the 'interface'. It is at the *dividing surface* that we would like to impose a boundary condition such as that given by Equation (5.2). In order to be definite about the location of the dividing surface, one might require that the area fraction of the solid phase be within 1% of the local volume fraction of the solid phase. If the porous medium is homogeneous, this represents an unambiguous definition; however, if significangaients in the local volume exist the definition becomes imprecise. Clearly there are different methods of locating a dividing surface, and at this point we will simply assume that some suitable method exists.

While the plane surface located at $z = H$ is suitable from the point of view of solving a boundary value problem, Equation (A.1) *does not apply* at the surface, but instead it can only be applied at the β–σ interface illustrated by the dashed line in Figure 6. This surface is represented by $\mathscr{A}^*_{\beta\sigma}$ and is the interfacial area located *above* the plane given by $z = H$. At a point on this surface we have illustrated an averaging volume of radius r_0, and it should be understood that the length-scale constraints $\ell_\beta \ll r_0$ and $\ell_\sigma \ll r_0$ are still in effect. However, the local volume fraction and the intrinsic phase average velocity will undergo significant

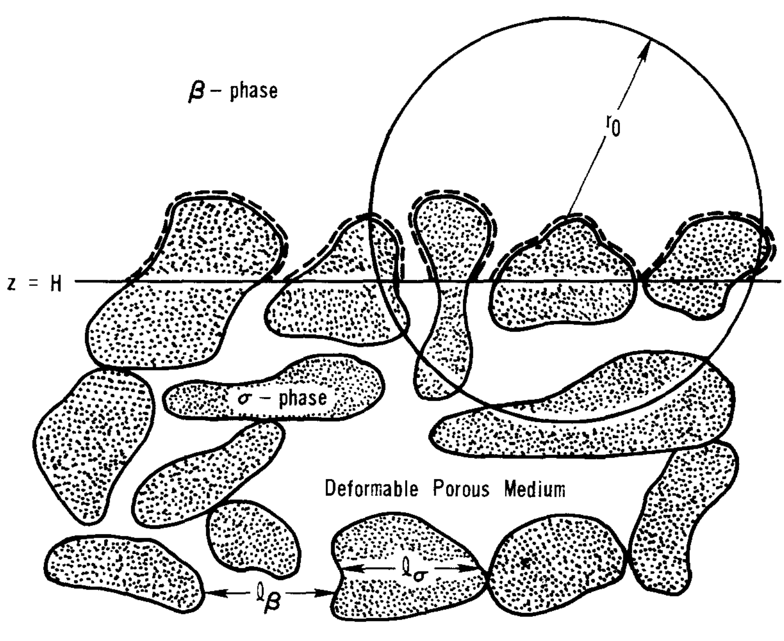

Fig. 6. interface between a fluid and a deformable porous medium.

variations over distances on the order of r_0 in the direction perpendicular to the dividing surface, thus we are faced with a situation in which $r_0 \sim L$.

The key idea in the development of boundary conditions between a porous medium and a surrounding fluid is the replacement of conditions that apply *at a point* on $\mathscr{A}^*_{\beta\sigma}$ with average conditions that apply at $z = H$. We begin the search for an averaged condition by integrating Equation (A.1) over $A^*_{\beta\sigma}$ to obtain

$$\int_{A^*_{\beta\sigma}} [-\mathbf{n}_{\beta\sigma} P_\beta + \mu_\beta(\mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta \cdot n_{\beta\sigma})]\, \mathrm{d}A = \int_{A^*_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \boldsymbol{\tau}_\sigma \, \mathrm{d}A. \tag{A.2}$$

Here $A^*_{\beta\sigma}$ is a portion of $\mathscr{A}^*_{\beta\sigma}$ bounded by a cylinder of radius r_0 which is parallel to the z-axis. We can use the divergence theorem and Equation (3.1) to alter the right-hand side of this result to obtain

$$\int_{A^*_{\beta\sigma}} \left[-\mathbf{n}_{\beta\sigma} P_\beta + \mu_\beta\left(\mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta \cdot \mathbf{n}_{\beta\sigma}\right)\right] \mathrm{d}A = -\int_{A^*_{\sigma e}} \mathbf{k} \cdot \boldsymbol{\tau}_\sigma \, \mathrm{d}A. \tag{A.3}$$

Here one should refer to Figure 7 and note that $\mathscr{A}^*_{\sigma e}$ represents the area of entrances and exits for the σ-phase associated with the area $A^*_{\beta\sigma}$. Since we seek a boundary condition for the displacement vector equation, it is convenient to use Equation (3.2) to obtain

$$\begin{aligned}\int_{A^*_{\beta\sigma}} &\left[-\mathbf{n}_{\beta\sigma} P_\beta + \mu_\beta\left(\mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta \cdot \mathbf{n}_{\beta\sigma}\right)\right] \mathrm{d}A \\ &= \int_{A^*_{\sigma e}} \left[\mu_\sigma\left(\mathbf{k} \cdot \nabla \mathbf{u}_\sigma + \nabla \mathbf{u}_\sigma \cdot \mathbf{k}\right) + \lambda_\sigma \mathbf{k} \nabla \cdot \mathbf{u}_\sigma\right] \mathrm{d}A.\end{aligned} \tag{A.4}$$

Here we must consider the fact that the left-hand side of Equation (A.4) is evaluated at a position $z > H$ while the right-hand side is evaluated at the *desired* position, $z = H$. If it were possible, we would make use of a Taylor series expansion of the form

$$P_\beta\big|_{\mathbf{r}=\mathbf{k}H+\boldsymbol{\eta}} = P_\beta\big|_{\mathbf{r}=\mathbf{k}H} + \boldsymbol{\eta} \cdot \nabla P_\beta + \cdots \tag{A.5}$$

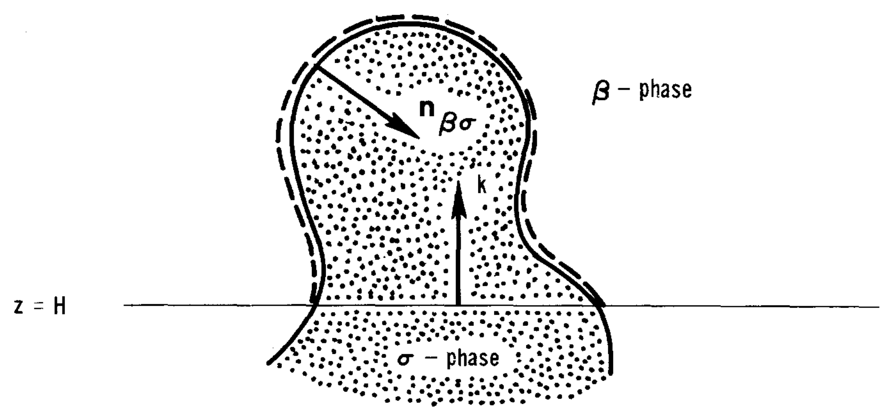

Fig. 7. Interfacial region.

in order to develop a boundary condition with all terms evaluated at $z = H$. This is the type of analysis that is successfully used with linear stability analysis (Taylor, 1950); however, in that case $\boldsymbol{\eta}$ is arbitrarily small and only the first term in the series is required. In this case, one expects to find significant variations in $\mathbf{v}_\beta$ over the distance $\boldsymbol{\eta}$ and it would appear that we are forced to proceed with Equation (A.4) in its present form.

At this point we introduce the decompositions

$$P_\beta = \langle P_\beta \rangle^\beta + \tilde{P}_\beta, \qquad \mathbf{v}_\beta = \langle \mathbf{v}_\beta \rangle^\beta + \tilde{\mathbf{v}}_\beta, \qquad \mathbf{u}_\sigma = \langle \mathbf{u}_\sigma \rangle^\sigma + \tilde{\mathbf{u}}_\sigma \tag{A.6}$$

in which $\langle P_\beta \rangle^\beta$ and $\langle \mathbf{v}_\beta \rangle^\beta$ are defined in terms of an averaging-volume located on $\mathscr{A}^*_{\beta\sigma}$ while $\langle \mathbf{u}_\sigma \rangle^\sigma$ is defined in terms of an averaging-volume located on $\mathscr{A}_{\sigma e}$. We use Equations (A.6) to express Equation (A.4) as

$$\begin{aligned}
&\int_{A^*_{\beta\sigma}} -\mathbf{n}_{\beta\sigma} \langle P_\beta \rangle^\beta \,\mathrm{d}A + \int_{A^*_{\beta\sigma}} \left[-\mathbf{n}_{\beta\sigma} \tilde{P}_\beta + \mu_\beta \left(\mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta + \nabla \tilde{\mathbf{v}}_\beta \cdot \mathbf{n}_{\beta\sigma} \right) \right] \mathrm{d}A \\
&= \int_{A^*_{\sigma e}} \left[\mu_\sigma \left(\mathbf{k} \cdot \nabla \langle u_\sigma \rangle^\sigma + \nabla \langle \mathbf{u}_\sigma \rangle^\sigma \cdot \mathbf{k} \right) + \lambda_\sigma \mathbf{k} \nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma \right] \mathrm{d}A - \\
&\quad - \int_{A^*_{\sigma e}} \left[\mu_\sigma \left(\mathbf{k} \cdot \nabla \tilde{\mathbf{u}}_\sigma + \nabla \tilde{\mathbf{u}}_\sigma \cdot \mathbf{k} \right) + \lambda_\sigma \mathbf{k} \nabla \cdot \tilde{\mathbf{u}}_\sigma \right] \mathrm{d}A.
\end{aligned} \tag{A.7}$$

Here we have used

$$\nabla \langle \mathbf{v}_\beta \rangle^\beta \ll \nabla \tilde{\mathbf{v}}_\beta \tag{A.8}$$

on the basis of the constraint $\ell_\beta \ll r_0 \sim L$. As in earlier parts of our analysis, we would like to remove volume-averaged quantities from within the area integrals; however, that step was based on the constraint, $r_0 \ll L$, which is no longer in force. For an averaged quantity, such as $\langle P_\beta \rangle^\beta$, we can write

$$\begin{aligned}
&\int_{A^*_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \left[\langle P_\beta \rangle^\beta + \boldsymbol{\eta} \cdot \nabla \langle P_\beta \rangle^\beta + \cdots \right] \mathrm{d}A \\
&\quad = \left\{ \int_{A^*_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \mathrm{d}A \right\} \langle P_\beta \rangle^\beta + \left\{ \int_{A^*_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \boldsymbol{\eta} \,\mathrm{d}A \right\} \cdot \nabla \langle P_\beta \rangle^\beta + \cdots
\end{aligned} \tag{A.9}$$

Here we have used $\langle P_\beta \rangle^\beta$ and $\nabla \langle P_\beta \rangle^\beta$ to indicate quantities evaluated at $z = H$ and we have in mind the idea that there are negligible variations in these quantities in the x-y plane.

At this point we must be careful to note that $A^*_{\beta\sigma}$ is *not* the interfacial area contained within an averaging-volume, thus we cannot draw upon the analysis of Section 2 in Part I (see especially Equations (2.26) and (2.27)) to estimate the magnitude of the area integrals on the right-hand side of Equation (A.9). The divergence theorem can be used to evaluate the first integral as

$$\int_{A^*_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \,\mathrm{d}A = -\mathbf{k} A_{\sigma e} \tag{A.10}$$

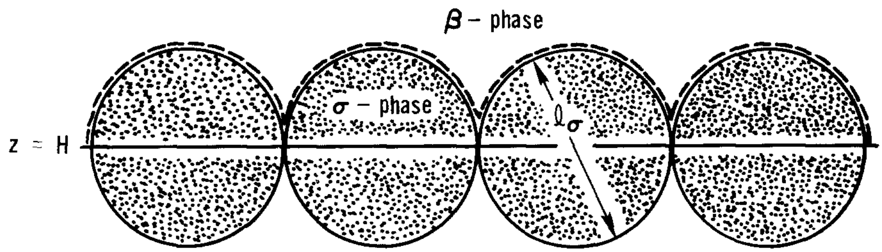

Fig. 8. Model of an interfacial region.

while the second integral can be determined using the model shown in Figure 8. A little analysis leads to

$$\int_{A^*_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\boldsymbol{\eta}\,\mathrm{d}A = -\frac{\ell_\sigma}{12}\mathbf{I}A_{\sigma e} \tag{A.11}$$

and this allows us to express the first term in Equation (A.7) as

$$\int_{A^*_{\beta\sigma}} -\mathbf{n}_{\beta\sigma}\langle P_\beta\rangle^\beta\,\mathrm{d}A = A_{\sigma e}\mathbf{k}\langle P_\beta\rangle^\beta|_{z=H} + \left(\frac{\ell_\sigma}{12}\right)A_{\sigma e}\,\nabla\langle P_\beta\rangle^\beta + \cdots \tag{A.12}$$

From Equation (2.48) we have the two estimates

$$\nabla\langle P_\beta\rangle^\beta|_{z=H} = \mathbf{O}\{\mu_\beta\mathbf{K}^{-1}\cdot\langle\mathbf{v}_\beta\rangle\}, \tag{A.13}$$

$$\langle P_\beta\rangle^\beta|_{z=H} = \mathbf{O}\{\mu_\beta H\mathbf{K}^{-1}\cdot\langle\mathbf{v}_\beta\rangle\}. \tag{A.14}$$

Here it becomes apparent that we can drop the last term in Equation (A.12) whenever $\ell_\sigma \ll H$. Under most circumstances this contraint is satisfied and we proceed to write Equation (A.7) in the form

$$\begin{aligned}
&\mathbf{k}A_{\sigma e}\langle P_\beta\rangle^\beta + \int_{A^*_{\beta\sigma}}\left[-\mathbf{n}_{\beta\sigma}\tilde{P}_\beta + \mu_\beta\left(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{v}}_\beta + \nabla\tilde{\mathbf{v}}_\beta\cdot\mathbf{n}_{\beta\sigma}\right)\right]\mathrm{d}A \\
&\quad = -\int_{A^*_{\sigma e}}\left[\mu_\sigma\left(\mathbf{k}\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + \nabla\langle\mathbf{u}_\sigma\rangle^\sigma\cdot\mathbf{k}\right) + \lambda_\sigma\mathbf{k}\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma\right]\mathrm{d}A - \\
&\quad\quad -\int_{A^*_{\sigma e}}\left[\mu_\sigma\left(\mathbf{k}\cdot\nabla\tilde{\mathbf{u}}_\sigma + \nabla\tilde{\mathbf{u}}_\sigma\cdot\mathbf{k}\right) + \lambda_\sigma\mathbf{k}\nabla\cdot\tilde{\mathbf{u}}_\sigma\right]\mathrm{d}A.
\end{aligned} \tag{A.15}$$

If one assumes that there are negligible variations of $\mathbf{k}\cdot\nabla\langle\mathbf{u}_\sigma\rangle^\sigma\cdot\mathbf{k}$ over the x-y plane, we can form the scalar product of Equation (A.15) with $\mathbf{k}$ and express the result as

$$\begin{aligned}
-\langle P_\beta\rangle^\beta &= 2\mu_\sigma\mathbf{kk}:\nabla\langle\mathbf{u}_\sigma\rangle^\sigma + \lambda_\sigma\nabla\cdot\langle\mathbf{u}_\sigma\rangle^\sigma - \\
&\quad -\frac{1}{A_{\sigma e}}\int_{A_{\sigma e}}\left(2\mu_\sigma\mathbf{kk}:\nabla\tilde{\mathbf{u}}_\sigma + \lambda_\sigma\nabla\cdot\tilde{\mathbf{u}}_\sigma\right)\mathrm{d}A + \\
&\quad +\frac{1}{A_{\sigma e}}\int_{A_{\sigma e}}\left[-\mathbf{k}\cdot\mathbf{n}_{\beta\sigma}\tilde{P}_\beta + \mu_\beta\left(\mathbf{n}_{\beta\sigma}\cdot\nabla\tilde{\mathbf{v}}_\beta\cdot\mathbf{k} + \mathbf{k}\cdot\nabla\tilde{\mathbf{v}}_\beta\cdot\mathbf{n}_{\beta\sigma}\right)\right]\mathrm{d}A.
\end{aligned} \tag{A.16}$$

Up to this point, the analysis is exact provided that the length scale constraints have been properly observed. However, we now encounter the problem that representations for $\tilde{\mathbf{v}}_\beta$ and $\tilde{\mathbf{u}}_\sigma$ are needed, and the representations that we currently have available are restricted in a very important manner. As we have pointed out in Section 3 of Part I of this paper and in Section 4 of this part, the representations for the spatial deviations are controlled by the nonhomogeneous terms in the boundary conditions at the β–σ interface, *provided* the boundary conditions at the entrances and exits of the macroscopic system can be ignored (see Equations (3.3) and (3.4) or Part I). The question at this point is: What is the representation for $\tilde{\mathbf{v}}_\beta$ at $\mathscr{A}^*_{\beta\sigma}$ and what is the representation for $\tilde{\mathbf{u}}_\sigma$ at $\mathscr{A}_{\sigma e}$? Certainly a plausible set of representations is

$$
\begin{aligned}
&\tilde{P}_\beta = \mu_\beta \mathbf{b}' \cdot \langle \mathbf{v}_\beta \rangle, \quad \tilde{\mathbf{v}}_\beta = \mathbf{B}' \cdot \langle \mathbf{v}_\beta \rangle, \\
&\tilde{\mathbf{u}}_\sigma = \mathbf{C}' \cdot \langle \mathbf{v}_\beta \rangle + \mathscr{D}' : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma.
\end{aligned}
\tag{A.17}
$$

Here we expect that $\mathbf{b}'$, $\mathbf{B}'$, $\mathbf{C}'$ and $\mathscr{D}'$ will have the same general structure as $\mathbf{b}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathscr{D}$, but they will not be equal to these tensor functions for which we have a method of determination.

If we make use of the plausible representations given by Equations (A.17) in Equation (A.16), we obtain

$$
\begin{aligned}
-\langle P_\beta \rangle^\beta = {} & 2\mu_\sigma \mathbf{kk} : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \lambda_\sigma \nabla \cdot \langle \mathbf{u}_\sigma \rangle^\sigma + \\
& + \frac{1}{A_{\sigma e}} \int_{A_{\sigma e}} \left(2\mu_\sigma \mathbf{kk} : \nabla \mathscr{D}' + \lambda_\sigma \nabla \cdot \mathscr{D}' \right) \mathrm{d}A : \nabla \langle \mathbf{u}_\sigma \rangle^\sigma + \\
& + \frac{1}{A_{\sigma e}} \int_{A_{\sigma e}} \left(2\mu_\sigma \mathbf{kk} : \nabla \mathbf{C}' + \lambda_\sigma \nabla \cdot \mathbf{C}' \right) \mathrm{d}A \cdot \langle \mathbf{v}_\beta \rangle + \\
& + \mu_\beta \frac{1}{A_{\sigma e}} \int_{A^*_{\beta\sigma}} \Big[-\mathbf{k} \cdot \mathbf{n}_{\beta\sigma} \mathbf{b}' + \Big(\mathbf{n}_{\beta\sigma} \cdot \nabla \bar{\mathbf{B}}' \cdot \mathbf{k} + \\
& + \mathbf{k} \cdot \nabla \bar{\mathbf{B}}' \cdot \mathbf{n}_{\beta\sigma} \Big) \Big] \mathrm{d}A \cdot \langle \mathbf{v}_\beta \rangle \Big\}.
\end{aligned}
\tag{A.18}
$$

Here we have been able to derive the same general *form* as was given earlier by Equation (5.2); however, we have no proof of the representations given by Equations (A.17) nor do we have a means of calculating the coefficients in those representations. Clearly the boundary conditions between a porous medium and a surrounding fluid deserves further study.

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