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Flow in Porous Media II: The Governing Equations for Immiscible, Two-Phase Flow

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Abstract. The Stokes flow of two immiscible fluids through a rigid porous medium is analyzed using the method of volume averaging. The volume-averaged momentum equations, in terms of averaged quantities and spatial deviations, are identical in form to that obtained for single phase flow; however, the solution of the closure problem gives rise to additional terms not found in the traditional treatment of two-phase flow. Qualitative arguments suggest that the nontraditional terms may be important when μ_β/μ_γ is of order one, and order of magnitude analysis indicates that they may be significant in terms of the motion of a fluid at very low volume fractions. The theory contains features that could give rise to hysteresis effects, but in the present form it is restricted to static contact line phenomena.

Key words. Volume averaging, interfacial phenomena, closure.

0. Nomenclature

Roman Letters (ω, $\eta = \beta$, γ, σ and $\omega \neq \eta$)

$\mathscr{A}_{\omega\eta}$ interfacial area of the ω–η interface contained within the macroscopic system, m^2
$\mathscr{A}_{\omega e}$ area of entrances and exits for the ω-phase contained within the macroscopic system, m^2
$A_{\omega\eta}$ interfacial area of the ω–η interface contained within the averaging volume, m^2
$A^*_{\omega\eta}$ interfacial area of the ω–η interface contained within a unit cell, m^2
$A^*_{\omega e}$ area of entrances and exits for the ω-phase contained within a unit cell, m^2
$\mathbf{g}$ gravity vector, m^2/s
H mean curvature of the β–γ interface, m^{-1}
$\langle H\rangle_{\beta\gamma}$ area average of the mean curvature, m^{-1}
$\tilde{H}$ $H - \langle H\rangle_{\beta\gamma}$, deviation of the mean curvature, m^{-1}
$\mathbf{I}$ unit tensor
$\mathbf{K}$ Darcy's law permeability tensor, m^2
$\mathbf{K}_\omega$ permeability tensor for the ω-phase, m^2
$\mathbf{K}_{\beta\gamma}$ viscous drag tensor for the β-phase equation of motion

$\mathbf{K}_{\gamma\beta}$	viscous drag tensor for the γ-phase equation of motion
L	characteristic length scale for volume averaged quantities, m
ℓ_ω	characteristic length scale for the ω-phase, m
$\mathbf{n}_{\omega\eta}$	unit normal vector pointing from the ω-phase toward the η-phase ($\mathbf{n}_{\omega\eta} = -\mathbf{n}_{\eta\omega}$)
p_c	$\langle p_\gamma\rangle^\gamma - \langle P_\beta\rangle^\beta$, capillary pressure, N/m^2
p_ω	pressure in the ω-phase, N/m^2
$\langle p_\omega\rangle^\omega$	intrinsic phase average pressure for the ω-phase, N/m^2
$\tilde{p}_\omega$	$p_\omega - \langle p_\omega\rangle^\omega$, spatial deviation of the pressure in the ω-phase, N/m^2
r_0	radius of the averaging volume, m
t	time, s
$\mathbf{v}_\omega$	velocity vector for the ω-phase, m/s
$\langle \mathbf{v}_\omega\rangle$	phase average velocity vector for the ω-phase, m/s
$\langle \mathbf{v}_\omega\rangle^\omega$	intrinsic phase average velocity vector for the ω-phase, m/s
$\tilde{\mathbf{v}}_\omega$	$\mathbf{v}_\omega - \langle \mathbf{v}_\omega\rangle^\omega$, spatial deviation of the velocity vector for the ω-phase, m/s
$\mathcal{V}$	averaging volume, m^3
V_ω	volume of the ω-phase contained within the averaging volume, m^3

Greek Letters

ϵ_ω	$V_\omega/\mathcal{V}$, volume fraction of the ω-phase
ρ_ω	mass density of the ω-phase, kg/m^3
μ_ω	viscosity of the ω-phase, Nt/m^2
σ	surface tension of the β–γ interface, N/m
$\boldsymbol{\tau}_\omega$	viscous stress tensor for the ω-phase, N/m^2
	μ/ρ, kinematic viscosity, m^2/s

1. Introduction

The classic examples of two-phase flow in porous media are associated with oil recovery processes and groundwater flows. The former may involve oil and gas, oil and water, or oil and solutions of polymers or surfactants, while the latter deals with air and water. The demands for predictive theories of two-phase flow in porous media are enormous, as are the complexities of the systems under consideration. The latter typically involve porous structures that are unknown or are difficult to characterize, along with large scale heterogeneities and sparse experimental data. Laboratory experiments are usually carried out in terms of steady, one-dimensional flows while practical problems are often multi-dimensional and transient. Laboratory studies of transient, multi-dimensional flows are hampered by the difficulty of measuring local saturations and volume-averaged velocities, and lack of experiments of this type may be the reason why the intuitive extension of Darcy's law by Richards (1931) and Muskat *et al.* (1937) has survived without modification.

In this work, the process of two-phase flow in a rigid porous medium is analyzed from a rigorous point of view leading to equations of motion of the form

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}_\beta}{\mu_\beta} \cdot (\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}) + \mathbf{K}_{\beta\gamma} \cdot \langle \mathbf{v}_\gamma \rangle, \tag{1.1}$$

$$\langle \mathbf{v}_\gamma \rangle = -\frac{\mathbf{K}_\gamma}{\mu_\gamma} \cdot (\nabla \langle p_\gamma \rangle^\gamma - \rho_\gamma \mathbf{g}) + \mathbf{K}_{\gamma\beta} \cdot \langle \mathbf{v}_\beta \rangle \tag{1.2}$$

along with the traditional form of the continuity equations

$$\frac{\partial \epsilon_\beta}{\partial t} + \nabla \cdot \langle \mathbf{v}_\beta \rangle = 0, \tag{1.3}$$

$$\frac{\partial \epsilon_\gamma}{\partial t} + \nabla \cdot \langle \mathbf{v}_\gamma \rangle = 0 \tag{1.4}$$

The details concerning $\mathbf{K}_{\beta\gamma}$ and $\mathbf{K}_{\gamma\beta}$ are available through the solution of very complex, coupled boundary-value problems for tensor quantities; however, from an intuitive point of view these two tensors simply represent the influence of the viscous drag that exists between the β-phase and the γ-phase. Because of this, we expect that these nontraditional terms will be important when μ_β/μ_γ is on the order of one. Here μ_β and μ_γ represent the viscosities of the β and γ-phases as illustrated in Figure 1. In groundwater flows one often thinks of the air (the γ-phase in Figure 1) as contributing nothing more than a constant pressure at the air-water interface, thus one is inclined to discard the last term in Equation (1.1). Since $\mu_\beta \gg \mu_\gamma$ in this case, this is a reasonable thing to do. On the other hand, if thin films of air ($\ell_\gamma \ll \ell_\beta$) were to exist during an imbibition experiment in which $\langle \mathbf{v}_\gamma \rangle^\gamma$ and $\langle \mathbf{v}_\beta \rangle^\beta$ were oppositely directed, it is possible that the last term in Equation (1.1) could be important. Given the wide range of flow configurations that can exist for two-phase flow in porous media (Wooding and Morel-Seytoux, 1976), it would seem worthwhile to consider the last two terms in Equations (1.1) and (1.2) carefully before neglecting them.

While the original intuitive extension of Darcy's law proposed by Richards and by Muskat *et al.* may suffice for many practical cases (Philip, 1972), there is a crucial question to be answered concerning the *phase permeabilities*, $\mathbf{K}_\beta$ and $\mathbf{K}_\gamma$, that appear in Equations (1.1) and (1.2). For single phase flow in a rigid porous medium, we have shown that

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}}{\mu_\beta} \cdot (\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}) \tag{1.5}$$

in which $\mathbf{K}$ should be referred to as the *Darcy's law permeability tensor*. For purely one-dimensional processes, the phase permeability is related to the Darcy's law permeability by the *relative permeability* according to

$$K_\beta = k_r^\beta K. \tag{1.6}$$

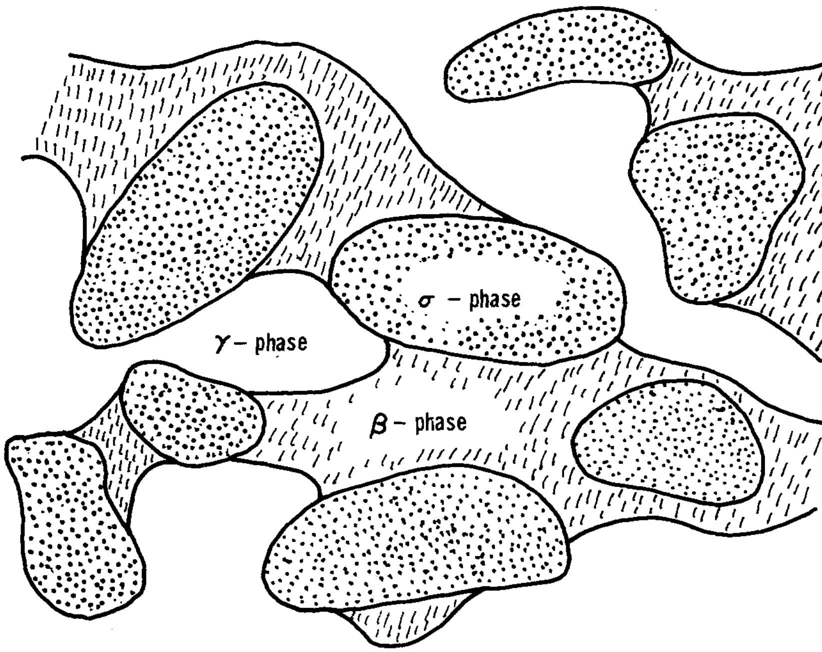

Fig. 1. Two-phase flow in porous media.

While there have been numerous experimental determinations of K and K_β to produce relative permeabilities as a function of saturation (Scheidegger, 1974; Bear, 1972; Greenkorn, 1984), there appears to be no information available concerning the relation between $\mathbf{K}_\beta$ and $\mathbf{K}$. It is important to keep in mind that even if the single-phase flow process can be treated as isotropic, i.e.,

$$\mathbf{K} = \mathbf{I}K, \quad \text{isotropic process} \tag{1.7}$$

there is no reason to believe that the two-phase flow process can be accorded the same simplification. Here one must keep in mind that there are *no isotropic porous media* of interest, while there are numerous *isotropic processes* of importance. The precise relation between $\mathbf{K}_\beta$ and $\mathbf{K}$ remains unknown at this time; however, many would agree that the relation

$$\mathbf{K}_\beta = k_r^\beta \mathbf{K} \tag{1.8}$$

would be a reasonable approximation. Although precise values of $\mathbf{K}_\beta$ and $\mathbf{K}$ are available through the solution of the closure equations, the computational problem is quite complex and no results are given at this time.

2. Theory

The system under consideration has been illustrated in Figure 1 and the process that we wish to investigate is described by

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta, \tag{2.1}$$

$$\nabla \cdot \mathbf{v}_\beta = 0, \tag{2.2}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{2.3}$$

$$\text{B.C.2} \quad \mathbf{v}_\beta = \mathbf{v}_\gamma, \quad \text{at } \mathscr{A}_{\beta\gamma}, \tag{2.4}$$

$$\text{B.C.3} \quad -p_\beta \mathbf{n}_{\beta\gamma} + \boldsymbol{\tau}_\beta \cdot \mathbf{n}_{\beta\gamma} = -p_\gamma \mathbf{n}_{\beta\gamma} + \boldsymbol{\tau}_\gamma \cdot \mathbf{n}_{\beta\gamma} + 2\sigma H \mathbf{n}_{\beta\gamma}, \quad \text{at } \mathscr{A}_{\beta\gamma}, \tag{2.5}$$

$$\text{B.C.4} \quad \mathbf{v}_\gamma = 0, \quad \text{at } \mathscr{A}_{\gamma\sigma}, \tag{2.6}$$

$$0 = -\nabla p_\gamma + \rho_\gamma \mathbf{g} + \mu_\gamma \nabla^2 \mathbf{v}_\gamma, \tag{2.7}$$

$$\nabla \cdot \mathbf{v}_\gamma = 0. \tag{2.8}$$

Here we have used σ to represent the interfacial tension, and H to represent the mean curvature of the interface. The total stress tensor has been decomposed according to

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \tag{2.9}$$

and on the basis of the analysis presented in Part I (Whitaker, 1986), the boundary conditions at entrances and exits of the macroscopic system have not been listed since they play no role in the development of the volume-averaged equations and the closure problem. It is worthwhile to note that the mean curvature is to be determined as part of the solution to the governing equations and that the stress condition given by Equation (2.5) is devoid of surfactant effects. The effect of surfactants, which are difficult to avoid in any natural system, can have a pronounced effect on the displacement process (Stoodt and Slattery, 1984); however, it seems wise to begin this type of study with the simplest possible problem and add complicating factors when the route to a solution has become more clear. In addition, the boundary conditions at the fluid-solid interfaces do not account for any colloidal forces and thus can not provide an accurate description for flow in clay systems (Mulla *et al.*, 1984).

To obtain the averaged forms of the governing equations, we need the spatial averaging theorem for a three-phase system and this is given by

$$\langle \nabla \psi_\beta \rangle = \nabla \langle \psi_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \psi_\beta \, \mathrm{d}A + \frac{1}{\mathscr{V}} \int_{A_{\beta\gamma}} \mathbf{n}_{\beta\gamma} \psi_\beta \, \mathrm{d}A. \tag{2.10}$$

Here $\langle \psi_\beta \rangle$ represents the phase average defined by

$$\langle \psi_\beta \rangle = \frac{1}{\mathscr{V}} \int_{V_\beta} \psi_\beta \, \mathrm{d}V. \tag{2.11}$$

A similar form of Equation (2.10) exists for the γ-phase and the analogous form for the σ-phase is not required since we are again dealing with a rigid porous medium.

It is convenient to begin the development of the averaged equations with Equation (2.2), and use of the averaging theorem leads to

$$\langle \nabla \cdot \mathbf{v}_\beta \rangle = \nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \mathbf{v}_\beta \, \mathrm{d}A + \frac{1}{\mathscr{V}} \int_{A_{\beta\gamma}} \mathbf{n}_{\beta\gamma} \cdot \mathbf{v}_\beta \, \mathrm{d}A = 0. \tag{2.12}$$

On the basis of the boundary condition given by Equation (2.3), this result simplifies to

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\gamma}} \mathbf{n}_{\beta\gamma} \cdot \mathbf{v}_\beta \, \mathrm{d}A = 0. \tag{2.13}$$

The general transport theorem (Truesdell and Toupin, 1960) can be used to obtain the geometrical relation

$$\frac{\partial \epsilon_\beta}{\partial t} = \frac{1}{\mathscr{V}} \int_{A_{\beta\gamma}} \mathbf{n}_{\beta\gamma} \cdot \mathbf{w} \, \mathrm{d}A \tag{2.14}$$

in which $\mathbf{n}_{\beta\gamma} \cdot \mathbf{w}$ is the normal component of the velocity of the β–γ interface. Use of this relation in Equation (2.13) leads to

$$\frac{\partial \epsilon_\beta}{\partial t} + \nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\gamma}} \mathbf{n}_{\beta\gamma} \cdot (\mathbf{v}_\beta - \mathbf{w}) \, \mathrm{d}A = 0. \tag{2.15}$$

When interfacial mass transfer is important (Whitaker, 1977) the last term in Equation (2.15) is nonzero; however, when the two fluids are immiscible we have

$$\mathbf{v}_\beta \cdot \mathbf{n}_{\beta\gamma} = \mathbf{w} \cdot \mathbf{n}_{\beta\gamma} = \mathbf{v}_\gamma \cdot \mathbf{n}_{\beta\gamma}, \quad \text{at } \mathscr{A}_{\beta\gamma} \tag{2.16}$$

and Equation (2.15) simplifies to

$$\frac{\partial \epsilon_\beta}{\partial t} + \nabla \cdot \langle \mathbf{v}_\beta \rangle = 0. \tag{2.17}$$

The volume-averaged form of the continuity equation for the γ-phase is obviously given by

$$\frac{\partial \epsilon_\gamma}{\partial t} + \nabla \cdot \langle \mathbf{v}_\gamma \rangle = 0 \tag{2.18}$$

and for completeness we list the constraint on the volume fractions as

$$\epsilon_\beta + \epsilon_\gamma + \epsilon_\sigma = 1. \tag{2.19}$$

The averaged forms of Equations (2.1) and (2.7) are obtained in precisely the same manner as for single-phase flow, and we simply list the results as

$$\begin{aligned} 0 = \nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \frac{1}{V_\beta} \int_{A_{\beta\sigma}} (-\mathbf{n}_{\beta\sigma} \tilde{p}_\beta + \mu_\beta \mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta) \, \mathrm{d}A + \\ + \frac{1}{V_\beta} \int_{A_{\beta\gamma}} (-\mathbf{n}_{\beta\gamma} \tilde{p}_\beta + \mu_\beta \mathbf{n}_{\beta\gamma} \cdot \nabla \tilde{\mathbf{v}}_\beta) \, \mathrm{d}A, \end{aligned} \tag{2.20}$$

$$\begin{aligned} 0 = -\nabla \langle p_\gamma \rangle^\gamma + \rho_\gamma \mathbf{g} + \frac{1}{V_\gamma} \int_{A_{\gamma\sigma}} (-\mathbf{n}_{\gamma\sigma} \tilde{p}_\gamma + \mu_\gamma \mathbf{n}_{\gamma\sigma} \cdot \nabla \tilde{\mathbf{v}}_\gamma) \, \mathrm{d}A + \\ + \frac{1}{V_\gamma} \int_{A_{\gamma\beta}} (-\mathbf{n}_{\gamma\beta} \tilde{p}_\gamma + \mu_\gamma \mathbf{n}_{\gamma\beta} \cdot \nabla \tilde{\mathbf{v}}_\gamma) \, \mathrm{d}A. \end{aligned} \tag{2.21}$$

These results are analogous to Equation (2.37) in Part I of this paper (Whitaker, 1986), and we have again made use of the length scale constraint $\ell \ll r_0 \ll L$ to discard the various Brinkman-like viscous terms that are evident in Equation (2.29) of Part I.

3. Closure

The governing equations for the spatial deviations can be extracted directly from the development given in Part I, and we list the results here as

$$\nabla \cdot \tilde{\mathbf{v}}_\beta = 0. \tag{3.1}$$

$$-\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta = \frac{1}{V_\beta} \int_{V_\beta} [-\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta] \, \mathrm{d}V, \tag{3.2}$$

$$\nabla \cdot \tilde{\mathbf{v}}_\gamma = 0, \tag{3.3}$$

$$-\nabla \tilde{p}_\gamma + \mu_\gamma \nabla^2 \tilde{\mathbf{v}}_\gamma = \frac{1}{V_\gamma} \int_{V_\gamma} [-\nabla \tilde{p}_\gamma + \mu_\gamma \nabla^2 \tilde{\mathbf{v}}_\gamma] \, \mathrm{d}V. \tag{3.4}$$

The boundary conditions for the spatial deviations are obtained by direct application of the decompositions

$$\mathbf{v}_\beta = \langle \mathbf{v}_\beta \rangle^\beta + \tilde{\mathbf{v}}_\beta, \qquad \mathbf{v}_\gamma = \langle \mathbf{v}_\gamma \rangle^\gamma + \tilde{\mathbf{v}}_\gamma \tag{3.5}$$

along with the analogous forms for the pressure. Use of these decompositions in Equations (2.3) through (2.6) leads to

$$\text{B.C.1} \quad \tilde{\mathbf{v}}_\beta = -\langle \mathbf{v}_\beta \rangle^\beta, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{3.6}$$

$$\text{B.C.2} \quad \tilde{\mathbf{v}}_\beta = \tilde{\mathbf{v}}_\gamma - (\langle \mathbf{v}_\beta \rangle^\beta - \langle \mathbf{v}_\gamma \rangle^\gamma), \quad \text{at } \mathscr{A}_{\beta\gamma}, \tag{3.7}$$

$$\begin{aligned} \text{B.C.3} \quad -\tilde{p}_\beta \mathbf{n}_{\beta\gamma} = & -\tilde{p}_\gamma \mathbf{n}_{\beta\gamma} + (\langle p_\beta \rangle^\beta - \langle p_\gamma \rangle^\gamma) \mathbf{n}_{\beta\gamma} - \\ & -[\mu_\beta (\nabla \tilde{\mathbf{v}}_\beta + \nabla \tilde{\mathbf{v}}_\beta^T) \cdot \mathbf{n}_{\beta\gamma} - \mu_\gamma (\nabla \tilde{\mathbf{v}}_\gamma + \nabla \tilde{\mathbf{v}}_\gamma^T) \cdot \mathbf{n}_{\beta\gamma}] + \\ & +2\sigma H \mathbf{n}_{\beta\gamma}, \quad \text{at } \mathscr{A}_{\beta\gamma}, \end{aligned} \tag{3.8}$$

$$\text{B.C.4} \quad \tilde{\mathbf{v}}_\gamma = -\langle \mathbf{v}_\gamma \rangle^\gamma, \quad \text{at } \mathscr{A}_{\gamma\sigma}. \tag{3.9}$$

Here we have made use of inequalities of the form

$$\mu_\alpha (\nabla \langle \mathbf{v}_\alpha \rangle^\alpha + \nabla \langle \mathbf{v}_\alpha \rangle^{\alpha^T}) \ll \mu_\alpha (\nabla \tilde{\mathbf{v}}_\alpha + \nabla \tilde{\mathbf{v}}_\alpha^T) \tag{3.10}$$

in which α represents both β and γ. Equation (3.10) is based on the same type of estimates that lead us from Equations (2.29) to (2.37) in Part I of this paper, and this type of simplification has already been imposed in order to obtain Equations (2.20) and (2.21). It is based on the estimates

$$\nabla \tilde{\mathbf{v}}_\alpha = \mathbf{O}(\langle \mathbf{v}_\alpha \rangle^\alpha / \ell_\alpha), \quad \nabla \langle \mathbf{v}_\alpha \rangle^\alpha = \mathbf{O}(\langle \mathbf{v}_\alpha \rangle^\alpha / L) \tag{3.11}$$

along with the length scale constraint, $\ell_\alpha \ll L$.

Clearly there are *four* nonhomogeneous terms ($\langle \mathbf{v}_\beta \rangle^\beta, \langle \mathbf{v}_\gamma \rangle^\gamma, \langle p_\beta \rangle^\beta - \langle p_\gamma \rangle^\gamma$, $2\sigma H$) in the boundary conditions for the closure problem, and these need to be considered in the representations for the spatial deviations. However, before moving on to that problem, we will find it worthwhile to consider the *area average* of the normal component of Equations (3.8). This can be expressed as

$$\begin{aligned} &-(\langle p_\beta \rangle^\beta - \langle p_\gamma \rangle^\gamma) \\ &\quad = 2\sigma \langle H \rangle_{\beta\gamma} + \langle \tilde{p}_\beta - \tilde{p}_\gamma \rangle_{\beta\gamma} - \qquad (3.12) \\ &\quad\; -\langle \mu_\beta \mathbf{n}_{\beta\gamma} \cdot (\nabla \tilde{\mathbf{v}}_\beta + \nabla \tilde{\mathbf{v}}_\beta^T) \cdot \mathbf{n}_{\beta\gamma} - \mu_\gamma \mathbf{n}_{\beta\gamma} \cdot (\nabla \tilde{\mathbf{v}}_\gamma + \nabla \tilde{\mathbf{v}}_\gamma^T) \cdot \mathbf{n}_{\beta\gamma} \rangle_{\beta\gamma} \quad \text{at } \mathscr{A}_{\beta\gamma} \end{aligned}$$

in which the area averages are defined according to

$$\langle H \rangle_{\beta\gamma} = \frac{1}{A_{\beta\gamma}} \int_{A_{\beta\gamma}} H \, \mathrm{d}A. \qquad (3.13)$$

In treating $\langle p_\beta \rangle^\beta - \langle p_\gamma \rangle^\gamma$ as a constant with respect to integration over $A_{\beta\gamma}$, we are making use of Equations (2.18) and (2.19) in Part I of this paper (Whitaker, 1986). If we make use of the second estimate given by Equation (3.11) along with the estimate for the pressure deviation given in Appendix A of Part I, we can write Equation (3.12) as

$$(\langle p_\beta \rangle^\beta - \langle p_\gamma \rangle^\gamma) = 2\sigma \langle H \rangle_{\beta\gamma} + \mathbf{O}\left(\frac{\mu_\alpha \langle \mathbf{v}_\alpha \rangle^\alpha}{\ell_\alpha}\right). \qquad (3.14)$$

Here we have used α to represent the *largest contribution* from either the β-phase or the γ-phase to the last two terms on Equation (3.12). If we impose the following restriction

$$\frac{\mu_\alpha \langle \mathbf{v}_\alpha \rangle^\alpha}{\sigma \langle H \rangle_{\beta\gamma} \ell_\alpha} \ll 1 \qquad (3.15)$$

we can express Equation (3.14) in the form

$$p_c = -(\langle p_\beta \rangle^\beta - \langle p_\gamma \rangle^\gamma) = 2\sigma \langle H \rangle_{\beta\gamma} \qquad (3.16)$$

where p_c is the *capillary pressure*. If we follow the usual custom, we must think of the β-phase as the phase which wets the solid so that the capillary pressure is a positive quantity. Restricting the analysis by Equation (3.15) is comparable to requiring that the *capillary number* be small compared to one, and this is a situation widely encountered in practical problems of two-phase flow in porous media. It is important to note that Equation (3.14) *overestimates* the effect of the pressure and velocity deviations in Equation (3.12) since the area-average of these quantities is likely to be significantly smaller than the order of magnitude estimates.

Use of Equation (3.16) in (3.8) allows us to express that boundary condition as

$$\text{B.C.3} \quad -\tilde{p}_\beta \mathbf{n}_{\beta\sigma} = -\tilde{p}_\gamma \mathbf{n}_{\beta\gamma} - \\ -[\mu_\beta(\nabla \tilde{\mathbf{v}}_\beta + \nabla \tilde{\mathbf{v}}_\beta^T) \cdot \mathbf{n}_{\beta\gamma} - \\ -\mu_\gamma(\nabla \tilde{\mathbf{v}}_\gamma + \nabla \tilde{\mathbf{v}}_\gamma^T) \cdot \mathbf{n}_{\beta\gamma}] + 2\sigma\tilde{H}\mathbf{n}_{\beta\gamma}, \quad \text{at } \mathscr{A}_{\beta\gamma}. \tag{3.17}$$

Here it is important to keep in mind that the *curvature deviation* is defined in terms of an area average

$$\tilde{H} = H - \langle H \rangle_{\beta\gamma} \tag{3.18}$$

and that $\tilde{H}$ (like H in Equation (2.5)) is to be determined as part of the solution of the problem. At this point we are faced with a linear problem containing two nonhomogeneous terms, $\langle \mathbf{v}_\beta \rangle^\beta$ and $\langle \mathbf{v}_\gamma \rangle^\gamma$, and an appropriate representation for $\tilde{H}$ is given by

$$\tilde{H} = \mathbf{h}_1 \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{h}_2 \cdot \langle \mathbf{v}_\gamma \rangle^\gamma, \quad \text{at } \mathscr{A}_{\beta\gamma}. \tag{3.19}$$

This result is consistent with the problem statement and with the idea that H can be treated as constant within the averaging volume for the static case. In their review of two-phase flow phenomena, Wooding and Morel-Seytoux (1976) conclude that the interface shape should be independent of velocity provided the capillary number is less than one, and in another review Philip (1972) reaches the same conclusion. If this is true, the solution to the closure problem will indicate that $\tilde{H} \ll \langle H \rangle_{\beta\gamma}$; however, this does not necessarily mean that the contribution to Equation (3.17) will be negligible. With this in mind we express B.C.3 as

$$\text{B.C.3} \quad -\tilde{p}_\beta \mathbf{n}_{\beta\gamma} \\ = -\tilde{p}_\gamma \mathbf{n}_{\beta\gamma} - [\mu_\beta(\nabla \tilde{\mathbf{v}}_\beta + \nabla \tilde{\mathbf{v}}_\beta^T) \cdot \mathbf{n}_\beta - \\ -\mu_\beta(\nabla \tilde{\mathbf{v}}_\gamma + \nabla \tilde{\mathbf{v}}_\gamma^T) \cdot \mathbf{n}_{\beta\gamma}] + \\ + 2\sigma \mathbf{n}_{\beta\gamma}[\mathbf{h}_1 \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{h}_2 \cdot \langle \mathbf{v}_\gamma \rangle^\gamma], \quad \text{at } \mathscr{A}_{\beta\gamma} \tag{3.20}$$

and propose the following representations for the spatial deviations of the pressure and velocity:

$$\tilde{\mathbf{v}}_\beta = \mathbf{A}_1^\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{A}_2^\beta \cdot \langle \mathbf{v}_\gamma \rangle^\gamma + \boldsymbol{\psi}^\beta, \tag{3.21}$$

$$\tilde{p}_\beta = \mu_\beta[\mathbf{a}_1^\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{a}_2^\beta \cdot \langle \mathbf{v}_\gamma \rangle^\gamma + \xi^\beta], \tag{3.22}$$

$$\tilde{\mathbf{v}}_\gamma = \mathbf{A}_1^\gamma \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{A}_2^\gamma \cdot \langle \mathbf{v}_\gamma \rangle^\gamma + \boldsymbol{\psi}^\gamma, \tag{3.23}$$

$$\tilde{p}_\gamma = \mu_\gamma[\mathbf{a}_1^\gamma \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{a}_2^\gamma \cdot \langle \mathbf{v}_\gamma \rangle^\gamma + \xi^\gamma], \tag{3.24}$$

Here one should think of $\boldsymbol{\psi}^\alpha$ and $\boldsymbol{\xi}^\alpha$ (where α represents both β and γ) as completely arbitrary functions so that we are free to specify the coefficients in Equations (3.21) through (3.24) in any manner that we wish. On the basis of the analysis given in Part I, we choose to specify these functions in terms of the following governing differential equations and boundary conditions*

$$-\nabla \mathbf{a}_i^\alpha + \nabla^2 \mathbf{A}_i^\alpha = \frac{1}{V_\alpha}\int_{V_\alpha} [-\nabla \mathbf{a}_i^\alpha + \nabla^2 \mathbf{A}_i^\alpha]\,\mathrm{d}V, \quad i = 1, 2, \tag{3.25}$$

$$\nabla \cdot \mathbf{A}_i^\alpha = 0, \quad i = 1, 2, \tag{3.26}$$

$$\text{B.C.1} \quad \mathbf{A}_1^\beta = -\mathbf{I}, \qquad \mathbf{A}_2^\beta = 0, \quad \text{at } \mathscr{A}_{\beta\sigma}, \tag{3.27}$$

$$\text{B.C.2} \quad \mathbf{A}_1^\beta = \mathbf{A}_1^\gamma - \mathbf{I}, \qquad \mathbf{A}_2^\beta = \mathbf{A}_2^\gamma + \mathbf{I}, \quad \text{at } \mathscr{A}_{\beta\gamma} \tag{3.28}$$

$$\begin{aligned} \text{B.C.3} \quad & \mu_\beta(-\mathbf{n}_{\beta\gamma}\mathbf{a}_i^\beta + \nabla \bar{\mathbf{A}}_i^\beta \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \mathbf{A}_i^\beta) \\ & \quad = \mu_\gamma(-\mathbf{n}_{\beta\gamma}\mathbf{a}_i^\gamma + \nabla \bar{\mathbf{A}}_i^\gamma \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \mathbf{A}_i^\gamma) + 2\sigma \mathbf{n}_{\beta\gamma}\mathbf{h}_i, \\ & \quad \text{at } \mathscr{A}_{\beta\gamma}, \quad i = 1, 2 \end{aligned} \tag{3.29}$$

$$\text{B.C.4} \quad \mathbf{A}_1^\gamma = 0, \qquad \mathbf{A}_2^\gamma = -\mathbf{I}, \quad \text{at } \mathscr{A}_{\gamma\sigma}, \tag{3.30}$$

$$\langle \mathbf{a}_i^\alpha \rangle^\alpha = 0, \qquad \langle \mathbf{A}_i^\alpha \rangle^\alpha = 0, \quad i = 1, 2. \tag{3.31}$$

One must keep in mind that no approximations have been made in this part of the closure scheme; however, we are now confronted with the problem of demonstrating that $\boldsymbol{\psi}^\alpha$ and ξ^α make negligible contributions to the velocity and pressure deviations. In developing the boundary value problem for $\boldsymbol{\psi}^\alpha$ and ξ^α we follow the analysis presented in Part I of this paper (Whitaker, 1986), but in this case we retain only the most important nonhomogeneous terms and represent the results as

$$\begin{aligned} -\nabla \xi^\alpha + \nabla \boldsymbol{\psi}^\alpha = & \frac{1}{V_\alpha}\int_{V_\alpha} [-\nabla \xi^\alpha + \nabla^2 \boldsymbol{\psi}^\alpha]\,\mathrm{d}V + \\ & + \mathbf{O}[\nabla \langle \mathbf{v}_\alpha \rangle^\alpha \cdot \mathbf{a}_i^\alpha, \nabla \langle \mathbf{a}_\alpha \rangle^\alpha : \nabla \mathbf{A}_i^\alpha]. \end{aligned} \tag{3.32}$$

$$\nabla \cdot \boldsymbol{\psi}^\alpha = \mathbf{O}[\nabla \langle \mathbf{v}_\alpha \rangle^\alpha : \mathbf{A}_i^\alpha], \tag{3.33}$$

$$\text{B.C.1} \quad \boldsymbol{\psi}^\beta = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{3.34}$$

$$\text{B.C.2} \quad \boldsymbol{\psi}^\beta = \boldsymbol{\psi}^\gamma, \quad \text{on } \mathscr{A}_{\beta\gamma}, \tag{3.35}$$

$$\begin{aligned} \text{B.C.3} \quad & \mu_\beta(-\mathbf{n}_{\beta\gamma}\xi^\beta + \nabla \boldsymbol{\psi}^\beta \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \boldsymbol{\psi}^\beta) \\ & \quad = \mu_\gamma(-\mathbf{n}_{\beta\gamma}\xi^\gamma + \nabla \boldsymbol{\psi}^\gamma \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \boldsymbol{\psi}^\gamma) + \\ & \quad + \mathbf{O}[\mu_\alpha \nabla \langle \mathbf{v}_\alpha \rangle^\alpha \cdot \mathbf{A}_i^\alpha], \quad \text{at } \mathscr{A}_{\beta\gamma} \end{aligned} \tag{3.36}$$

$$\text{B.C.4} \quad \boldsymbol{\psi}^\gamma = 0, \quad \text{on } \mathscr{A}_{\gamma\sigma}, \tag{3.37}$$

$$\langle \xi^\alpha \rangle^\alpha = 0, \qquad \langle \boldsymbol{\psi}^\alpha \rangle^\alpha = 0. \tag{3.38}$$

Here we have used α to represent both β and γ when it is possible to do so.

* For compactness we have used α to represent both β and γ, and for neatness the transpose of $\mathbf{A}_i^\alpha$ has been indicated by $\bar{\mathbf{A}}_i^\alpha$.

In Appendix A it is shown that the solution of the homogeneous problem for ξ^α and $\boldsymbol{\psi}^\alpha$ in a spatially periodic system is the null solution, thus we feel comfortable in estimating the magnitude of ξ^α and $\boldsymbol{\psi}^\alpha$ on the basis of the nonhomogeneous terms in Equations (3.32), (3.33) and (3.36). The estimates are of the same form as those given in Part I of this paper and we express them as

$$\boldsymbol{\psi}^\alpha = \mathbf{O}\left\{\mathbf{A}_i^\alpha \cdot \langle \mathbf{v}_\alpha \rangle^\alpha \left(\frac{\ell_\alpha}{L}\right), \quad \mathbf{a}_i^\alpha \cdot \langle \mathbf{v}_\alpha \rangle^\alpha \left(\frac{\ell_\alpha^2}{L}\right)\right\}, \tag{3.39}$$

$$\xi^\alpha = \mathbf{O}\left\{\mathbf{a}_i^\alpha \cdot \langle \mathbf{v}_\alpha \rangle^\alpha \left(\frac{\ell_\alpha}{L}\right), \quad \mathbf{A}_i^\alpha \cdot \langle \mathbf{v}_\alpha \rangle^\alpha / L\right\}. \tag{3.40}$$

From this point one can follow the same line of reasoning presented in Part I to conclude that $\boldsymbol{\psi}^\alpha$ and ξ^α make negligible contributions to the pressure and velocity deviations and Equations (3.21) through (3.24) can be simplified to

$$\tilde{\mathbf{v}}_\beta = \mathbf{A}_1^\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{A}_2^\beta \cdot \langle \mathbf{v}_\gamma \rangle^\gamma, \tag{3.41}$$

$$\tilde{p}_\beta = \mu_\beta \left(\mathbf{a}_1^\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{a}_2^\beta \cdot \langle \mathbf{v}_\gamma \rangle^\gamma\right), \tag{3.42}$$

$$\tilde{\mathbf{v}}_\gamma = \mathbf{A}_1^\gamma \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{A}_2^\gamma \cdot \langle \mathbf{v}_\gamma \rangle^\gamma, \tag{3.43}$$

$$\tilde{p}_\gamma = \mu_\gamma \left(\mathbf{a}_1^\gamma \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{a}_2^\gamma \cdot \langle \mathbf{v}_\gamma \rangle^\gamma\right). \tag{3.44}$$

These expressions are to be used in Equations (2.20) and (2.21) to provide us with the correct *form* of the volume-averaged momentum equations for two-phase flow. In doing this, we make repeated use of the approximations indicated by Equations (2.18), (3.33) and (3.34) in Part I of this paper in order to obtain

$$0 = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} - \mu_\beta \mathbf{M}_1^\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta - \mu_\beta \mathbf{M}_2^\beta \cdot \langle \mathbf{v}_\gamma \rangle^\gamma, \tag{3.45}$$

$$0 = -\nabla \langle p_\gamma \rangle^\gamma + \rho_\gamma \mathbf{g} - \mu_\gamma \mathbf{M}_1^\gamma \cdot \langle \mathbf{v}_\gamma \rangle^\gamma - \mu_\gamma \mathbf{M}_2^\gamma \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{3.46}$$

Here the $\mathbf{M}_i^\alpha$ are defined by

$$\mathbf{M}_i^\alpha = -\frac{1}{V_\alpha}\left\{\int_{A_{\alpha\sigma}} [-\mathbf{n}_{\alpha\sigma}\mathbf{a}_i^\alpha + \mathbf{n}_{\alpha\sigma} \cdot \nabla \mathbf{A}_i^\alpha]\,\mathrm{d}V + \right.$$
$$\left. + \int_{A_{\alpha_\eta}} [-\mathbf{n}_{\alpha\eta}\mathbf{a}_i^\alpha + \mathbf{n}_{\alpha\eta} \cdot \nabla \mathbf{A}_i^\alpha]\,\mathrm{d}V\right\} \tag{3.47}$$

Once again we have used $i = 1, 2$ and $\alpha = \beta, \gamma$, and in this case we have also used $\eta = \beta, \gamma$ with the restriction that $\eta \neq \alpha$. At this point we need only *assume* that the inverses of $\mathbf{M}_1^\beta$ and $\mathbf{M}_2^\gamma$ exist in order to obtain*

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}_\beta}{\mu_\beta} \cdot [\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}] + \mathbf{K}_{\beta\gamma} \cdot \langle \mathbf{v}_\gamma \rangle \tag{3.48}$$

* Similar forms have been postulated by Raats and Klute (1968) and more recently by Baveye and Sposito (1984).

$$\langle \mathbf{v}_\gamma \rangle = -\frac{\mathbf{K}_\gamma}{\mu_\gamma} \cdot [\nabla \langle p_\gamma \rangle^\gamma - \rho_\gamma \mathbf{g}] + \mathbf{K}_{\gamma\beta} \cdot \langle \mathbf{v}_\beta \rangle \tag{3.49}$$

Here we have represented Equations (3.48) and (3.49) in terms of the *phase average* velocities, and the four tensors in this result are defined by

$$\mathbf{K}_\beta = \epsilon_\beta (\mathbf{M}_1^\beta)^{-1}, \qquad \mathbf{K}_{\beta\gamma} = -(\mathbf{M}_1^\beta)^{-1} \cdot \mathbf{M}_2^\beta (\epsilon_\beta / \epsilon_\gamma) \tag{3.50}$$

$$\mathbf{K}_\gamma = \epsilon_\gamma (\mathbf{M}_2^\gamma)^{-1}, \qquad \mathbf{K}_{\gamma\beta} = -(\mathbf{M}_2^\gamma)^{-1} \cdot \mathbf{M}_1^\gamma (\epsilon_\gamma / \epsilon_\beta) \tag{3.51}$$

It is clear that the nontraditional terms in Equations (3.48) and (3.49) represent the viscous drag of one fluid upon the other, and the importance of these terms depends on their magnitude relative to the viscous drag exerted by the solid and relative to the surface tension forces. From an intuitive point of view, one would expect that the term, $\mathbf{K}_{\beta\gamma} \cdot \langle \mathbf{v}_\gamma \rangle$, in Equation (3.48) could be neglected when $\mu_\gamma \gg \mu_\beta$. Under these circumstances the γ-phase acts as a 'solid' relative to the β-phase, and the motion of the β-phase should be governed by the traditional modification of Darcy's law. Under these same conditions, one could argue that the β-phase would exert a negligible viscous stress on the γ-phase, thus allowing us to discard the last term in Equation (3.49). This line of reasoning would lead one to conclude that the traditional *form* proposed by Richards (1931) and Muskat *et al.* (1937) is satisfactory except when the ratio μ_γ / μ_β is on the order of one. Philip (1972) takes a more definitive position based on several case studies and concludes that the matter of momentum exchange between fluids is, for most practical purposes, a nonproblem. This point of view is supported by the fact that surfactants are always present in real systems, thus the β–γ interface is highly resistant to shear. Clearly some detailed calculations that would produce values for the tensors in Equations (3.48) and (3.49) would be of value in assessing the importance of fluid-fluid momentum exchange.

4. Experiments

A theoretical determination of the importance of the nontraditional terms in Equations (3.48) and (3.49) would require the solution of Equations (3.25) through (3.31). This represents an extremely difficult computational problem, and before engaging in such an effort one is inclined to ask the question: If the nontraditional terms in Equations (3.48) and (3.49) are important, why has this not been discovered experimentally? Clearly such a question deserves an answer.

The traditional scheme for determining relative permeabilities makes use of a steady, uniform flow in the absence of gravitational effects. If the gravitational terms in Equations (3.48) and (3.49) are neglected and Equation (3.16) is used, these equations can be expressed as

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}_\beta}{\mu_\beta} \cdot \nabla \langle p_\gamma \rangle^\gamma + \mathbf{K}_{\beta\gamma} \cdot \langle \mathbf{v}_\gamma \rangle + \frac{\mathbf{K}_\beta}{\mu_\beta} \cdot \nabla p_c, \tag{4.1}$$

$$\langle \mathbf{v}_\gamma \rangle = -\frac{\mathbf{K}_\gamma}{\mu_\gamma} \cdot \nabla \langle p_\gamma \rangle^\gamma + \mathbf{K}_{\gamma\beta} \cdot \langle \mathbf{v}_\beta \rangle \tag{4.2}$$

Elimination of $\langle \mathbf{v}_\gamma \rangle$ from Equation (4.1) and $\langle \mathbf{v}_\beta \rangle$ from Equation (4.2) leads to

$$\begin{aligned} \langle \mathbf{v}_\beta \rangle = &-(\mathbf{I} - \mathbf{K}_{\beta\gamma} \cdot \mathbf{K}_{\gamma\beta})^{-1} \cdot \left(\frac{\mathbf{K}_\beta}{\mu_\beta} + \frac{\mathbf{K}_{\beta\gamma} \cdot \mathbf{K}_\gamma}{\mu_\gamma} \right) \cdot \nabla \langle p_\gamma \rangle^\gamma + \\ &+(\mathbf{I} - \mathbf{K}_{\beta\gamma} \cdot \mathbf{K}_{\gamma\beta})^{-1} \cdot \left(\frac{\mathbf{K}_\beta}{\mu_\beta} \cdot \nabla p_c \right) \end{aligned} \tag{4.3}$$

$$\begin{aligned} \langle \mathbf{v}_\gamma \rangle = &-(\mathbf{I} - \mathbf{K}_{\gamma\beta} \cdot \mathbf{K}_{\beta\gamma})^{-1} \cdot \left(\frac{\mathbf{K}_\gamma}{\mu_\gamma} + \frac{\mathbf{K}_{\gamma\beta} \cdot \mathbf{K}_\beta}{\mu_\beta} \right) \cdot \nabla \langle p_\gamma \rangle^\gamma + \\ &+(\mathbf{I} - \mathbf{K}_{\gamma\beta} \cdot \mathbf{K}_{\beta\gamma})^{-1} \cdot \left(\frac{\mathbf{K}_{\gamma\beta} \cdot \mathbf{K}_\beta}{\mu_\beta} \right) \cdot \nabla p_c \, . \end{aligned} \tag{4.4}$$

In the traditional experiment used to measure relative permeabilities (Bear, 1972), great care is taken to insure that conditions are *uniform*. This means that ϵ_β (and ϵ_γ) are constant along with the capillary pressure, and Equations (4.3) and (4.4) reduce to

$$\langle \mathbf{v}_\beta \rangle = -(\mathbf{I} - \mathbf{K}_{\beta\gamma} \cdot \mathbf{K}_{\gamma\beta})^{-1} \cdot \left(\frac{\mathbf{K}_\beta}{\mu_\beta} + \frac{\mathbf{K}_{\beta\gamma} \cdot \mathbf{K}_\gamma}{\mu_\gamma} \right) \cdot \nabla \langle p_\gamma \rangle^\gamma, \tag{4.5}$$

$$\langle \mathbf{v}_\gamma \rangle = -(\mathbf{I} - \mathbf{K}_{\gamma\beta} \cdot \mathbf{K}_{\beta\gamma})^{-1} \cdot \left(\frac{\mathbf{K}_\gamma}{\mu_\gamma} + \frac{\mathbf{K}_{\gamma\beta} \cdot \mathbf{K}_\beta}{\mu_\beta} \right) \cdot \nabla \langle p_\gamma \rangle^\gamma. \tag{4.6}$$

Here we see that the general *form* of the equations for uniform flow is identical to that obtained from Equations (3.48) and (3.49) with $\mathbf{K}_{\beta\gamma} = \mathbf{K}_{\gamma\beta} = 0$; however, the *exact form* including the coefficients of viscosity would only be the same if

$$\mathbf{K}_{\beta\gamma} \sim \frac{\mu_\gamma}{\mu_\beta}, \qquad \mathbf{K}_{\gamma\beta} \sim \frac{\mu_\beta}{\mu_\gamma} \tag{4.7}$$

That this is actually the case will be demonstrated in the following paragraphs.

4.1. ORDER OF MAGNITUDE ANALYSIS

In order to develop some idea about the functional dependence of $\mathbf{K}_\beta$, $\mathbf{K}_\gamma$, $\mathbf{K}_{\beta\gamma}$ and $\mathbf{K}_{\gamma\beta}$, we need to consider Equation (3.47) and the closure problem in some detail. To begin with, we can use Equation (3.47) to express $\mathbf{M}_1^\beta$ as

$$\mathbf{M}_1^\beta = -\frac{1}{V_\beta} \int_{A_{\beta\sigma} + A_{\beta\gamma}} \mathbf{n} \cdot [\nabla \mathbf{A}_1^\beta - \mathbf{I}\mathbf{a}_1^\beta] \, \mathrm{d}A \tag{4.8}$$

in which $\mathbf{n}$ represents the outwardly directed unit normal vector for the β-phase. From the first of Equations (3.27) we put forth the estimate

$$\mathbf{A}_1^\beta = \mathbf{O}(1) \tag{4.9}$$

with the idea that the no-slip boundary condition at the β–σ interface dominates this field. In estimating $\mathbf{a}_1^\beta$ we ignore the boundary condition given by Equation (3.29) and use the governing differential equation given by Equation (3.25) to obtain

$$\mathbf{a}_1^\beta = \mathbf{O}(\ell_\beta^{-1}). \tag{4.10}$$

Use of these two estimates in Equation (4.8) yields

$$\mathbf{M}_1^\beta = \mathbf{O}(\ell_\beta^{-2}). \tag{4.11}$$

Here we have used the idea that the characteristic length for the β-phase is given by

$$\ell_\beta^{-1} = \mathbf{O}[(A_{\beta\sigma} + A_{\beta\gamma})/V_\beta]. \tag{4.12}$$

For $\mathbf{M}_2^\gamma$ we center our attention on Equations (3.30) and (3.25) to conclude that

$$\mathbf{M}_2^\gamma = \mathbf{O}(\ell_\gamma^{-2}) \tag{4.13}$$

and from the definition of $\mathbf{K}_\beta$ and $\mathbf{K}_\gamma$ given by Equations (3.50) and (3.51) we find

$$\mathbf{K}_\beta = \mathbf{O}(\epsilon_\beta \ell_\beta^2), \qquad \mathbf{K}_\gamma = \mathbf{O}(\epsilon_\gamma \ell_\gamma^2). \tag{4.14}$$

If we now let ℓ represent the characteristic length for the void space, we can estimate ℓ_β and ℓ_γ by

$$\ell_\beta \sim \epsilon_\beta \ell, \qquad \ell_\gamma \sim \epsilon_\gamma \ell. \tag{4.15}$$

Use of these relations in Equations (4.14) leads to

$$\mathbf{K}_\beta = \mathbf{O}(\epsilon_\beta^3 \ell^2), \qquad \mathbf{K}_\gamma = \mathbf{O}(\epsilon_\gamma^3 \ell^2). \tag{4.16}$$

These estimates are in reasonably good agreement with experimental values of the Darcy's law permeability and the relative permeabilities for the two phases. To estimate $\mathbf{K}_{\beta\gamma}$ we need $\mathbf{M}_2^\beta$ from Equation (3.47) and this is given by

$$\mathbf{M}_2^\beta = -\frac{1}{V_\beta} \int_{A_{\beta\sigma}+A_{\beta\gamma}} \mathbf{n} \cdot [\nabla \mathbf{A}_2^\beta - \mathbf{I}\mathbf{a}_2^\beta] \, \mathrm{d}A. \tag{4.17}$$

If we *ignore the surface tension term* in Equation (3.29), we can use that result to form the estimate

$$\mathbf{n}_{\beta\gamma} \cdot [\nabla \mathbf{A}_2^\beta - \mathbf{I}\mathbf{a}_2^\beta] = \mathbf{O}\left\{\left(\frac{\mu_\gamma}{\mu_\beta}\right) \mathbf{n}_{\beta\gamma} \cdot [\nabla \mathbf{A}_2^\gamma - \mathbf{I}\mathbf{a}_2^\gamma]\right\}, \quad \text{at } \mathscr{A}_{\beta\gamma}. \tag{4.18}$$

The estimates for $\mathbf{A}_2^\gamma$ and $\mathbf{a}_2^\gamma$ are comparable to those given by Equations (4.9) and (4.10) and we list them here as

$$\mathbf{A}_2^\gamma = \mathbf{O}(1), \qquad \mathbf{a}_2^\gamma = \mathbf{O}(\ell_\gamma^{-1}). \tag{4.19}$$

Use of these relations in Equation (4.18) leads to

$$\mathbf{n}_{\beta\gamma} \cdot [\nabla \mathbf{A}_2^\beta - \mathbf{Ia}_2^\beta] = \mathbf{O}\left\{\left(\frac{\mu_\gamma}{\mu_\beta}\right) \ell_\gamma^{-1}\right\}, \quad \text{at } \mathscr{A}_{\beta\gamma} \tag{4.20}$$

and if we use this result as an estimate in Equation (4.17) we obtain

$$\mathbf{M}_2^\beta = \mathbf{O}\left[\left(\frac{\mu_\gamma}{\mu_\beta}\right)(\ell_\beta \ell_\gamma)^{-1}\right]. \tag{4.21}$$

From the second of Equation (3.50) and from Equations (4.11) and (4.21) we obtain the estimate for $\mathbf{K}_{\beta\gamma}$ given by

$$\mathbf{K}_{\beta\gamma} = \mathbf{O}\left[\left(\frac{\mu_\gamma}{\mu_\beta}\right)\left(\frac{\ell_\beta}{\ell_\gamma}\right)\left(\frac{\epsilon_\beta}{\epsilon_\gamma}\right)\right]. \tag{4.22}$$

A similar line of reasoning for $\mathbf{K}_{\gamma\beta}$ yields

$$\mathbf{K}_{\gamma\beta} = \mathbf{O}\left[\left(\frac{\mu_\beta}{\mu_\gamma}\right)\left(\frac{\ell_\gamma}{\ell_\beta}\right)\left(\frac{\epsilon_\gamma}{\epsilon_\beta}\right)\right] \tag{4.23}$$

and we can use Equation (4.15) to obtain our final estimates

$$\mathbf{K}_{\beta\gamma} = \mathbf{O}\left[\left(\frac{\mu_\gamma}{\mu_\beta}\right)\left(\frac{\epsilon_\beta}{\epsilon_\gamma}\right)^2\right], \qquad \mathbf{K}_{\gamma\beta} = \mathbf{O}\left[\left(\frac{\mu_\beta}{\mu_\gamma}\right)\left(\frac{\epsilon_\gamma}{\epsilon_\beta}\right)^2\right]. \tag{4.24}$$

Although these results are only *estimates*, they support the suggestion indicated by Equations (4.7). This means that the coefficient of $\nabla \langle p_\gamma \rangle^\gamma$ in Equation (4.5) is essentially *inversely proportional* to μ_β and the coefficient of $\nabla \langle p_\gamma \rangle^\gamma$ in Equation (4.6) is essentially *inversely proportional* to μ_γ. Under these circumstances, it would seem to be virtually impossible to detect the *form* of Equations (3.48) and (3.49) using steady-state, uniform flow conditions such as are traditionally used in the determination of relative permeabilities. Keeping in mind that we have only estimates of $\mathbf{K}_\beta$, $\mathbf{K}_\gamma$, $\mathbf{K}_{\beta\gamma}$ and $\mathbf{K}_{\gamma\beta}$, it is of some interest to use Equations (4.16) and (4.24) in Equations (4.5) to obtain

$$\langle \mathbf{v}_\beta \rangle = -\frac{\ell^2}{\mu_\beta}\left\{\mathbf{O}(\epsilon_\beta^3) + \mathbf{O}(\epsilon_\gamma \epsilon_\beta^2)\right\} \nabla \langle p_\gamma \rangle^\gamma \tag{4.25}$$

This would suggest that the last term in Equation (3.48), which gives rise to the term $\epsilon_\gamma \epsilon_\beta^2$, may be of considerable importance as $\epsilon_\beta \to 0$. In recent studies of the liquid phase motion during the drying of granular media (Whitaker, 1984), it was found that nonzero values of the relative permeability existed below the so-called *critical saturation*. This phenomenon may, in fact, be a manifestation of the additional flow caused by the last two terms in Equations (3.48) and (3.49); however, the nonuniformities that exist for transient, two-phase flows (Jacquin and Adler, 1985) in conjunction with the concepts suggested by Baveye and Sposito (1984) are a more likely explanation. It seems clear that an experimental exploration would benefit from both uniform flow experiments and imbibition experiments; however, the latter are difficult to carry out in a manner that gives rise to a one-dimensional flow (Lefebvre du Prey, 1978).

5. Solution of the Closure Problem

In the formulation of the closure problem presented in Section 3, no mention was made of the boundary conditions at the entrances and exits of the macroscopic system, $\mathscr{A}_{\beta e}$ and $\mathscr{A}_{\gamma e}$ in the notation of Part I of this paper. Since the closure problem is a *local problem* to be solved in a representative region, the boundary conditions at entrances and exits are to be replaced with periodic conditions and the closure problem takes the form

$$-\nabla \mathbf{a}_i^\alpha + \nabla^2 \mathbf{A}_i^\alpha = \frac{1}{V_\alpha}\int_{V_\alpha} [-\nabla \mathbf{a}_i^\alpha + \nabla^2 \mathbf{A}_i^\alpha]\,\mathrm{d}V, \quad i = 1, 2, \tag{5.1}$$

$$\nabla \cdot \mathbf{A}_i^\alpha = 0, \quad i = 1, 2, \tag{5.2}$$

$$\text{B.C.1} \quad \mathbf{A}_1^\beta = -\mathbf{I}, \qquad \mathbf{A}_2^\beta = 0, \quad \text{at } A_{\beta\sigma}^* \tag{5.3}$$

$$\text{B.C.2} \quad \mathbf{A}_1^\beta = \mathbf{A}_1^\gamma - \mathbf{I}, \qquad \bar{\mathbf{A}}_2^\beta = \mathbf{A}_2^\gamma + \mathbf{I}, \quad \text{at } A_{\beta\gamma}^* \tag{5.4}$$

$$\begin{aligned} \text{B.C.3} \quad & \mu_\beta(-\mathbf{n}_{\beta\gamma}\mathbf{a}_i^\beta + \nabla \bar{\mathbf{A}}_i^\beta \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \mathbf{A}_i^\beta) \\ & = \mu_\gamma(-\mathbf{n}_{\beta\gamma}\mathbf{a}_i^\gamma + \nabla \bar{\mathbf{A}}_i^\gamma \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \mathbf{A}_i^\gamma) + 2\sigma \mathbf{n}_{\beta\gamma}\mathbf{h}_i, \\ & \text{at } A_{\beta\gamma}^*, \quad i = 1, 2, \end{aligned} \tag{5.5}$$

$$\text{B.C.4} \quad \mathbf{A}_1^\gamma = 0, \qquad \mathbf{A}_2^\gamma = -\mathbf{I}, \quad \text{at } A_{\gamma\sigma}^*, \tag{5.6}$$

$$\text{B.C.5} \quad \mathbf{a}_i^\alpha(\mathbf{r} + \ell_j) = \mathbf{a}_i^\alpha(\mathbf{r}), \qquad \mathbf{A}_i^\alpha(\mathbf{r} + \ell_j) = \mathbf{A}_i^\alpha(\mathbf{r}), \quad i = 1, 2, \quad j = 1, 2, 3, \tag{5.7}$$

$$\langle \mathbf{a}_i^\alpha \rangle^\alpha = 0, \qquad \langle \mathbf{A}_i^\alpha \rangle^\alpha = 0, \quad i = 1, 2. \tag{5.8}$$

Here $A_{\beta\sigma}^*$, $A_{\beta\gamma}^*$ and $A_{\gamma\sigma}^*$ represent the interfacial areas contained within the unit cell illustrated in Figure 2. There the β-phase is shown as the fluid that wets the solid phase, and one would begin the solution procedure by a search for an acceptable solution of the static problem. This would require the specification of the curvature, $H = \langle H \rangle_{\beta\gamma}$, and a contact angle. Solution of the static problem would yield the volume fractions, ϵ_β and ϵ_γ, and thus a point on the capillary pressure-saturation curve. The velocities $\langle \mathbf{v}_\beta \rangle^\beta$ and $\langle \mathbf{v}_\gamma \rangle^\gamma$, would then be specified so that the curvature deviation could be computed in an iterative manner in conjunction with the solution of Equations (5.1) through (5.8). Any strong coupling between the velocity and the curvature would surely lead to numerical difficulties and to the prediction of significant hysteresis effects. However, when the restriction given by Equation (3.15) is in effect, the coupling should be weak if not non-existent. This would allow for the solution of Equations (3.25) through (3.31) in a straightforward manner once the position of the β–γ interface is

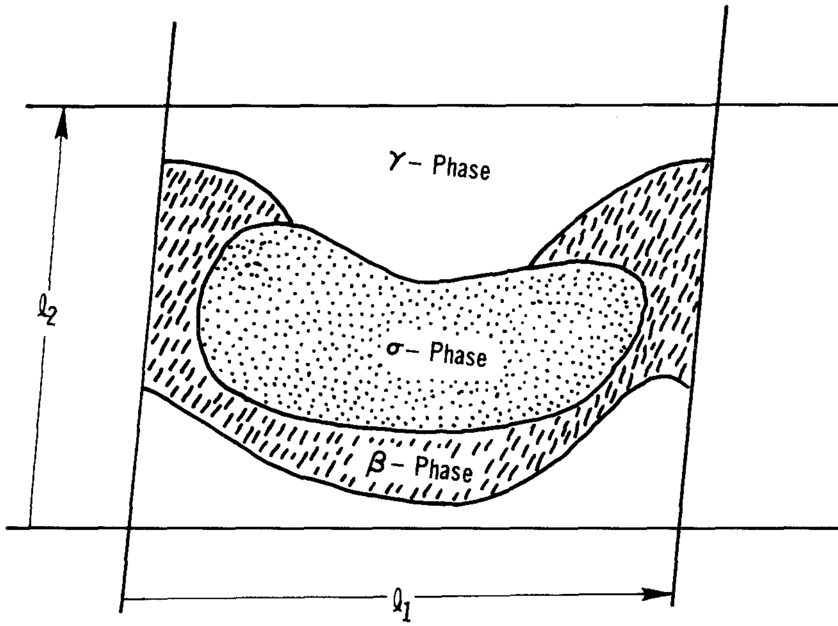

Fig. 2. Two-phase flow in a spatially periodic porous medium.

located. With calculated values for the $\mathbf{a}_i^\alpha$ and $\mathbf{A}_i^\alpha$ fields, the four tensors in Equations (3.38) and (3.39) could be determined for the particular saturation in question. Repeated calculations for different values of the curvature would yield values of $\mathbf{K}_\beta$, $\mathbf{K}_\gamma$, $\mathbf{K}_{\beta\gamma}$ and $\mathbf{K}_{\gamma\beta}$ for the entire range of saturations. One could easily argue that the system shown in Figure 2 will not be capable of capturing the essential features of two-phase flow phenomena; however, more complex structures, such as that shown in Figure 3 of Part I of this paper, can be used if sufficient computer power is available. On the other hand, imaginative models (Legait and Jacquin, 1984) offer an attractive alternative to extensive numerical computation.

While this computational procedure seems appropriate (but very difficult) for a steady flow, the question of moving contact lines (Huh and Scriven, 1971; Dussan, 1979) presents another problem. Certainly the closure problem associated with moving-front or imbibition processes must be a *local problem*; however, a moving contact line is incompatible with the spatially periodic conditions given by Equation (5.7) and with the form of the governing differential equations given by Equations (2.1) and (2.7). In single-phase flow, one argues that local phenomena are quasi-steady since the time constraint given by

$$\frac{\nu t}{\ell^2} \gg 1 \tag{5.9}$$

is so easily satisfied. In two-phase flow this constraint is still valid on the average; however, it is not at all clear how moving contact line phenomena can be incorporated into the framework of the current analysis without considering transient effects in the closure problem. Studies of this type have been carried out by Giordana and Slattery (1980).

6. Conclusions

The method of volume averaging has been extended to immiscible flow of two fluids in a rigid porous medium. The analysis yields a pair of momentum equations that contain additional terms over and above the obvious intuitive extension of Darcy's law, along with a scheme for computing the coefficients that appear in these equations when the flow is steady. The theory contains the possibility of predicting hysteresis phenomena, but only detailed calculations will indicate whether this phenomena can be captured without including moving contact line phenomena.

Appendix: Uniqueness of the Closure Scheme

The purpose of this appendix is to prove that the solution of the homogeneous problem for $\boldsymbol{\psi}^\alpha$ and ξ^α in a spatially periodic system is

$$\boldsymbol{\psi}^\alpha = \xi^\alpha = 0 \tag{A.1}$$

The proof is nothing more than a modest extension of the development given in Appendix B of Part I of this paper. The system under consideration is illustrated in Figure 2 and the boundary value problem for $\boldsymbol{\psi}^\alpha$ and ξ^α is

$$-\nabla \xi^\alpha + \nabla^2 \boldsymbol{\psi}^\alpha = \frac{1}{V_\alpha}\int_{V_\alpha} [-\nabla \xi^\alpha + \nabla^2 \boldsymbol{\psi}^\alpha]\,\mathrm{d}V, \quad \alpha = \beta, \gamma, \tag{A.2}$$

$$\nabla \cdot \boldsymbol{\psi}^\alpha = 0, \quad \alpha = \beta, \gamma, \tag{A.3}$$

$$\text{B.C.1} \quad \boldsymbol{\psi}^\beta = 0, \quad \text{on } A^*_{\beta\sigma}, \tag{A.4}$$

$$\text{B.C.2} \quad \boldsymbol{\psi}^\beta = \boldsymbol{\psi}^\gamma, \quad \text{on } A^*_{\beta\gamma}, \tag{A.5}$$

$$\begin{aligned}\text{B.C.3} \quad & \mu_\beta(-\mathbf{n}_{\beta\gamma}\xi^\beta + \nabla \boldsymbol{\psi}^\beta \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \boldsymbol{\psi}^\beta) \\ & = \mu_\gamma(-\mathbf{n}_{\beta\gamma}\xi^\gamma + \nabla \boldsymbol{\psi}^\gamma \cdot \mathbf{n}_{\beta\gamma} + \mathbf{n}_{\beta\gamma} \cdot \nabla \boldsymbol{\psi}^\gamma), \quad \text{on } A^*_{\beta\gamma},\end{aligned} \tag{A.6}$$

$$\text{B.C.4} \quad \boldsymbol{\psi}^\gamma = 0, \quad \text{on } A^*_{\gamma\sigma}, \tag{A.7}$$

$$\text{B.C.5} \quad \xi^\alpha(\mathbf{r} + \ell_i) = \xi^\alpha(\mathbf{r}),\ \boldsymbol{\psi}^\alpha(\mathbf{r} + \ell_i) = \boldsymbol{\psi}^\alpha(\mathbf{r}), \quad i = 1, 2, 3, \tag{A.8}$$

$$\langle \xi^\alpha \rangle^\alpha = 0, \qquad \langle \boldsymbol{\psi}^\alpha \rangle^\alpha = 0. \tag{A.9}$$

Following the arguments given in the analysis for single-phase flow, we form the scalar product of Equation (A.2) (with $\alpha = \beta$) and $\boldsymbol{\psi}^\beta$ and integrate over the

β-phase contained in a unit cell to obtain

$$-\int_{V_\beta^*} \nabla\cdot(\boldsymbol{\psi}^\beta \xi^\beta)\,\mathrm{d}V + \tfrac{1}{2}\int_{V_\beta^*} \nabla^2(\boldsymbol{\psi}^\beta\cdot\boldsymbol{\psi}^\beta)\,\mathrm{d}V = \int_{V_\beta^*} \nabla\boldsymbol{\psi}^\beta : (\nabla\boldsymbol{\psi}^\beta)^T\,\mathrm{d}V \tag{A.10}$$

The divergence theorem and Equation (A.4) allow us to express this result as

$$\begin{aligned}-\int_{A_{\beta e}^*} \mathbf{n}_{\beta e}\cdot\boldsymbol{\psi}^\beta\xi^\beta\,\mathrm{d}A - \int_{A_{\beta\gamma}^*} \mathbf{n}_{\beta\gamma}\cdot\boldsymbol{\psi}^\beta\xi^\beta\,\mathrm{d}A + \int_{A_{\beta e}^*} (\mathbf{n}_{\beta e}\cdot\nabla\boldsymbol{\psi}^\beta)\cdot\boldsymbol{\psi}^\beta\,\mathrm{d}A + \\ + \int_{A_{\beta\gamma}^*} (\mathbf{n}_{\beta\gamma}\cdot\nabla\boldsymbol{\psi}^\beta)\cdot\boldsymbol{\psi}^\beta\,\mathrm{d}A = \int_{V_\beta^*} \nabla\boldsymbol{\psi}^\beta : (\nabla\boldsymbol{\psi}^\beta)^T\,\mathrm{d}V\end{aligned} \tag{A.11}$$

while the spatial periodicity of ξ^β and $\boldsymbol{\psi}^\beta$ allow us to simplify Equation (A.11) to obtain

$$-\int_{A_{\beta\gamma}^*} \mathbf{n}_{\beta\gamma}\cdot\boldsymbol{\psi}^\beta\xi^\beta\,\mathrm{d}A + \int_{A_{\beta\gamma}^*} (\mathbf{n}_{\beta\gamma}\cdot\nabla\boldsymbol{\psi}^\beta)\cdot\boldsymbol{\psi}^\beta\,\mathrm{d}A = \int_{V_\beta^*} \nabla\boldsymbol{\psi}^\beta : (\nabla\boldsymbol{\psi}^\beta)^T\,\mathrm{d}V. \tag{A.12}$$

The result for ξ^γ and $\boldsymbol{\psi}^\gamma$ is given by

$$-\int_{A_{\beta\gamma}^*} \mathbf{n}_{\gamma\beta}\cdot\boldsymbol{\psi}^\gamma\xi^\gamma\,\mathrm{d}A + \int_{A_{\beta\gamma}^*} (\mathbf{n}_{\gamma\beta}\cdot\nabla\boldsymbol{\psi}^\beta)\cdot\boldsymbol{\psi}^\gamma\,\mathrm{d}A = \int_{V_\gamma^*} \nabla\boldsymbol{\psi}^\gamma : (\nabla\boldsymbol{\psi}^\gamma)^T\,\mathrm{d}V \tag{A.13}$$

in which $\mathbf{n}_{\gamma\beta} = -\mathbf{n}_{\beta\gamma}$. At this point we turn our attention to Equations (A.5) and (A.6) and note that they can be used to obtain the relation

$$\begin{aligned}&\mu_\beta[-\mathbf{n}_{\beta\gamma}\cdot\boldsymbol{\psi}^\beta\xi^\beta + \boldsymbol{\psi}^\beta\cdot\nabla\boldsymbol{\psi}^\beta\cdot\mathbf{n}_{\beta\gamma} + (\mathbf{n}_{\beta\gamma}\cdot\nabla\boldsymbol{\psi}^\beta)\cdot\boldsymbol{\psi}^\beta] \\ &\quad = \mu_\gamma[-\mathbf{n}_{\beta\gamma}\cdot\boldsymbol{\psi}^\gamma\xi^\gamma + \boldsymbol{\psi}^\gamma\cdot\nabla\boldsymbol{\psi}^\gamma\cdot\mathbf{n}_{\beta\gamma} + (\mathbf{n}_{\beta\gamma}\cdot\nabla\boldsymbol{\psi}^\gamma)\cdot\boldsymbol{\psi}^\gamma], \quad \text{at } A_{\beta\gamma}.\end{aligned} \tag{A.14}$$

If we multiply Equation (A.12) by μ_β and Equation (A.13) by μ_γ, we can add the results and make use of the integral of Equation (A.14) over $A_{\beta\gamma}^*$ to obtain

$$\begin{aligned}&\int_{A_{\beta\gamma}^*} \left(\mu_\gamma\boldsymbol{\psi}^\gamma\cdot\nabla\boldsymbol{\psi}^\gamma\cdot\mathbf{n}_{\beta\gamma} - \mu_\beta\boldsymbol{\psi}^\beta\cdot\nabla\boldsymbol{\psi}^\beta\cdot\mathbf{n}_{\beta\gamma}\right)\mathrm{d}A \\ &\quad = \mu_\beta\int_{V_\beta^*} \nabla\boldsymbol{\psi}^\beta : (\nabla\boldsymbol{\psi}^\beta)^T\,\mathrm{d}V + \mu_\gamma\int_{V_\gamma^*} \nabla\boldsymbol{\psi}^\gamma : (\nabla\boldsymbol{\psi}^\gamma)^T\,\mathrm{d}V.\end{aligned} \tag{A.15}$$

Since $\boldsymbol{\psi}^\beta$ and $\boldsymbol{\psi}^\gamma$ are solenoidal, we can prove that

$$\int_{A_{\beta\sigma}^*} \boldsymbol{\psi}^\beta\cdot\nabla\boldsymbol{\psi}^\beta\cdot\mathbf{n}_{\beta\sigma}\,\mathrm{d}A + \int_{A_{\beta\gamma}^*} \boldsymbol{\psi}^\beta\cdot\nabla\boldsymbol{\psi}^\beta\cdot\mathbf{n}_{\beta\gamma}\,\mathrm{d}A = \int_{V_\beta^*} \nabla\boldsymbol{\psi}^\beta : \nabla\boldsymbol{\psi}^\beta\,\mathrm{d}V, \tag{A.16}$$

$$\int_{A_{\gamma\sigma}^*} \boldsymbol{\psi}^\gamma\cdot\nabla\boldsymbol{\psi}^\gamma\cdot\mathbf{n}_{\gamma\sigma}\,\mathrm{d}A + \int_{A_{\beta\gamma}^*} \boldsymbol{\psi}^\gamma\cdot\nabla\boldsymbol{\psi}^\gamma\cdot\mathbf{n}_{\gamma\beta}\,\mathrm{d}A = \int_{V_\gamma^*} \nabla\boldsymbol{\psi}^\gamma : \nabla\boldsymbol{\psi}^\gamma\,\mathrm{d}V. \tag{A.17}$$

These results, along with Equations (A.4) and (A.7), can be used to reduce Equation (A.15) to the form

$$\mu_\beta \int_{V_\beta^*} \nabla \boldsymbol{\psi}^\beta : [\nabla \boldsymbol{\psi}^\beta + (\nabla \boldsymbol{\psi}^\beta)^T] \,\mathrm{d}V + \mu_\gamma \int_{V_\gamma^*} \nabla \boldsymbol{\psi}^\gamma : [\nabla \boldsymbol{\psi}^\gamma + (\nabla \boldsymbol{\psi}^\gamma)^T] \,\mathrm{d}V = 0. \tag{A.18}$$

A little thought will indicate that this result can be expressed as

$$\begin{aligned} &\mu_\beta \int_{V_\beta^*} [\nabla \boldsymbol{\psi}^\beta + (\nabla \boldsymbol{\psi}^\beta)^T] : [\nabla \boldsymbol{\psi}^\beta + (\nabla \boldsymbol{\psi}^\beta)^T] \,\mathrm{d}V + \\ &\quad + \mu_\gamma \int_{V_\gamma^*} [\nabla \boldsymbol{\psi}^\gamma + (\nabla \boldsymbol{\psi}^\gamma)^T] : [\nabla \boldsymbol{\psi}^\gamma + (\nabla \boldsymbol{\psi}^\gamma)^T] \,\mathrm{d}V = 0 \end{aligned} \tag{A.19}$$

because the irreducible parts of a second order tensor are orthogonal. Since both integrands consist of a sum of squared terms, they must both be identically zero in order that Equation (A.19) be satisfied. Under these circumstances one can deduce that $\nabla \boldsymbol{\psi}^\beta$ and $\nabla \boldsymbol{\psi}^\gamma$ must be skew-symmetric. This means that

$$\nabla \boldsymbol{\psi}^\beta = -(\nabla \boldsymbol{\psi}^\beta)^T, \qquad \nabla \boldsymbol{\psi}^\gamma = -(\nabla \boldsymbol{\psi}^\gamma)^T \tag{A.20}$$

and the *motion* described by $\boldsymbol{\psi}^\beta$ and $\boldsymbol{\psi}^\gamma$ is *rigid body motion*. Under these circumstances, the existence of a no-slip condition is sufficient to conclude that

$$\boldsymbol{\psi}^\beta = \boldsymbol{\psi}^\gamma = 0. \tag{A.21}$$

The proof that ξ^β and ξ^γ are also zero follows the development in Appendix B of Part I of this paper (Whitaker, 1986).

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