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Yangian Double*

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Abstract. Studying the algebraic structure of the double $\mathcal{D}Y(g)$ of the Yangian $Y(g)$, we present the triangular decomposition of $\mathcal{D}Y(g)$ and a factorization for the canonical pairing of the Yangian with its dual inside $Y^0(g)$. As a consequence, we describe a structure of the universal R-matrix R for $\mathcal{D}Y(g)$ which is complete for $\mathcal{D}Y(sl_2)$. We demonstrate how this formula works in evaluation representations of $Y(sl_2)$. We interpret the one-dimensional factor arising in concrete representations of R as a bilinear form on highest-weight polynomials of irreducible representations of $Y(g)$ and express this form in terms of Γ-functions.

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1. Introduction

The Yangian $Y(g)$ of a simple Lie algebra g was introduced by Drinfeld [5] as a deformation of the universal enveloping algebra $U(g[t])$ of a current algebra $g[t]$. The Yangians $Y(g)$ and quantum affine algebras $U_q(\hat{g})$ play the role of dynamical symmetries in quantum field theories [1, 24]. Tensor products of finite-dimensional representations of the Yangians produce rational solutions of the Yang–Baxter equation; tensor products of finite-dimensional representations of quantum affine algebras produce trigonometric solutions of the Yang–Baxter equation. One can find out other deep parallels in representation theories of Yangians and of quantum affine algebras. Nevertheless, both of them have their own original features. The Yangian $Y(g)$ is much more closer to classical Lie algebras, at least it contains the universal enveloping algebra $U(g)$ as a subalgebra; moreover the Yangian $Y(gl_n)$ could be defined entirely in terms of classical representation theory [21]. The structure of quantum affine algebra $U_q(\hat{g})$ is more complicated. On the other hand, $U_q(\hat{g})$ inhabit main properties of the contragradient algebras. For example, Chevalley generators and q-deformed Serre relations are permanent participants of the games with the quantum affine algebras.

The theory of the Cartan–Weyl basis (See [16, 17]) allows us to describe explicitly the universal R-matrix, one of the main objects in physical applications. We have

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nothing of this for the Yangians, although suitable modifications of classical methods of representation theory work for $\mathrm{Y}(g)$ and $\mathrm{U}_q(\hat{g})$ as well (see [4, 5, 13, 20], for instance). We want to make this gap smaller.

It is more reasonable to work with quantum double $\mathscr{D}\mathrm{Y}(g)$ of the Yangian, if we keep in mind physical applications. We present here an algebraic description of $\mathscr{D}\mathrm{Y}(g)$ in terms of the 'second' Drinfeld realization of $\mathrm{Y}(g)$. Unfortunately, we can give rigorous proof of this presentation only for $g = \mathrm{sl}_n$ (see Theorem 5.1 and Section 6.2). For the general case, we have a number of indirect arguments; the arguments of Section 6.2 can be also generalized for arbitrary simple g, but technically it looks too cumbersome.

Finite-dimensional representations of $\mathscr{D}\mathrm{Y}(g)$ do not differ much from the representations of the Yangian $\mathrm{Y}(g)$: the extension of a representation of $\mathrm{Y}(g)$ to a representation of $\mathscr{D}\mathrm{Y}(g)$ can be achieved just by re-expansion of the currents from $\mathrm{Y}(g)$ in other points of the projective line. We present here a study of some algebraic properties of $\mathscr{D}\mathrm{Y}(g)$, with the accent to the canonical pairing in the double.

We prove that $\mathscr{D}\mathrm{Y}(g)$ itself and a Hopf pairing of $\mathrm{Y}(g)$ with its dual inside $\mathscr{D}\mathrm{Y}(g)$ admit a triangular decomposition analogous to a Gauss decomposition of ordinary matrices. This property gives the possibility to describe the pairing quite explicitly. As a consequence, we obtain an explicit factorized expression for the universal R-matrix of $\mathscr{D}\mathrm{Y}(g)$ (completely proved for $\mathscr{D}\mathrm{Y}(\mathrm{sl}_2)$ and partially in the general case). To make the formulas more transparent, we present detailed calculations for $\mathscr{D}\mathrm{Y}(\mathrm{sl}_2)$, including the action of the universal R-matrix on evaluation representations.

The most interesting factor R_0 of the universal R-matrix is concerned with a (zero-charge) Heisenberg subalgebra of $\mathscr{D}\mathrm{Y}(g)$ which is a deformation of the currents to the Cartan subalgebra h of g. Analogously to the case of $\mathrm{U}_q(\hat{g})$ [14, 16], the structure of R_0 is governed by the q-analog of the invariant scalar product in h; whenever R_0 acts on representations of $\mathscr{D}\mathrm{Y}(g)$, a variable q becomes a shift operator T: $Tf(x) = f(x-1)$ (for quantum affine algebras, the parameter q goes to multiplicative shift $T_q f(x) = f(qx)$ in an analogous situation). After substitution of the universal R-matrix into the tensor product of concrete representations of $\mathscr{D}\mathrm{Y}(g)$, we obtain a more than usual rational R-matrix; some additional information is concentrated in a scalar-phase factor (scalar S-matrix) which we interpret as a bilinear multiplicative form on highest-weight polynomials of finite-dimensional representations V of $\mathrm{Y}(g)$ (or, equivalently, on $K_0(\mathrm{Rep}\,\mathrm{Y}(g))$. This form is a deformation of skewsymmetric form $\langle , \rangle/(a-b)$ on irreducible evaluation representations of $g[t]$, where $\langle , \rangle$ is an invariant scalar product in h^* and a and b are the points where the evaluation representations are living. We present an explicit expression of this form as some ratio of Γ-functions defined by the structure of the q-analog of the invariant scalar product in h.

2. Yangian Y(g) and its Quantum Double

The Yangian Y(g) as a quantum deformation of the universal enveloping algebra $Ug[t]$, where $g[t]$ is the polynomial currents over a simple Lie algebra g, was introduced by Drinfeld [5] firstly in terms of generators which actually are not associated to the choice of a concrete basis in g. Later in [6–8], Drinfeld gave another realization of the Yangians in terms of generators connected with the Cartan–Weyl basis in g. In this Letter, we use this second Drinfeld realization. Let us recall the definition of this realization.

Let g be a simple Lie algebra with a standard Cartan matrix $A = (a_{ij})_{i,j=0}^{r}$, a system of simple roots $\Pi := \{\alpha_1, \ldots, \alpha_2\}$ and a system of positive roots $\Delta_+(g)$. Let

$$e_i := e_{\alpha_i}, \qquad h_i := h_{\alpha_i}, \qquad f_i := f_{\alpha_i} := e_{-\alpha_i} \quad (i = 1, \ldots, r),$$

be Chevalley generators and $\{e_\gamma, f_\gamma\}$, $(\gamma \in \Delta)$, be a basis Cartan–Weyl in g, normalized so that $(e_\alpha, f_\alpha) = 1$.

DEFINITION 2.1. The Yangian $\mathrm{Y} := \mathrm{Y}(g)$ associated to g is the Hopf algebra over C generated (as an associative algebra) by the elements

$$e_{ik} := e_{\alpha_i k}, \qquad h_{ik} := h_{\alpha_i k}, \qquad f_{ik} := f_{\alpha_i k} \quad (i = 1, \ldots, r;\ k = 0, 1, 2, \ldots),$$

with the relations:

$$[h_{ik}, h_{jl}] = 0, \qquad [e_{ik}, f_{jl}] = \delta_{ij} h_{i,k+l}, \tag{2.1}$$

$$[h_{i0}, e_{jl}] = (\alpha_i, \alpha_j) e_{jl}, \qquad [h_{i0}, f_{jl}] = -(\alpha_i, \alpha_j) f_{jl}, \tag{2.2}$$

$$[h_{i,k+1}, e_{jl}] - [h_{ik}, e_{j,l+1}] = \tfrac{1}{2}(\alpha_i, \alpha_j)\{h_{ik}, e_{jl}\}, \tag{2.3}$$

$$[h_{i,k+1}, f_{jl}] - [h_{ik}, f_{j,l+1}] = -\tfrac{1}{2}(\alpha_i, \alpha_j)\{h_{ik}, f_{jl}\}, \tag{2.4}$$

$$[e_{i,k+1}, e_{jl}] - [e_{ik}, e_{j,l+1}] = \tfrac{1}{2}(\alpha_i, \alpha_j)\{e_{ik}, e_{jl}\}, \tag{2.5}$$

$$[f_{i,k+1}, f_{jl}] - [f_{ik}, f_{j,l+1}] = -\tfrac{1}{2}(\alpha_i, \alpha_j)\{f_{ik}, f_{jl}\}, \tag{2.6}$$

$$\begin{aligned} &\mathrm{Sym}_{\{k\}}[e_{i,k_1}[e_{i,k_2} \cdots [e_{i,k n_{ij}}, e_{jl}] \cdots]] = 0, \\ &\mathrm{Sym}_{\{k\}}[f_{i,k_1}[f_{i,k_2} \cdots [f_{i,k n_{ij}}, f_{jl}] \cdots]] = 0, \quad \text{for } i \neq j, \end{aligned} \tag{2.7}$$

where $\{a, b\} := ab + ba$, $n_{ij} := 1 - A_{ij}$, the symbol 'Sym'$_{\{k\}}$ denotes a symmetrization on $k_1, k_2, \ldots, k_{ij}$. The comultiplication map of Y is given for basic generators e_{ik}, h_{ik}, f_{ik}, $(k = 0, 1)$, by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad x \in g, \tag{2.8}$$

$$\Delta(e_{i1}) = e_{i1} \otimes 1 + 1 \otimes e_{i1} + h_{i0} \otimes e_{i0} - \sum_{\gamma \in \Delta_+(g)} f_\gamma \otimes [e_{\alpha_i}, e_\gamma], \tag{2.9}$$

$$\Delta(f_{i1}) = f_{i1} \otimes 1 + 1 \otimes f_{i1} + f_{i0} \otimes h_{i0} + \sum_{\gamma \in \Delta_+(g)} [f_{\alpha_i}, f_\gamma] \otimes e_\gamma, \tag{2.10}$$

$$\Delta(h_{i1}) = h_{i1} \otimes 1 + 1 \otimes h_{i1} + h_{i0} \otimes h_{i0} - \sum_{\gamma \in \Delta_+(g)} (\alpha_i, \gamma) f_\gamma \otimes e_\gamma. \tag{2.11}$$

Remarks. (i) The universal enveloping algebra $U(g)$ is embeded in $Y(g)$: $U(g) \hookrightarrow Y(g)$.

(ii) One can show that $Y(g)$ is generated only by the elements e_{i0}, f_{i0}, f_{i1} $(i = 1, \ldots, r)$ and, therefore, we can obtain the comultiplication map for all generators e_{ik}, f_{ik}, h_{ik}, f_{ik} $(i = 1, \ldots, r;\ k \in \mathbb{Z})$.

(iii) If we replace the right parts of (2.3)–(2.6) by zeros, then we obtain the algebra isomorphic to $U(g[t])$.

In this Letter, we study a quantum double $\mathscr{D}Y(g)$ of $Y(g)$ (see [7] for definitions). In the following, we describe that algebraic structure of $\mathscr{D}Y(g)$.

Let $C(g)$ be an algebra generated by the elements e_{ik}, f_{ik}, h_{ik}, $(i = 1, \ldots, r,\ k \in \mathbb{Z})$, with relations (2.1)–(2.7). Algebra $C(g)$ admits $\mathbb{Z}$-filtration

$$\cdots \subset C_{-n} \subset \cdots \subset C_{-1} \subset C_0 \subset C_1 \cdots \subset C_n \cdots \subset C(g) \tag{2.12}$$

defined by the condition

$$\deg e_{ik} = \deg f_{ik} = \deg h_{ik} = k; \ \deg C_m = m.$$

Let $\bar{C}(g)$ be the corresponding formal completion of $C(g)$. The generators e_{ik}, f_{ik}, h_{ik} $(i = 1, \ldots, r, k \geqslant 0)$ define an inclusion $Y(g) \hookrightarrow \bar{C}(g)$. We denote sometimes its image by $Y^+(g)$ or shortly by Y^+ when we need short notation. In the next sections, we prove that dual to $Y(g)$ Hopf algebra with opposite comultiplication $Y^0(g)$ is isomorphic to the subalgebra $Y^- := Y^-(g)$ which is generated by formal series $\sum_{m<0} a_m$, $\deg a_m = m$.

An algebra $\bar{C}(g)$ was introduced by Drinfeld [6, 7] as a quantum double of a Hopf algebra quantizing the currents to a Borel subalgebra with rational r-matrix. In the trigonometric situation, he proved that an algebra analogous to $\bar{C}(g)$ is isomorphic to the double of $U_q(\hat{b}_+)$ modulo Cartan elements (i.e., to $U_q(\hat{g})$). This proof does not fit for the rational case, since $Y_-(g)$ is not finitely generated and $\mathscr{D}Y(g)$ has no alternative description in terms of Chevalley generators. We assume, nevertheless, that algebras $\mathscr{D}Y(g)$ and $\bar{C}(g)$ are isomorphic and give a sketch of the proof of this for $Y(sl_n)$.

For a description of $\mathscr{D}Y(g)$, it is more convenient to use the generating functions ('fields') $e_i^\pm(u)$, $h_i^\pm(u)$ and $f_i^\pm(u)$ of $Y_\pm(g)$:

$$e_i^+(u) := \sum_{k \geqslant 0} e_{ik} u^{-k-1}, \qquad f_i^+(u) := \sum_{k \geqslant 0} f_{ik} u^{-k-1},$$
$$h_i^+(u) := 1 + \sum_{k \geqslant 0} h_{ik} u^{-k-1}, \tag{2.13}$$

$$e_i^-(u) := - \sum_{k<0} e_{ik} u^{-k-1}, \qquad f_i^-(u) := - \sum_{k<0} f_{ik} u^{-k-1},$$
$$h_i^-(u) := 1 - \sum_{k<0} h_{ik} u^{-k-1}. \tag{2.14}$$

In the case of $Y^{\pm}(sl_2)$, we omit for simplicity everywhere an index of a simple root, e.g. for the fields we use the notations $e_{\pm}(u)$, $h_{\pm}(u)$ and $f_{\pm}(u)$. Moreover, we put $(\alpha, \alpha) = 2$. In this case, we can prove the following proposition.

PROPOSITION 2.1. (i) *The defining relations (2.1)–(2.7) of the algebra $\bar{C}(sl_2)$ are equivalent to the following relations for the fields (2.13) and (2.14):*

$$[h^{\pm}(u), h^{\pm}(v)] = 0, \qquad [h^{+}(u), h^{-}(v)] = 0, \tag{2.15}$$

$$\begin{aligned} [e^{\pm}(u), f^{\pm}(v)] &= -\frac{h^{\pm}(u) - h^{\pm}(v)}{u - v}, \\ [e^{\pm}(u), f^{\mp}(v)] &= -\frac{h^{\mp}(u) - h^{\pm}(v)}{u - v}, \end{aligned} \tag{2.16}$$

$$[h^{\pm}(u), e^{\pm}(v)] = -\frac{\{h^{\pm}(u), (e^{\pm}(u) - e^{\pm}(v))\}}{u - v}, \tag{2.17}$$

$$[h^{\pm}(u), e^{\mp}(v)] = -\frac{\{h^{\pm}(u), (e^{\pm}(u) - e^{\mp}(v))\}}{u - v}, \tag{2.18}$$

$$[h^{\pm}(u), f^{\pm}(v)] = \frac{\{h^{\pm}(u), f^{\pm}(u) - f^{\pm}(v)\}}{u - v}, \tag{2.19}$$

$$[h^{\pm}(u), f^{\mp}(v)] = \frac{\{h^{\pm}(u), (f^{\pm}(u) - f^{\mp}(v))\}}{u - v}, \tag{2.20}$$

$$\begin{aligned} [e^{\pm}(u), e^{\pm}(v)] &= -\frac{(e^{\pm}(u) - e^{\pm}(v))^2}{u - v}, \\ [e^{+}(u), e^{-}(v)] &= -\frac{(e^{+}(u) - e^{-}(v))^2}{u - v}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} [f^{\pm}(u), f^{\pm}(v)] &= \frac{(f^{\pm}(u) - f^{\pm}(v))^2}{u - v}, \\ [f^{+}(u), f^{-}(v)] &= \frac{(f^{+}(u) - f^{-}(v))^2}{u - v}. \end{aligned} \tag{2.22}$$

(ii) *An algebra generated by $e^{\pm}(u)$, $h^{\pm}(u)$, $f^{\pm}(u)$ with the relations (2.15)–(2.22) and comultiplication map (2.23)–(2.25) is a Hopf algebra isomorphic to $\mathcal{D}Y(sl_2)$. (The co-multiplication map for the positive fields $e^{+}(u)$, $f^{+}(u)$* and *$h^{+}(u)$ were communicated to us by Molev [19].) The comultiplication in $\mathcal{D}Y(sl_2)$ looks as follows:*

$$\begin{aligned} \Delta(e^{\pm}(u)) &= e^{\pm}(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k (f^{\pm}(u+1))^k h^{\pm}(u) \otimes (e^{\pm}(u))^{k+1} \\ &= e^{\pm}(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k h^{\pm}(u) (f^{\pm}(u-1))^k \otimes (e^{\pm}(u))^{k+1}, \end{aligned} \tag{2.23}$$

$$\begin{aligned}\Delta(f^{\pm}(u)) &= 1\otimes f^{\pm}(u) + \sum_{k=0}^{\infty}(-1)^k(f^{\pm}(u))^{k+1}\otimes h^{\pm}(u)(e^{\pm}(u+1))^k\\ &= 1\otimes f^{\pm}(u) + \sum_{k=0}^{\infty}(-1)^k(f^{\pm}(u))^{k+1}\otimes(e^{\pm}(u-1)^k h^{\pm}(u)),\end{aligned}\tag{2.24}$$

$$\begin{aligned}\Delta(h^{\pm}(u)) &= \sum_{k=0}^{\infty}(-1)^k(k+1)(f^{\pm}(u+1))^k h^{\pm}(u)\otimes h^{\pm}(u)(e^{\pm}(u+1))^k\\ &= \sum_{k=0}^{\infty}(-1)^k(k+1)h^{\pm}(u)(f^{\pm}(u-1))^k\otimes(e^{\pm}(u-1))^k h^{\pm}(u).\end{aligned}\tag{2.25}$$

Proof. Part (i) and compatibility of comultiplication with (2.15)–(2.22) is derived by direct calculations; an isomorphism with $\mathscr{D}Y(sl_2)$ follows from the existence of the universal R-matrix (see Remark to Theorem 5.1).

For arbitrary g, we have the following analog of Proposition 2.1:

PROPOSITION 2.2. *The defining relations (2.1)–(2.7) of the algebra $\bar{C}(g)$ are equivalent to the following relations for the fields (2.13) and (2.14):*

$$[h_i^{\pm}(u), h_j^{\pm}(v)] = 0, \qquad [h_i^{+}(u), h_j^{-}(v)] = 0, \tag{2.26}$$

$$\begin{aligned}&[e_i^{\pm}(u), f_j^{\pm}(v)] = -\delta_{i,j}\frac{h_i^{\pm}(u) - h_j^{\pm}(v)}{u-v},\\ &[e_i^{\pm}(u), f_j^{\mp}(v)] = -\delta_{i,j}\frac{h_i^{\mp}(u) - h_j^{\pm}(v)}{u-v},\end{aligned}\tag{2.27}$$

$$[h_i^{\pm}(u), e_j^{\pm}(v)] = -\tfrac{1}{2}(\alpha_i,\alpha_j)\frac{\{h_i^{\pm}(u), (e_j^{\pm}(u) - e_j^{\pm}(v))\}}{u-v}, \tag{2.28}$$

$$[h_i^{\pm}(u), e_j^{\mp}(v)] = -\tfrac{1}{2}(\alpha_i,\alpha_j)\frac{\{h_i^{\pm}(u), (e_j^{\pm}(u) - e_j^{\mp}(v))\}}{u-v}, \tag{2.29}$$

$$[h_i^{\pm}(u), f_j^{\pm}(v)] = \tfrac{1}{2}(\alpha_i,\alpha_j)\frac{\{h_i^{\pm}(u), f_j^{\pm}(u) - f_j^{\pm}(v)\}}{u-v}, \tag{2.30}$$

$$[h_i^{\pm}(u), f_j^{\mp}(v)] = \tfrac{1}{2}(\alpha_i,\alpha_j)\frac{\{h_i^{\pm}(u), (f_j^{\pm}(u) - f_j^{\mp}(v))\}}{u-v}, \tag{2.31}$$

$$\begin{aligned}&[e_i^{\pm}(u), e_j^{\pm}(v)] + [e_j^{\pm}(u), e_i^{\pm}(v)]\\ &\quad= -\tfrac{1}{2}(\alpha_i,\alpha_j)\frac{\{(e_i^{\pm}(u) - e_i^{\pm}(v)), (e_j^{\pm}(u) - e_j^{\pm}(v))\}}{u-v},\end{aligned}\tag{2.32}$$

$$\begin{aligned}&[e_i^{+}(u), e_j^{-}(v)] + [e_j^{+}(u), e_i^{-}(v)]\\ &\quad= -\tfrac{1}{2}(\alpha_i,\alpha_j)\frac{\{(e_i^{+}(u) - e_i^{-}(v)), (e_j^{+}(u) - e_j^{-}(v))\}}{u-v},\end{aligned}\tag{2.33}$$

$$[f_i^{\pm}(u), f_j^{\pm}(v)] + [f_j^{\pm}(u), f_i^{\pm}(v)]$$
$$= \tfrac{1}{2}(\alpha_i, \alpha_j) \frac{\{(f_i^{\pm}(u) - f_i^{\pm}(v)), (f_j^{\pm}(u) - f_j^{\pm}(v))\}}{u - v}, \tag{2.34}$$

$$[f_i^{+}(u), f_j^{-}(v)] + [f_j^{+}(u), f_i^{-}(v)]$$
$$= \tfrac{1}{2}(\alpha_i, \alpha_j) \frac{\{(f_i^{+}(u) - f_i^{-}(v)), (f_j^{+}(u) - f_j^{-}(v))\}}{u - v}, \tag{2.35}$$

$$\begin{gathered} \mathrm{Sym}_{\{k\}} [e_i^{\varepsilon_{k_1}}(u_{k_1}) [e_i^{\varepsilon_{k_2}}(u_{k_2}) \cdots [e_i^{\varepsilon_{k_{n_{ij}}}}(u_{k_{n_{ij}}}, e_j^{\varepsilon_0}(v)] \cdots]] = 0, \\ \mathrm{Sym}_{\{k\}} [f_i^{\varepsilon_{k_1}}(u_{k_1}) [f_i^{\varepsilon_{k_2}}(u_{k_2}) \cdots [f_i^{\varepsilon_{k_{n_{ij}}}}(u_{k_{n_{ij}}}, f_j^{\varepsilon_0}(v)] \cdots]] = 0, \quad \text{for } i \neq j, \end{gathered} \tag{2.36}$$

where $\varepsilon_m = \pm$.

CONJECTURE. Yangian double $\mathscr{D}\mathrm{Y}(g)$ is isomorphic to the algebra described in Proposition 2.2.

In the next section we describe certain decompositions in the Yangians $\mathrm{Y}(g)$ and its quantum double $\mathscr{D}\mathrm{Y}(g)$.

3. Triangular Decomposition of $\mathscr{D}\mathrm{Y}(g)$

Let Y_+', Y_0' and Y_-' be a nonunital (without unit element) subalgebras of $\mathrm{Y}(g)$, generated by the elements e_{ik} $(i = 1, \ldots, r, ; k \geqslant 0)$; h_{ik} $(i = 1, \ldots, r,\ k \geqslant 0)$; f_{ik}, $i = 1, \ldots, r$, $k \geqslant 0)$ correspondingly. We denote also by Y_+, Y_0, and Y_- the algebras Y_+', Y_0', and Y_-' with an added unit element. Following [2], one can deduce from (2.1)–(2.7) the following decomposition of $\mathrm{Y}(g)$:

PROPOSITION 3.1. *A multiplication in* $\mathrm{Y}(g)$ *induces an isomorphism of vector spaces*

$$Y_+ \otimes Y_0 \otimes Y_- \rightsquigarrow \mathrm{Y}(g). \tag{3.1}$$

We are going to extend this decomposition to the double $\mathscr{D}\mathrm{Y}(g)$ and factorize the natural pairing of $\mathrm{Y}(g)$ and $Y^0(g)$ with respect to this decomposition. First, we summarize properties of the comultiplication in $\mathrm{Y} := \mathrm{Y}(g)$, which easily generalizes by induction of formulas (2.8)–(2.11) (see also [2]).

LEMMA 3.1. *The following relations hold:*

$$\Delta(e) = e \otimes 1 \pmod{Y \otimes Y_+'}, \tag{3.2}$$

for any $e \in Y_+'$;

$$\Delta(f) = 1 \otimes f \pmod{Y_-' \otimes Y}, \tag{3.3}$$

for any $f \in Y_-'$.

In particular, we conclude that Y_+ is a right coideal $(\Delta(Y_+) \subset Y \otimes Y_+)$ and Y_- is a right coideal $(\Delta(Y_-) \subset Y_- \otimes Y)$ of $\mathrm{Y}(g)$.

Also let $\bar{Y}'_+$ be a nonunital subalgebra of Y(g) generated by the elements e_{ik} and h_{jl} ($i, j = 1, \ldots, r, k, l \geqslant 0$); $\bar{Y}'_-$ be a nonunital subalgebra of Y(g) generated by the elements f_{ik} and h_{jl} ($i, j = 1, \ldots, r, k, l \geqslant 0$).

LEMMA 3.2. *The following properties take place:*

$$\Delta(e) = e \otimes 1 \quad (\text{mod } Y \otimes \bar{Y}'_+), \tag{3.4}$$

for any $e \in \bar{Y}'_+$;

$$\Delta(f) = 1 \otimes f \quad (\text{mod } \bar{Y}'_- \otimes Y), \tag{3.5}$$

for any $f \in \bar{Y}'_-$;

$$\begin{aligned} \Delta(h) &= h \otimes 1 \quad (\text{mod } Y \otimes \bar{Y}'_+) \\ &= 1 \otimes h \quad (\text{mod } \bar{Y}'_- \otimes Y), \end{aligned} \tag{3.6}$$

for any $h \in Y'_0$.

Let $\langle , \rangle$ denote the canonical Hopf pairing of $\mathrm{Y} := \mathrm{Y}(g)$ and its dual $\mathrm{Y}^0 := \mathrm{Y}^0(g)$. The Hopf property of $\langle , \rangle$ in this case can be read as

$$\langle ab, c^* d^* \rangle = \langle \Delta(ab), c^* \otimes d^* \rangle = \langle b \otimes a, \Delta(c^* d^*) \rangle$$

for any $a, b \in Y$, and for any $c^*, d^* \in Y^0$. Here, $\langle a \otimes b, c^* \otimes d^* \rangle = \langle a, c^* \rangle \langle b, d^* \rangle$. Let $(Y\bar{Y}'_+)^*_\perp$ be an annulator of $Y\bar{Y}'_-$, i.e.

$$(Y\bar{Y}'_-)^*_\perp = \{e^* \in Y^0 \colon \langle yf, e^* \rangle = 0, \forall y \in Y, \forall f \in \bar{Y}'_-\}. \tag{3.7}$$

Analogously, we define $(\bar{Y}'_+ Y)^*_\perp$, $(YY'_-)^*_-$, and $(Y'_+ Y)^*_\perp$. We shall also use the following short notations

$$\begin{aligned} Y^*_+ &:= (Y\bar{Y}'_-)^*_\perp, \qquad Y^*_- := (\bar{Y}_+ Y)^*_\perp, \\ \bar{Y}^*_+ &:= (YY'_-)^*_\perp, \qquad \bar{Y}^*_- := (Y'_+ Y)^*_\perp, \end{aligned} \tag{3.8}$$

and

$$Y^*_0 := \bar{Y}^*_+ \cap \bar{Y}^*_-. \tag{3.9}$$

PROPOSITION 3.2. *The dual subsets* Y^*_+, Y_-, $\bar{Y}^*_+$, $\bar{Y}^*_-$ *are subalgebras of* $\mathrm{Y}^0(g)$.

The validity of the proposition follows directly from Lemmas (3.1), (3.2), and the following simple lemma.

LEMMA 3.3. *Let* A *and* A^* *be two Hopf algebras with a Hopf pairing* $\langle , \rangle$: $A \otimes A^* \to \mathbb{C}$. *Let a subset* $X \subset A$ *satisfy the condition* $\Delta(X) \subset X \otimes A + A \otimes X$. *Then both* $(AX)^*_\perp$ *and* $(XA)^*_\perp$ *are subalgebras of* A^*.

The main result of this section is the following theorem.

THEOREM 3.1. (i) *For any* $e \in Y_+$, $h \in Y_0$, $f \in Y_-$, $e^* \in Y_+^*$, $h^* \in Y_0^*$, $f^* \in Y_-^*$ *the canonical pairing is factorized as*

$$\langle ehf, e^* h^* f^* \rangle = \langle e, e^* \rangle \langle h, h^* \rangle \langle f, f^* \rangle. \tag{3.10}$$

(ii) *A multiplication in* $Y^0(g)$ *induces the isomorphism of the vector spaces:*

$$Y_+^* \otimes Y_0^* \otimes Y_-^* \rightsquigarrow Y^0(g). \tag{3.11}$$

Proof. (i) First using Lemmas 3.1 and 3.2, we easily check that for any $e \in Y_+'$, $e^* \in Y_+^*$, $f \in \mathscr{F}$, $f^* \in \mathscr{F}^*$, we have

$$\begin{aligned}\langle ef, e^* f^* \rangle &= \langle \Delta(e)\Delta(f), e^* \otimes f^* \rangle \\ &= \langle (e \otimes 1 + \sum y_n \otimes e_n)(1 \otimes f + \sum f_m \otimes y_m), (e^* \otimes f^*) \rangle \\ &= \langle e, e^* \rangle \langle f, f^* \rangle,\end{aligned} \tag{3.12}$$

where $e_i \in Y_+'$, $f_j \in Y_-'$ by virtue of (3.2)–(3.5). Furthermore, we can prove analogously that

$$\langle eh, e^* h^* \rangle = \langle e, e^* \rangle \langle h, h^* \rangle \tag{3.13}$$

for any $e \in Y_+'$, $h \in \mathscr{H}'$, $e^* \in Y_+^*$, $h^* \in H^*$. Thus, we have (3.10).

(ii) Let us choose a basis in $Y(g)$ in accordance with the decomposition (3.1), i.e. each basic vector has the form ehf, where $e \in Y_-$, $h \in \mathscr{H}$, $f \in Y_-$. Then vectors biorthogonal to these basic vectors make up a basis in $Y^0(g)$ and, in accordance with (3.11), they have the form $e^* h^* f^*$, where $e^* \in Y_+^*$, $h^* \in \mathscr{H}^*$, $f^* \in Y_-^*$. This means (3.11).

Now we proceed with a more detailed study of the pairing in the Yangian double. In the next section, we compute explicitly the pairing between the generators of $Y(g)$ and $Y_0(g)$.

4. Basic Pairing for $\mathscr{D}Y(g)$

The aim of this section is to compute explicitly the pairing between generators e_{ik}, h_{ik}, and f_{ik} of $Y^+(g)$ and of $Y^-(g) \simeq Y^0(g)$. The answer will be written in terms of generating functions ('fields') (2.22) and (2.23). Explicit calculations will be done for the case of $Y(sl_2)$ where, for simplicity, we omit everywhere the index of a simple root, e.g. for generating functions we use the notations $e_\pm(u)$, $h_\pm(u)$, and $f_\pm(u)$.

Let

$$\{e^*_{-k-1}\} := \{e_{-1}, e_{-2}, \dots\}, \qquad \{f^*_{-k-1}\} := \{f_{-1}, f_{-2}, \dots\}$$
$$\text{and} \quad \{h^*_{-k-1}\} := \{h_{-1}, h_{-2}, \dots\}$$

be some sets of elements from Y^*_+, Y^*_+, and Y^*_+, correspondingly. Let us construct the generating functions (the fields) of these sets:

$$\begin{aligned} e^*(u) &:= \sum_{k<0} e^*_k u^{-k-1}, \qquad f^*(u) := \sum_{k<0} f^*_k u^{-k-1}, \\ h^*(u) &:= 1 + \sum_{k<0} h^*_k u^{-k-1}. \end{aligned} \tag{4.1}$$

We are interested in the pairings of these fields with the fields $e^+(u)$, $f^+(u)$, and $h^+(u)$, i.e. $\langle e^+(u), e^*(v)\rangle$, $\langle f^+(u), f^*(v)\rangle$ and $\langle h^+(u), h^*(v)\rangle$, where $|u| \gg 1 \gg |u|$. The following proposition is valid.

PROPOSITION 4.1. *If the fields* $e^*(u)$, $f^*(u)$ and $h^*(u)$ *are such that*

$$\langle e^+(u_1)\cdots e^+(u_n), e^*(v)\rangle = \langle f^+(u_1)\cdots f^+(u_n), f^*(v)\rangle = 0, \tag{4.2}$$

for any $n > 1$ *and*

$$\begin{aligned} &\langle h^+(u_1)\cdots e^+(u_n), h^*(v)\rangle \\ &\quad = \langle h^+(u_1), h^*(v)\rangle \cdots \langle h^+(u_2), h^*(v)\rangle \end{aligned} \tag{4.3}$$

for $n > 1$. *Then the conditions*

$$\langle e^+(u), e^*(v)\rangle = \frac{\alpha}{u-v}, \qquad \langle f^+(u), f^*(v)\rangle = \frac{\alpha^{-1}}{u-v} \tag{4.4}$$

for some $\alpha \in \mathbb{C}$,

$$\langle h^+(u), h^*(v)\rangle = \frac{u-v+1}{u-v-1}, \tag{4.5}$$

are realized if and only if the relations

$$[h^*(u), h^*(v)] = 0, \qquad [e^*(u), f^*(v)] = -\frac{h^*(u) - h^*(v)}{u-v}, \tag{4.6}$$

$$[h^*(u), e^*(v)] = -\frac{\{h^*(u), (e^*(u) - e^*(v))\}}{u-v} \tag{4.7}$$

$$[h^*(u), f^*(v)] = \frac{\{h^*(u), (f^*(u) - f^*(v))\}}{u-v}. \tag{4.8}$$

$$\begin{aligned} [e^*(u), e^*(v)] &= -\frac{(e^*(u) - e^*(v)^2}{u-v}, \\ [f^*(u), f^*(v)] &= \frac{(f^*(u) - f^*(v)^2}{u-v}, \end{aligned} \tag{4.9}$$

are satisfied.

Proof. Using the expressions (2.23)–(2.25) and the commutation relations (2.15)–(2.22) for the fields $e^+(u)$, $f^+(u)$, $h^+(u)$, we have

$$\Delta(e_+(u)) = e_+(u) \otimes 1 + h_+(u) \otimes e_+(u) \qquad (\text{mod } YY_- \otimes Y_+'), \tag{4.10}$$

$$\Delta(f_+(u)) = 1 \otimes f_+(u) + f_+(u) \otimes h_+(u) \quad (\text{mod } Y_-' \otimes Y_+' Y), \tag{4.11}$$

$$\Delta(h_+(u)) = h_+(u) \otimes h_+(u) \qquad (\text{mod } YY_-' \otimes Y_+' Y), \tag{4.12}$$

$$\begin{aligned} &\Delta(e^+(u)e^+(v)) \\ &\quad = e^+(u)e^+(v) \otimes 1 + 1 \otimes e^+(u)e^+(v) + \\ &\quad + \frac{u-v+1}{u-v-1} e^+(v) \otimes e^+(u) - \\ &\quad - \frac{2}{u-v-1} e^+(u-1) \otimes e^+(u) \quad (\text{mod } YY_- \otimes Y_+'), \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\Delta(f^+(u)f^+(v)) \\ &\quad = f^+(u)f^+(v) \otimes 1 + 1 \otimes f^+(u)f^+(v) + \\ &\quad + \frac{u-v-1}{u-v+1} f^+(v) \otimes f^+(u) - \\ &\quad - \frac{2}{u-v+1} f^+(v) \otimes f^+(v-1) \quad (\text{mod } Y_-' \otimes Y_+' Y), \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\Delta(h^+(u)f^+(v)) \\ &\quad = h^+(u) \otimes h^+(u)f^+(v) + \\ &\quad + h^+(u)f^+(v) \otimes h^+(u)h^+(v) \quad (\text{mod } YY_-' \otimes Y_+' Y), \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\Delta(e^+(u)h^+(v)) \\ &\quad = e^+(u)h^+(v) \otimes h^+(v) + \\ &\quad + h^+(u)h^+(v) \otimes e^+(u)h^+(v) \quad (\text{mod } YY_-' \otimes Y_+' Y). \end{aligned} \tag{4.16}$$

Let now

$$\langle e^+(u), e^*(v) \rangle = E(u,v), \qquad \langle f^+(u), f^*(v) \rangle = F(u,v) \quad \text{and}$$
$$\langle h^+(u), h^*(v) \rangle = H(u,v).$$

The computation of the pairing of the relations (4.6)–(4.9) with the r.h.s. of (4.13)–(4.16) gives the functional equations on the functions E, F, H which determine them to be equal to the r.h.s. of (4.4)–(4.5). The arguments can be reversed if we use conditions (4.2)–(4.3).

The main result of this section for $\mathcal{D}Y(\mathrm{sl}_2)$ may be formulated in the following theorem:

THEOREM 4.1. (i) *Subalgebras* Y_{+}^{*}, $\mathscr{H}^{*}$ *and* Y_{-}^{*} *(see (3.7)–(3.9)) of* $Y_{-}(sl_2)$ *are generated by the fields* $e^{-}(u)$, $h^{-}(u)$, *and* $f^{-}(u)$, *correspondingly;*

(ii) *The pairing of the generators of* $Y_{+}(sl_2)$ *is given by the relations* ($|v| \ll 1 \ll |u|$)

$$\langle e^{+}(u), f^{-}(v)\rangle = \langle f^{+}(u), e^{-}(v)\rangle = \frac{1}{u-v},$$
$$\langle h^{+}(u), h^{-}(v)\rangle = \frac{u-v+1}{u-v-1} \tag{4.17}$$

or, in terms of the generators,

$$\langle e_k, f_{-l-1}\rangle = \langle f_k, e_{-l-1}\rangle = -\delta_{kl},$$
$$\langle h_k, h_{-l-1}\rangle = -\frac{2k!}{l!(k-l)!} \tag{4.18}$$

for $k, l \geqslant 0$.

Proof. Proposition 4.1 shows that the relations (4.17) are satisfied modulo some constant α:

$$\langle e^{+}(u), f^{-1}(v)\rangle = \frac{\alpha}{u-v}, \qquad \langle f^{+}(u), e^{-}(v)\rangle = \frac{\alpha^{-1}}{u-v},$$
$$\langle h^{+}(u), h^{-}(v)\rangle = \frac{u-v+1}{u-v-1}.$$

We can find this constant from the Yangian R-matrix $R = 1 + (P/a-b)$ acting in the tensor product $V(a) \otimes V(b)$ of two-dimensional representations of $\mathscr{D}Y(sl_2)$. The action of the generators of $\mathscr{D}Y(sl_2)$ in $V(c)$ with a basis v_1, v_2 can be described by the formulas

$$e_i(v_1) = 0, \qquad e_i(v_2) = c^i v_1, \qquad f_i(v_2) = 0, \qquad f_i(v_1) = c^i v_2,$$
$$h_i(v_1) = c^i v_1, \qquad h_i(v_2) = -c^i v_2.$$

According to Theorem 3.1 (see also Proposition 5.1, the reformulation of the theorem in terms of the universal R-matrix), we take the Gauss decomposition of the Yangian R-matrix: $R = R_E R_H R_F$ and find that

$$R_E = 1 + \frac{1}{a-b} e_0 \otimes f_0 = 1 - \sum_{i \geqslant 0} e_i \otimes f_{-i-1},$$

which gives $\langle e_i, f_{-i-1}\rangle = -1$, $i \geqslant 0$, $\alpha = 1$.

In the general case, we have the following analogous theorem.

THEOREM 4.2. (i) *Subalgebras* Y_{+}^{*}, $\mathscr{H}^{*}$ *and* Y_{-}^{*} *(see (3.7)–(3.9)) of* $Y_{-}(g)$ *are generated by the fields* $e_i^{-}(u)$, $h_i^{-}(u)$, *and* $f_i^{-}(u)$ *(*$i = 1,\ldots,r$*), correspondingly;*

(ii) *The pairing of the generators of* $Y_{\pm}(g)$ *is given by the relations* ($|x| \ll 1 \ll |u|$):

$$\langle e_i^{+}(u), f_j^{-}(v)\rangle = \langle f_i^{+}(u), e_j^{-}(v)\rangle = \frac{\delta_{ij}}{u-v}, \tag{4.19}$$

$$\langle h_i^+(u), h_j^-(v)\rangle = \frac{u - v + \frac{1}{2}(\alpha_i, \alpha_j)}{u - v - \frac{1}{2}(\alpha_i, \alpha_j)}. \tag{4.20}$$

The proof is analogous to the case of sl_2 with the use of the relations from Proposition 2.2.

Remarks. (i) Actually, a variant of pairing (4.17), (4.18) was computed by Drinfeld [9]. It appeared as one of the basic points of his quantization of $g[t]$.

(ii) The pairing (4.17), (4.18) may be considered as a deformation of the classical pairing in $g[[t^{-1}, t]$ given by the rational r-matrix $r = \Omega/u$, where Ω is a split Casimir operator. Formulas (4.17), (4.18) show that this pairing remains unchanged for the currents to nilpotent subalgebras and changes by shifts $\pm\frac{1}{2}(\alpha_i, \alpha_j)$ in (de)nominators of the pairing functions of the current to Cartan subalgebras.

5. The Universal R-Matrix for $\mathscr{D}\mathrm{Y}(g)$

Let us recall that the universal R-matrix [7] for a quasitriangular Hopf algebra A is an invertible element R of some extension of $A \otimes A$ satisfying the conditions

$$\Delta'(x) = R\Delta(x)R^{-1} \quad \forall x \in \mathscr{A}, \tag{5.1}$$

$$(\Delta \otimes \mathrm{id})R = R^{13}R^{23}, \quad (\mathrm{id} \otimes \Delta)R = R^{13}R^{12}, \tag{5.2}$$

where $\Delta' = \sigma\Delta$, ($\sigma(x \otimes y) = y \otimes x$ is an opposite comultiplication in A. If A is a quantum double of a Hopf algebra A^+, $A \simeq A^+ \otimes A^-$, $A^- := A^0$ being dual to A^+ with an opposite comultiplication, then A admits a canonical realization of the universal R-matrix: $R = \Sigma_n \xi_{(n)} \otimes \xi^{(n)}$, where $\xi_{(n)}$ and $\xi^{(n)}$ are dual bases in A^+ and A^-. In our case $A^+ = \mathrm{Y}(g)$ and the canonical element R is the universal R-matrix in $\mathscr{D}\mathrm{Y}(g)$.

Let $Y_+^\pm$, $Y_0^\pm$ and $Y_-^\pm$ be subalgebras of $\mathscr{D}\mathrm{Y}(g)$ generated by the fields $e_i^\pm(u)$, $h_i^\pm(u)$, and $f_i^\pm(u)$, $(i = 1, \ldots, r)$, correspondingly. As a consequance of Theorems 3.1 and 4.2, we obtain the following proposition.

PROPOSITION 5.1. *The universal R-matrix R of the Yangian double $\mathscr{D}\mathrm{Y}(g)$ can be realized in the factorable form*

$$R = R_+ R_0 R_-, \tag{5.3}$$

where

$$R_+ \in Y_+^+ \otimes Y_-^-, \qquad R_0 \in Y_0^+ \otimes Y_0^-, \qquad R_- \in Y_-^+ \otimes Y_+^-.$$

In the case of $\mathscr{D}\mathrm{Y}(sl_2)$ we easily find by induction the general formulas for the pairing Y_+^+ with Y_-^-, and Y_-^+ with Y_+^- (cf. [23]):

$$\langle e_0^{n_0} e_1^{n_1} \cdots e_k^{n_k}, f_{-1}^{m_1} f_{-2}^{m_2} \cdots f_{-k-1}^{m_{k+1}}\rangle = (-1)^{n_0 + \cdots + n_k} \delta_{n_0, m_1} \cdots \delta_{n_k, m_{k+1}} n_0! n_1! \cdots n_k!, \tag{5.4}$$

$$\langle f_k^{n_k}\cdots f_1^{n_1} f_0^{n_0}, e_{-k-1}^{m_{k+1}}\cdots e_{-2}^{m_2} e_{-1}^{m_1}\rangle$$
$$= (-1)^{n_0+\cdots+n_k}\delta_{n_0,m_1}\cdots\delta_{n_k,m_{k+1}} n_0! n_1!\cdots n_k!, \tag{5.5}$$

The following lemma is an immediate corollary of these relations.

LEMMA 5.1. *The elements R_+ and R_- in (5.3) for the universal R-matrix of $\mathcal{D}$Y(sl$_2$) can be presented as*

$$R_+ = \prod_{k\geq 0}^{\rightarrow} \exp(-e_k\otimes f_{-k-1}) := \exp(-e_0\otimes f_{-1})\exp(-e_1\otimes f_{-2})\cdots, \tag{5.6}$$

$$R_- = \prod_{k\geq 0}^{\leftarrow} \exp(-f_k\otimes e_{-k-1}) := \cdots\exp(-f_1\otimes e_{-2})\exp(-f_0\otimes e_{-1}). \tag{5.7}$$

(See formulas (5.30), (5.31) at the end of this section for the general case of this Lemma.)

The middle term R_0 in (5.3) has a more complicated structure. One can find it directly by cumbersome calculations but we prefer here to use another argument for the connection between two realizations of $\mathcal{D}$Y(g). The general scheme is as follows.

Let $\widehat{\mathcal{D}\mathrm{Y}}(g)$ be a Hopf algebra isomorphic to $\mathcal{D}$Y(g) (as an associative algebra) with the following comultiplication [7] (it naturally appears in a quantization of the current algebra $g[t]$):

$$\tilde\Delta(h_i^{\pm}(u)) = h_i^{\pm}(u)\otimes h_i^{\pm}(u), \tag{5.8}$$

$$\begin{aligned}\tilde\Delta(e_i(u)) &= e_i(u)\otimes 1 + h_i^-(u)\otimes e_i(u),\\ \tilde\Delta(f_i(u)) &= 1\otimes f_i(u) + f_i(u)\otimes h_i^+(u),\end{aligned} \tag{5.9}$$

where

$$e_i(u) := e_i^+(u) - e_i^-(u) = \sum_{k\in Z} e_{i,k}u^{-k-1},$$
$$f_i(u) := f_i^+(u) - f_i^-(u) = \sum_{k\in Z} f_{i,k}u^{-k-1}.$$

The arguments of [18] show that just as for a case of quantum affine algebras $U_q(\hat g)$, the coalgebraic sector of $\widehat{\mathcal{D}\mathrm{Y}}(g)$ is connected with a coalgebraic sector of $\mathcal{D}$Y(g) via twisting by a limit translation automorphism (as an action of an analog of a 'virtual' longest element of an affine Weyl group). The elements of $Y_0^{\pm}$ are stable under this action and therefore the pairing $\langle h_i^+(u), h_j^-(v)\rangle$ is the same in $\mathcal{D}$Y(g) and in $\widehat{\mathcal{D}\mathrm{Y}}(g)$. Formulas (5.8) show that the components of $\ln h_i^{\pm}(u)$ are primitive elements of $\widehat{\mathcal{D}\mathrm{Y}}(g)$. This allows us to obtain immediately the pairing for the whole subalgebras Y_0^+ and Y_0^-. The calculation reduces to a diagonalization of the form $\langle \ln h_i^+(u), \ln h_j^-(v)\rangle$. An explicit diagonalization will be done later, and now we want to formulate a general statement about the structure of R_0.

PROPOSITION 5.2. *Let*

$$\varphi_i^+(u) := \sum_{k \geq 0} \varphi_{i,k} u^{-k-1} = \ln h_i^+(u),$$

$$\varphi_i^-(u) := \sum_{k \leq -1} \varphi_{i,k} u^{-k-1} = \ln h_i^+(u)$$

and let Φ^+ *and* Φ^- *be linear spaces with the bases* $\{\varphi_{i,0}, \varphi_{i,1}, \ldots\}$ *and* $\{\varphi_{i,-1}, \varphi_{i,-2}, \ldots\}$*, correspondingly. The element* R_0 *in (5.3) has the form*

$$R_0 = \exp\left(\sum_a \phi_k \otimes \phi^k\right), \tag{5.10}$$

where $\Sigma_a \phi_k \otimes \phi^k$ *is a canonical tensor in* $\Phi^+ \otimes \Phi^-$ *with respect to the pairing*

$$\langle \varphi_i^+(u), \varphi_j^-(v) \rangle = \ln \frac{u - v + \frac{1}{2}(\alpha_i, \alpha_j)}{u - v - \frac{1}{2}(\alpha_i, \alpha_j)}. \tag{5.11}$$

Here u *and* v *satisfy the constraint* $|v| \ll 1 \ll |u|$.

Proof. It should be noted that there is no action of the affine Weyl group on $\mathscr{D}Y(g)$ (natural analogs of simple reflections map $\mathscr{D}Y(g)$ into another algebra). Nevertheless, the affine shifts in $\mathscr{D}Y(g)$ are well defined. Let, for instance, $\hat{t}$ be the following 'translation' automorphism of $\mathscr{D}Y(g)$:

$$\hat{t}(e_{i,k}) = e_{i,k+1}, \qquad \hat{t}(f_{i,k}) = f_{i,k-1}, \qquad \hat{t}_0(h_{i,k}) = h_{i,k}. \tag{5.12}$$

for all $k \in \mathbb{Z}$. The arguments of [18] applied to $\mathscr{D}Y(g)$ give the following lemma.

LEMMA 5.2. *The comultiplication maps of the Hopf algebras* $\widehat{\mathscr{D}Y}(g)$ *and* $\mathscr{D}Y(g)$ *(which are isomorphic as algebras) are connected via twisting by the limit translation automorphism:* $\hat{t}_\infty = \lim_{n\to\infty} \hat{t}^n$, *i.e.*

$$\tilde{\Delta}(x) = \Delta^{(t_\infty)}(x) := \lim_{n \to \infty} (\hat{t}^n \otimes \hat{t}^n)\Delta(\hat{t}^{-n}(x)), \tag{5.13}$$

for any $x \in \mathscr{D}Y(g)$ $(\widehat{\mathscr{D}Y}(g))$ *in a suitable topology of* $\mathscr{D}Y(g) \otimes \mathscr{D}Y(g)$ *(see [18]).*

The Hopf algebra $\widehat{\mathscr{D}Y}(g)$ is by definition a double of the subalgebra $(\widehat{\mathscr{D}Y}(g))^+$ generated by the elements $e_{i,k}$, $k \in \mathbb{Z}$, and $h_{i,k}$, $k \geq 0$. Let $(\widehat{\mathscr{D}Y}(g))^-$ be a subalgebra generated by $f_{i,k}$, $k \in \mathbb{Z}$, and $h_{i,k}, k < 0$. Then $(\widehat{\mathscr{D}Y}(g))^-$ is isomorphic to a dual of $(\widehat{\mathscr{D}Y}(g))^+$ with an opposite comultiplication. Lemma 5.2 allows us to compute the pairing $(\widehat{\mathscr{D}Y}(g))^+ \otimes (\widehat{\mathscr{D}Y}(g))^- \mapsto \mathbb{C}$. Before computing this pairing, let us first say some general words about Hopf pairing and automorphisms.

Let A_1 and A_2 be two Hopf algebras with a Hopf pairing $\langle , \rangle : A_1 \otimes A_2 \mapsto \mathbb{C}$ and let $w_1 : A_1 \mapsto A_1'$ and $w_2 : A_2 \mapsto A_2'$ be some isomorphisms of algebras. Then the algebras A_1' and A_2' can be canonically equipped with a structure of Hopf algebras if we define comultiplication maps $\Delta^{(w_1)} : A_1' \mapsto A_1' \otimes A_1'$ and $\Delta^{(w_2)} : A_2' \mapsto A_2' \otimes A_2'$ by the rules

$$\Delta^{w_1}(a) = (w_1 \otimes w_1)\Delta(w_1^{-1}(a)), \qquad \Delta^{w_2}(b) = (w_2 \otimes w_2)\Delta(w_2^{-1}(b)) \tag{5.14}$$

for any $a \in A_1'$ and any $b \in A_2'$.

LEMMA 5.3. *The pairing* $\langle , \rangle : A'_1 \otimes A'_2 \mapsto \mathbb{C}$ *defined by*

$$\langle a, b \rangle = \langle w_1^{-1}(a), w_2^{-1}(b) \rangle, \quad \forall a \in A'_1, \forall b \in A'_2 \tag{5.15}$$

is a Hopf pairing.

Taking $A_1 = Y^+$, $A_2 = Y^-$, $w_1 = w_2 = \bar{t}^n$, we obtain a Hopf pairing between $(Y^+)^{(n)}$ and $(Y^-)^{(n)}$, where $(Y^+)^{(n)} := \bar{t}^n(Y^+)$, $(Y^-)^{(n)} := \bar{t}^n(Y^-)$. One can easily see from (4.19), (4.20) that this pairing stabilizes when $n \to \infty$ and defines a Hopf pairing between $(\widehat{\mathscr{D}\mathrm{Y}}(g))^+$ and $(\widehat{\mathscr{D}\mathrm{Y}}(g))^-$ (which should actually coincide with that from a double structure of $\widehat{\mathscr{D}\mathrm{Y}}(g)$). This pairing looks like

$$\langle e_{i,k}, f_{j,-l-1} \rangle = -\delta_{ij}\delta_{kl} \tag{5.16}$$

and

$$\langle h_i^+(u), h_j^-(v) \rangle = \frac{u - v + \frac{1}{2}(\alpha_i, \alpha_j)}{u - v - \frac{1}{2}(\alpha_i, \alpha_j)}, \tag{5.17}$$

for $|x| \ll 1 \ll |u|$. The pairing (5.16) coincides with (4.19), since the elements $h_{i,k}$ remain stable under the action of automorphism $\bar{t}$. From (5.10) we see that

$$\tilde{\Delta}(\varphi_i^\pm(u)) = \varphi_i^\pm(u) \otimes 1 + 1 \otimes \varphi_i^\pm(u), \tag{5.18}$$

$$\langle \varphi_i^+(u), \varphi_j^-(v) \rangle = \ln \frac{u - v + \frac{1}{2}(\alpha_i, \alpha_j)}{u - v - \frac{1}{2}(\alpha_i, \alpha_j)} \tag{5.19}$$

for $|v| \ll 1 \ll |u|$. The relation (5.18) means that the coefficients of $\varphi_i^\pm(u)$ are primitive elements with respect to $\tilde{\Delta}$. Now we have the condition of the following simple general statement.

Let A_1 and A_2 be two dual Hopf algebras isomorphic (as algebras) to free commutative algebras $A_1 \simeq \mathbb{C}[\Phi^+]$, $A_2 \simeq \mathbb{C}[\Phi^-]$, where Ψ^+ and Φ^- are vector spaces of generators, such that all $\phi^+ \in \Psi^+$ (or all $\phi^- \in \Phi^-$) are primitive elements. Let $\{\phi_k\}$ and $\{\phi^k\}$ be dual bases of Φ^+ and Φ^- with respect to a (nondegenerated) restriction of the pairing to $\Phi^+ \otimes \Phi^-$. Then, once we choose some order of basic vectors, we have

$$\langle (\phi_1)^{n_1} \cdots (\psi_k)^{n_k}, (\phi^1)^{m_1} \cdots (\phi^k)^{m_k} \rangle = \delta_{n_1 m_1} \cdots \delta_{n_k m_k} n_1! \cdots n_k!$$

or, in other words, the canonical tensor $R^{A_1 \otimes A_2} = \Sigma\, a_k \otimes b^k$ of the pairing of A_1 and A_2 is an exponential of the canonical tensor $\Omega = \Sigma\, \phi_k \otimes \phi^k$ of the pairing of Φ and Φ:

$$R^{A_1 \otimes A_2} = \exp \Omega \tag{5.20}$$

which proves Proposition 5.2.

We further describe the canonical tensor $\Omega = \Sigma\, \psi_k \otimes \phi^k$ from Proposition 5.2 more explicitly, in other words we present concrete diagonalization of the bilinear form (5.11). For illustration, we first do this for $\mathscr{D}\mathrm{Y}(\mathrm{sl}_2)$.

Let $(\varphi^+(u))' := (d/du)\varphi^+(u)$. From (5.16) we have

$$\langle (\varphi^+(u))', \varphi^-(v)\rangle = \frac{1}{u-v+1} - \frac{1}{u-v-1}. \tag{5.21}$$

If

$$\phi^+(u) = \sum_{i\geq 0} \phi_k u^{-k-1},\ \phi^-(u) = \sum_{k\leq -1} \phi_k u^{-k-1}$$

are arbitrary fields from Φ^+ and Φ^-, then the diagonal pairing $\langle \psi_k, \phi_{-l-1}\rangle_{\mathrm{diag}} = \delta_{ij}$, in terms of generating functions, looks like

$$\langle \phi^+(u), \phi^-(v)\rangle_{\mathrm{diag}} = \frac{1}{u-v}$$

for $|x| \ll 1 \ll |u|$. Let $\hat{T}$ be a linear operator in the space Φ^-. If we use the notation $\langle \psi^+(u), \hat{T}\phi^-(v)\rangle_{\mathrm{diag}}$ as a pairing of $\phi^+(u)$ and $\hat{T}\phi^-(v)$ under the condition that the pairing of $\phi^+(u)$ and $\phi^-(v)$ is known to be diagonal, then (5.21) can be read as

$$\langle (\varphi^+(u))', \varphi^-(v)\rangle = \langle (\varphi^+(u))', (T - T^{-1})\varphi^-(v)\rangle_{\mathrm{diag}}, \tag{5.22}$$

where T: $Tf(v) = f(v-1)$ is a shift operator. From (5.22), we have

$$\langle (\varphi^+(u))', (T - T^{-1})^{-1}\varphi^-(v)\rangle = \langle (\varphi^+(u))', \varphi^-(v)\rangle_{\mathrm{diag}} = \frac{1}{u-v}.$$

Now we can formally invert an operator $(T - T^{-1})$ as

$$(T - T^{-1})^{-1} = T^{-1} + T^{-3} + T^{-5} + \cdots \tag{5.23}$$

which gives the diagonalization of the bilinear form (5.11):

$$\sum_{n\geq 0} \langle (\varphi^+(u))', \varphi^-(v + 1 + 2n)\rangle = \frac{1}{u-v}. \tag{5.24}$$

Remark. The inverse (5.23) to a difference derivative $T - T^{-1}$ is only a right-inverse operator (just as usual integral). We can define it in a different manner like, e.g.,

$$(T - T^{-1})^{-1} = -T - T^3 - T^5 - \cdots \tag{5.25}$$

and obtain the diagonalization form

$$\sum_{n\geq 0} \langle -(\varphi^+(u))', \varphi^-(v - 1 - 2n)\rangle = \frac{1}{u-v}. \tag{5.26}$$

Both formulas work for proper regions of finite-dimensional representations of $\mathscr{D}\mathrm{Y}(g)$.

Using the notations $(\psi(u))_k = \psi_k$ for $\psi(u) = \Sigma\, \psi_k u^{-k-1}$ and

$$\mathrm{Res}_{u=v}\, \psi(u)(\otimes\phi(v)) = \sum_k \psi_k \otimes \phi_{-k-1},$$

we interpret (5.24) as the following expression of the factor R_0 of the universal R-matrix for $\mathcal{D}Y(sl_2)$:

$$R_0 = \prod_{n\geq 0} \exp\left(\sum_k ((\varphi^+(u))'_k \otimes (\varphi^-(v+2n+1))_{-k-1}\right)$$
$$= \prod_{n\geq 0} \exp \operatorname{Res}_{u=v}(\varphi^+(u))' \otimes \varphi^-(v+2n+1) \tag{5.27}$$

or, if we use (5.26),

$$R_0 = \prod_{n\geq 0} \exp\left(-\sum_{k\geq 0} ((\varphi^+(u))_k \otimes (\varphi^-(v-2n-1))_{-k-1}\right)$$
$$= \prod_{n\geq 0} \exp \operatorname{Res}_{u=v}(-(\varphi^+(u))' \otimes \varphi^-(v-2n-1) \tag{5.28}$$

We can summarize the calculations of the universal R-matrix for $\mathcal{D}Y(sl_2)$ in the following theorem.

THEOREM 5.1. *The universal R-matrix for* $\mathcal{D}Y(sl_2)$ *can be presented in the factorized form*

$$R = R_+ R_0 R_-, \tag{5.29}$$

where

$$R_+ = \prod_{k\geq 0}^{\rightarrow} \exp(-e_k \otimes f_{-k-1}), \qquad R_- = \prod_{k\geq 0}^{\leftarrow} \exp(-f_k \otimes e_{-k-1}), \tag{5.30}$$

$$R_0 = \prod_{n\geq 0} \exp \operatorname{Res}_{u=v}(\varphi^+(u))' \otimes \varphi^-(v+2n+1). \tag{5.31}$$

Here $\varphi^{\pm}(u) = \ln h^{\pm}(u)$.

Remark. Actually, we simultaneously prove statement (ii) of Proposition 2.1, since we have checked the properties (5.2) of the element R and (5.1) for the fundamental representation of $\mathcal{D}Y(sl_2)$, which is sufficient.

Now we return to the general case. Just as for $Y(sl_2)$-case, we have the following description of the pairing (5.11) in terms of the derivative of $\varphi_i^+(u)$:

$$\langle (\varphi_i^+(u))', \varphi_j^-(v)\rangle = \frac{1}{u-v+\frac{1}{2}(\alpha_i,\alpha_j)} - \frac{1}{u-v-\frac{1}{2}(\alpha_i,\alpha_j)}. \tag{5.32}$$

It is more convenient to collect fields $\varphi_i^{\pm}(u)$ to vector-valued generating functions

$$\varphi^{\pm}(u) = \begin{pmatrix} \varphi_1^{\pm}(u) \\ \varphi_2^{\pm}(u) \\ \cdots \\ \varphi_r^{\pm}(u) \end{pmatrix},$$

where $r = \operatorname{rank} g$. In terms of vector-valued fields $\phi^+(u)$ and $\phi^-(v)$, the diagonal pairing $\langle \phi_{i,k}, \phi_{j,-m-1} \rangle = \delta_{ij}\delta_{km}$ looks like

$$\langle \psi(u), \phi(v) \rangle_{\text{diag}} = \frac{1}{u - v} I,$$

where I is an $r \times r$ identity matrix.

Let $B = (b_{ij})_{i,j=0}^r$ be a symmetrized Cartan matrix of g with matrix elements being integers without a common divisor, $b_{ij} = (\alpha_i, \alpha_j)$, and $B(q)$ be a q-analog of B:

$$B_{ij}(q) = [(\alpha_i, \alpha_j)]_q = \frac{q^{(\alpha_i, \alpha_j)} - q^{-(\alpha_i, \alpha_j)}}{q - q^{-1}}. \tag{5.33}$$

Here we use the standard notation

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

Let $D(q)$ be an inverse matrix to $B(q)$. One can see that $D(q)$ can be presented in the form

$$D(q) = \frac{1}{[l(g)]_q} C(q), \tag{5.34}$$

where $C(q)$ is a matrix with matrix coefficients $c_{ij}(q)$ being polynomials of q and of q^{-1} with positive integer coefficients and $l(g)$ being a positive integer. Actually, the calculation of $\det B(q)$ shows that $l(g)$ is proportional to a dual Coxeter number $h^\vee$ of g (see also Table (7.6) below:

$$\begin{aligned} &l(g) = h^\vee(\hat{g}) \quad \text{for } g = A_n, E_6, E_7, E_8; \\ &l(g) = 2h^\vee(\hat{g}) \quad \text{for } g = B_n, D_n, F_4; \\ &l(g) = 3h^\vee(\hat{g}) \quad \text{for } g = G_2; \qquad l(g) = 4h^\vee(\hat{g}) \quad \text{for } g = C_n. \end{aligned} \tag{5.35}$$

In these notations, the pairing (5.32) can be written as

$$\langle (\varphi^+(u))', \varphi^-(v) \rangle = \langle (\varphi^+(u))', ((q - q^{-1})B(q))|_{q = \hat{T}^{-1/2}} \varphi^-(v) \rangle_{\text{diag}}, \tag{5.36}$$

where a shift operator $\hat{T}$: $\hat{T}f(v) = f(v - 1)$ is substituted inside the r.h.s. of (5.36) instead of q^2. Next, we deduce that

$$\langle (\varphi^+(u))', (\hat{T}^{l(g)/2} - \hat{T}^{-(l(g)/2)})^{-1} C(\hat{T}^{-(1/2)}) \varphi^-(v) \rangle = \frac{1}{u - v} I. \tag{5.37}$$

Returning to the original notations, we get the following diagonalization of the pairing (5.11):

$$\sum_{n \geq 0} \langle (\varphi_i^+(u))', \sum_l c_{lj}(\hat{T}^{-(1/2)}) \varphi_l^-(v + (n + \tfrac{1}{2}) l(g)) \rangle = \frac{\delta_{ij}}{u - v}. \tag{5.38}$$

Let us note once more that (just as in $Y(sl_2)$ case (5.28) it is possible to write down another diagonalization of the pairing (5.16) for any $Y(g)$. We can summarize the calculations in the following theorem.

THEOREM 5.2. *The element R_0 in (5.3) for the universal R-matrix of $\mathscr{D}Y(g)$ can be presented as*

$$R_0 = \prod_{n\geqslant 0} \exp \sum_{i,j=1,\ldots,r} \sum_{k\geqslant 0} (\varphi_i^+(u))'_k \otimes (c_{ji}(\hat{T}^{-(1/2)})\varphi_j^-(v+(n+\tfrac{1}{2})l(g)))_{-k-1}$$
$$= \prod_{n\geqslant 0} \exp \sum_{i,j=1,\ldots,r} \mathrm{Res}_{u=v}(\varphi_i^+(u))' \otimes c_{ji}(\hat{T}^{-(1/2)})\varphi_j^-(v+(n+\tfrac{1}{2})l(g)). \tag{5.39}$$

In order to complete the description of the universal R-matrix for $\mathscr{D}Y(g)$, we should extend the description of their factors R_+ and R_- from the sl_2 case to the case of arbitrary simple Lie algebra g. Let us first change the notations for generators of $\mathscr{D}Y(g)$. Instead of $e_{i,k}$, we use $e_{\alpha_i+k\delta}$ and instead of $f_{i,k}$, we use $e_{\alpha_i+k\delta}$. We denote also by $\hat{\Delta}^{re}$ the set of all real roots of corresponding affine nontwisted Lie algebra. Let Σ be a subset of $\hat{\Delta}^{re}$. Recall that a total linear ordering $\prec$ of Σ is called *normal* (or convex) [27] if, for any three roots, $\alpha, \beta, \gamma \in \Sigma$, $\gamma = \alpha + \beta$, we have $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$.

Let Σ_+ and Σ_- be the following subsets of $\hat{\Delta}^{re}$:

$$\begin{aligned} \Sigma_+ &= \{\gamma + k\delta \mid \gamma \in \Delta_+(g),\ k \geqslant 0\},\\ \Sigma_- &= \{-\gamma + k\delta \mid \gamma \in \Delta_+(g),\ k \geqslant 0\}. \end{aligned} \tag{5.40}$$

Here δ is a minimal imaginary root of $\hat{g}$. Let us equip Σ_+ and Σ_- with two arbitrary normal orderings $\prec_+$ and $\prec_-$ satisfying the additional constraint

$$\gamma + k\delta \prec_+ \gamma + l\delta \quad \text{and} \quad -\gamma + l\delta \prec_- -\gamma + l\delta \quad \text{if } n > m, \tag{5.41}$$

for any $\gamma \in \Delta_+(g)$. We can define the 'root vectors' $e_{\pm\beta}, \beta \in \Sigma_+ \cup \Sigma_-$ by induction following the instruction [17]

$$e_{\beta_3} = [e_{\beta_1}, e_{\beta_2}], \qquad e_{-\beta_3} = [e_{-\beta_2}, e_{-\beta_1}] \tag{5.42}$$

if $\beta_1 < \beta_3 < \beta_2$ and (β_1, β_2) is a minimal segment in a sense of chosen orderings containing β_3, and e_{β_1}, e_{β_2} are already being constructed. Analogously to the case $U_q(\hat{g})$, [17, 18] one can prove that the procedure (5.42) is correctly defined and the monomials

$$e_{\beta_1}^{n_1} e_{\beta_2}^{n_2} \cdots e_{\beta_k}^{n_k}, \quad \beta_1 \prec_+ \beta_2 \prec_+ \cdots \prec_+ \beta_k, \quad \beta_i \in \Sigma_+$$

form a basis of the subalgebra $Y_+ \subset Y(g)$ and the monomials

$$e_{\beta_1}^{n_1} e_{\beta_2}^{n_2} \cdots e_{\beta_k}^{n_k}, \quad \beta_1 \prec_- \beta_2 \prec_- \cdots \prec_- \beta_k, \quad \beta_i \in \Sigma_F$$

form a basis of subalgebra $Y_- \subset Y(g)$.

Various arguments [14, 15] show that the factors R_+ and R_- of the universal R-matrix for $\mathcal{D}Y(g)$ should have the following form which we state here as a conjecture, since we did not check the rigorous proof.

CONJECTURE. The factors R_+ and R_- of the universal R-matrix for $\mathcal{D}Y(g)$ have the following form

$$R_+ = \prod_{\beta\in\Sigma_+}^{\rightarrow} \exp(-a(\beta)e_\beta \otimes e_{-\beta}), \qquad R_- = \prod_{\beta\in\Sigma_-}^{\rightarrow} \exp(-a(\beta)e_\beta \otimes e_{-\beta}), \tag{5.43}$$

where the products are taken in the given normal orderings $\prec_+$ and $\prec_-$ satisfying (5.41). Normalizing constants $a(\beta)$ are taken from the relations

$$[e_\beta, e_{-\beta}] = (a(\beta))^{-1} h_\gamma \quad \text{if } \beta = \gamma + n\delta \in \Sigma_+, \quad \gamma \in \Delta_+(g),$$
$$[e_{-\beta}, e_\beta] = (a(\beta))^{-1} h_\gamma \quad \text{if } \beta = -\gamma + n\delta \in \Sigma_-, \quad \gamma \in \Delta_+(g).$$

6. An Example of $Y(sl_2)$

6.1. R-MATRIX FOR TENSOR PRODUCTS OF EVALUATION REPRESENTATIONS OF $Y(sl_2)$

Here we demonstrate how the general formulas (5.29)–(5.32) for the universal R-matrix work in evaluation representations of the Yangian $Y(sl_2)$. Analogous calculations for $U_q(\widehat{sl_2})$ are presented in [14].

One can easily check that the assignment

$$\Theta: e_0 \mapsto e, \quad h_0 \mapsto h, \quad f_0 \mapsto f, \qquad e_1 \mapsto \frac{h-1}{2}e, \quad f_1 \mapsto f\,\frac{h-1}{2}, \tag{6.1}$$

extends to an epimorphism of algebras $Y(sl_2) \to U(sl_2)$. In terms of generating functions, morphism Θ can be written as

$$\Theta e^+(u) = (u - \tfrac{1}{2}(h-1))^{-1} e, \qquad \Theta f^+(u) = f(u - \tfrac{1}{2}(h-1))^{-1},$$
$$\Theta h^+(u) = 1 + (u - \tfrac{1}{2}(h-1))^{-1} ef - (u - \tfrac{1}{2}(h+1))^{-1} fe, \quad |u| \gg 1. \tag{6.2}$$

Morphism ϕ is also properly defined for $\mathcal{D}Y(sl_2)$, since one can interpret the r.h.s. of (6.2) as an expansion near zero:

$$\Theta e^-(u) = (u - \tfrac{1}{2}(h-1))^{-1} e, \qquad \Theta f^-(u) = f(u - \tfrac{1}{2}(h-1))^{-1},$$
$$\Theta h^-(u) = 1 + (u - \tfrac{1}{2}(h-1))^{-1} ef - (u - \tfrac{1}{2}(h+1))^{-1} fe, \quad |u| \ll 1. \tag{6.3}$$

Let V_λ and V_μ be two representations of Lie algebra sl_2 with highest weights λ and μ (Verma modules or their finite-dimensional quotions, for instance), let $V_\lambda(a)$ and $V_\mu(b)$ be corresponding evaluation representations. In a concrete example, when V_n is finite-dimensional representation ($\dim V_n = n + 1$) with highest-weight vector v_0 and a basis $v_0, v_1, \ldots, v_n$, we have for $V_n(a)$ by (6.2):

$$e_i v_k = k\left(a + \frac{n-2k+1}{2}\right)^i v_{k-1}, \qquad f_i v_k = (n-k)\left(a + \frac{n-2k-1}{2}\right)^i v_{k+1}$$

$$h_i v_k = \left((k+1)(n-k)\left(a + \frac{n-2k-1}{2}\right)^i - \right.$$

$$\left. - k(n-k+1)\left(a + \frac{n-2k+1}{2}\right)^i\right) v_k. \tag{6.4}$$

For the calculation of R-matrix in $V_\lambda(a) \otimes V_\mu(b)$ it is sufficient to compute $(\phi \otimes \phi)(T_a \otimes T_b)R$ as a function of $a - b$ with values in $U(sl_2) \otimes U(sl_2)$. Here T_d is a shift operator in $\mathscr{D}Y(sl_2)$,

$$T_d e_\pm(u) = e_\pm(u-d), \qquad T_d h_\pm(u) = h_\pm(u-d), \qquad T_d f_\pm(u) = f_\pm(u-d).$$

We do this first for the factors R_+ and R_-. Let

$$y = a - b + \frac{h \otimes 1 - 1 \otimes h}{2}. \tag{6.5}$$

We think of y as of diagonal matrix acting in $V_\lambda(a) \otimes V_\mu(b)$. Substitution of (6.2) into (5.6) gives the following answer:

$$(\Theta \otimes \Theta)(T_a \otimes T_b)R_+$$

$$= 1 + e \otimes f\,\frac{1}{y+1} + e^2 \otimes f^2\,\frac{1}{2(y+1)(y+2)} + \cdots +$$

$$+ e^n \otimes f^n\,\frac{1}{n!(y+1)\cdots(y+n)} + \cdots \tag{6.6}$$

One can consider r.h.s. of (6.6) as a difference analog of the ordered exponential:

$$(\Theta \otimes \Theta)(T_a \otimes T_b)(R_+) = :\exp e \otimes f(y+1)^{-1}:_{T^{-1}}$$

where T: $Tf(x) = f(x-1)$ is again a shift operator and

$$:\exp f(y):_{T^{-1}} = 1 + f(y) + \tfrac{1}{2} f(y) f(y+1) + \cdots +$$

$$+ \frac{1}{n!} f(y) f(y+1) \cdots f(y+(n-1)) + \cdots$$

Analogously,

$$(\Theta \otimes \Theta)(T_a \otimes T_b)(R_-)$$

$$= :\exp(y+1)^{-1} f \otimes e:_{T^{-1}}$$

$$= 1 + \frac{1}{y+1} f \otimes e + \frac{1}{2(y+1)(y+2)} f^2 \otimes e^2 + \cdots +$$

$$+ \frac{1}{n!(y+1)\cdots(y+n)} f^n \otimes e^n + \cdots \tag{6.7}$$

Note that, for any weight vector of $V_\lambda \otimes V_\mu$, the series (6.6) and (6.7) is finite and has a form of operator from $U(sl_2) \otimes U(sl_2)$ with rational coefficients.

The calculation of $(\Theta \otimes \Theta)(T_a \otimes T_b)(R_0)$ is more complicated. We perform this calculation directly in $V_\lambda \otimes V_\mu$ for simplicity. We can rewrite the action of $h^+(u)$ in $V_\lambda(a)$ (6.4) as

$$h^+(u) = \frac{(u - a - \frac{\lambda+1}{2})(u - a + \frac{\lambda+1}{2})}{(u - a - \frac{h+1}{2})(u - a - \frac{h-1}{2})}, \quad |u| \gg 1 \tag{6.8}$$

and, analogously, the action of $h^-(v)$ in $V_\mu(b)$ as

$$h^-(v) = \frac{(v - b - \frac{\mu+1}{2})(v - b + \frac{\mu+1}{2})}{(v - b - \frac{h+1}{2})(v - b - \frac{h-1}{2})}, \quad |v| \ll 1. \tag{6.9}$$

From (6.8) and (6.9), we get

$$\frac{d}{du}\varphi^+(u) = d \log h^+(u)$$

$$= \frac{1}{(u - a - \frac{\lambda+1}{2})} + \frac{1}{(u - a + \frac{\lambda+1}{2})} -$$

$$- \frac{1}{(u - a - \frac{h+1}{2})} - \frac{1}{(u - a - \frac{h-1}{2})}, \quad |u| \gg 1 \tag{6.10}$$

and

$$\varphi^-(v) = \log \frac{(v - b - \frac{\mu+1}{2})(v - b + \frac{\mu+1}{2})}{(v - b - \frac{h+1}{2})(v - b - \frac{h-1}{2})}, \quad |v| \ll 1. \tag{6.11}$$

The rest of the computations reduces to the calculation of the residues for the functions with simple poles. Denoting

$$h_1 = h \otimes 1, \qquad h_2 = 1 \otimes h, \qquad c = \frac{a-b}{2} \tag{6.12}$$

we finally get the following horrible answer:

$$R_0|_{V_\lambda(a) \otimes V_\mu(b)}$$

$$= \frac{\Gamma(c + \frac{\lambda - h_2}{4} + \frac{1}{2})\Gamma(c - \frac{\lambda + h_2}{4} + \frac{1}{2})\Gamma(c + \frac{\lambda - h_2}{4} + 1)\Gamma(c - \frac{\lambda + h_2}{4})}{\Gamma(c + \frac{\lambda - \mu}{4} + \frac{1}{2})\Gamma(c - \frac{\lambda - \mu}{4} + \frac{1}{2}))\Gamma(c + \frac{\lambda + \mu}{4} + 1)\Gamma(c - \frac{\lambda + \mu}{4})} \times$$

$$\times \frac{\Gamma(c+\frac{h_1-\mu}{4}+\frac{1}{2})\Gamma(c+\frac{h_1+\mu}{4}+\frac{1}{2})\Gamma(c+\frac{h_1+\mu}{4}+1)\Gamma(c+\frac{h_1-\mu}{4})}{\Gamma(c+\frac{h_1-h_2}{4})(\Gamma(c+\frac{h_1-h_2}{4}+\frac{1}{2}))^2\,\Gamma(c+\frac{h_1-h_2}{4}+1)}. \tag{6.13}$$

If we suppose λ and μ to be integers and V_λ, V_μ be finite-dimensional ($\dim V_\lambda = \lambda + 1$, $\dim V_\mu = \mu + 1$), then the whole R-matrix has rational coefficients up to a scalar factor. We can easily find this factor $\chi^{(R)}_{\lambda,\mu}$ normalizing the R-matrix in such a way that the matrix coefficient of the tensor product of the highest-weight vector to itself is equal to one. This gives

$$\chi^{(R)}_{\lambda,\mu} = \frac{\Gamma(\frac{a-b}{2}+\frac{\lambda+\mu}{4}+\frac{1}{2})\Gamma(\frac{a-b}{2}-\frac{\lambda+\mu}{4}+\frac{1}{2})}{\Gamma(\frac{a-b}{2}+\frac{\lambda-\mu}{4}+\frac{1}{2})\Gamma(\frac{a-b}{2}+\frac{\mu-\lambda}{4}+\frac{1}{2})}. \tag{6.14}$$

In the next section we describe this scalar factor for arbitrary finite-dimensional representations of $Y(g)$.

6.2. L-OPERATOR PRESENTATION OF $\mathscr{D}Y(sl_2)$

An explicit expression for the universal R-matrix for $\mathscr{D}Y(sl_2)$ gives the possibility to present $\mathscr{D}Y(sl_2)$ in the form of L-operators [11, 22]. Let

$$L^-(z) = (\rho_1(z) \otimes id)R, \qquad L^+(z) = (\rho_1(z) \otimes id)(R^{21})^{-1}, \tag{6.15}$$

where $\rho_1(z)$ is a two-dimensional representation $V_1(z)$ of $\mathscr{D}Y(sl_2)$ (see (6.4)). Substitution of (5.28)–(5.31) into (6.15) gives the following Gauss decomposition of $L^\pm(z)$ (compare [10] for $U_q(\widehat{gl}_n)$):

$$L^\pm(z) = \begin{pmatrix} 1 & f^\pm(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1^\pm(z) & 0 \\ 0 & k_2^\pm(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^\pm(z) & 1 \end{pmatrix}, \tag{6.16}$$

where

$$(k_2^\mp(z))^{-1} k_1^\pm(z) = h^\pm(z), \qquad k_1^\pm(z) k_2^\mp(z-1) = 1,$$

$$k_1^+(z) = \prod_{k \geq 0} \frac{h^+(z-2n-1)}{h^+(z-2n-2)}, \qquad k_2^+(z) = \prod_{k \geq 0} \frac{h^+(z-2n-1)}{h^+(z-2n)}.$$

$$k_1^-(z) = \prod_{k \geq 0} \frac{h-(z+2n)}{h^-(z+2n+1)}, \qquad k_2^-(z) = \prod_{k \geq 0} \frac{h^-(z+2n+2)}{h^-(z+2n+1)}$$

L-operators $L^\pm(z)$ satisfy the equations

$$R_{12}(z-w) L_1^\pm(z) L_2^\pm(w) = L_2^\pm(w) L_1^\pm(z) R_{12}(z-w),$$

$$R_{12}(z-w) L_1^-(z) L_2^+(w) = L_2^+(w) L_1^-(z) R_{12}(z-w), \qquad q \det L^\pm(z) = 1,$$

where

$$R_{12}(z-w) = 1 + \frac{P}{z-w}$$

is a rational R-matrix in $V_1(z) \otimes V_1(w)$,

$$L_1^{\pm}(z) = L^{\pm}(z) \otimes \mathrm{id}, \qquad L_2^{\pm}(z) = \mathrm{id} \otimes L^{\pm}(z),$$

$$q\det A(z) = a_{11}(z)a_{22}(z-1) - a_{21}(z)a_{12}(z-1).$$

Note that if we apply $\rho_1 \otimes \mathrm{id}$ to the current realization of R or, equivalently, take the current realization of generators in (6.16), then we get $\mathrm{sl}_2 \otimes \mathbb{C}(z)$ matrix

$$L^{\pm}(z) = c^{\pm}(z, \lambda)\begin{pmatrix} z + \frac{h+1}{2} & f \\ e & z - \frac{h-1}{2} \end{pmatrix},$$

where $c^{\pm}(z, \lambda)$ is the coefficient

$$c^{\pm}(z, \lambda) = \frac{\Gamma(\mp z' - \frac{\lambda}{4})\Gamma(\mp z' + \frac{\lambda+2}{4})}{\Gamma(\mp z' + \frac{2-\lambda}{4})\Gamma(\mp z' + \frac{\lambda+4}{4})}, \quad z' = \frac{z}{2} + \frac{1}{4},$$

with λ being the value of the highest weight on which $L(z)$ acts (actually $c(z, \lambda)$ depends on the Laplace operator, since $c^{\pm}(z, \lambda) = c^{\pm}(z, -2 - \lambda)$).

7. The Character of Universal R-Matrix

Let V and W be two irreducible finite-dimensional representations of the Yangian $\mathrm{Y}(g)$. It can be proved by fusion procedure and by studying the R-matrix for fundamental representations of the Yangian, that the R-matrix $R_{V,W}$ intertwining two coproducts Δ and Δ' in $V(a) \otimes V(b)$, can be presented as a matrix with rational coefficients of $a - b$ (here $V(a)$ and $W(b)$ are obtained from V and W by means of a natural shift automorphism of $\mathrm{Y}(g)$). On the other hand, one can apply $\rho_{V(a)} \otimes \rho_{W(b)}$ to the universal R-matrix. The results should differ by a scalar phase factor $\chi_{V,W}^{(R)}$ which appears due to nonlinear conditions (5.2) on the universal R-matrix. The factor $\chi_{V,W}^{(R)}$ is, by definition, a unique modulo rational function of $(a - b)$ and plays an important role in scattering theory.

Unfortunately, the intriguing theory of finite-dimensional representations of Yangians is too young and does not says such about representations: there is a classification of irreducible modules but their structure is almost unknown (including the dimensions and characters). Nevertheless, the theory of highest weight developed by Drinfeld [7] (see also [25, 26] coupled with our description of the universal R-matrix, allows us to compute the factor $\chi_{V,W}^{(R)}$ for arbitrary irreducible finite-dimensional representations of Yangian $\mathrm{Y}(g)$.

Let us recall the basic definitions of highest-weight polynomials of finite-dimensional representation of $\mathrm{Y}(g)$.

DEFINITION 7.1. Let V be a $\mathrm{Y}(g)$-module. A vector $v \in V$ is a highest weight vector if

(i) $e_i^+(u)v = 0, \quad i = 1, \ldots, r$
(ii) $h_i^+(u)v = H_i(u)v$ (i.e., v is an eigenvector for all $h_{i,n}, n \geqslant 0$).

The functions $H_i(u)$, $i = 1, \ldots, r$, are called eigenfunctions of v.

THEOREM 7.1 [6]. (a) *Any irreducible finite-dimensional* $Y(g)$*-module is generated by a highest-weight vector;* (b) *An irreducible finite-dimensional* $Y(g)$*-module* V *with a highest-weight vector* v *is finite-dimensional iff the eigenfunctions* $H_i(u)$ *of* v *can be presented as a ratio of polynomials*

$$H_i(u) = \frac{P_i(u + \frac{1}{2}(\alpha_i, \alpha_i))}{P_i(u)}. \tag{7.1}$$

DEFINITION 7.2. (a) A polynomial-tuple (vector polynomial) $\mathbf{P}(u) := (P_1(u), \ldots, P_r(u))$ defined by the condition (7.1) is called a highest-weight vector polynomials of a highest-weight vector v (and of finite-dimensional representation V);

(b) Finite-dimensional representation of $Y(g)$ with a highest-weight vector polynomial

$$\mathbf{P}_i(u) = (1, \ldots, 1, P_i(u) = u - a, 1, \ldots, 1), \quad a \in \mathbb{C},$$

is called ith fundamental representation of $Y(g)$.

We denote the i-th fundamental representation of $Y(g)$ by $\omega_i(a)$.

Componentwise multiplication of the weight polynomials endows the set $E(g)$ of all irreducible finite-dimensional representations of $Y(g)$ with a structure of Abelian (multiplicative) semigroup generated by fundamental representations. An element V of $E(g)$ can be presented as

$$V = \omega_1(a_{1,1}) \cdots \omega_1(a_{1,i_1}) \cdot \cdots \cdot \omega_r(a_{r,1}) \cdots \omega_r(a_{r,i_r}), \tag{7.2}$$

which means that the j-component of the highest-weight vector polynomial of V is $P_j(u) = (u - a_{j,1}) \cdots (u - a_{j,i_j})$ and V can be realized as a subfactor of the tensor product of fundamental representations $\omega(a_{ij})$ containing the tensor product of highest-weight vectors of fundamental representations. Analogously, the multiplication law $U = V \cdot W$ implies that an irreducible module U is a subfactor of $V \otimes W$ generated by the image of the tensor product of the highest-weight vectors of V and W.

Now let V and W be finite-dimensional irreducible representations of $Y(g)$ generated by highest-weight vectors v and w. They can be endowed with a structure of $Y_-(g)$ (and of $\mathcal{D}Y(g)$)-module just by re-expanding of matrix coefficients of $e_i^+(u), h_i^+(u), f_i^+(u)$, in $u = 0$. The general structure (5.3) of the universal R-matrix shows that $v \otimes u$ is an eigenvector of R.

DEFINITION 7.3. A scalar function $\chi^{(R)}_{V,W}$ defined by the condition

$$R(v \otimes w) = \chi^{(R)}_{V,W}\, v \otimes w, \tag{7.3}$$

where R is a universal R-matrix for $\mathcal{D}Y(g)$ and v and w are highest-weight vectors of V and W, is called the character of the universal R-matrix R corresponding to highest-weight representations V and W.

The following lemma explains that the character of the universal R-matrix may be considered as a (multiplicative) bilinear form on $E(g)$.

LEMMA 7.1. *Let* $V, V_1, V_2;\ W, W_1, W_2 \in E(g)$. *Then*

$$\chi^{(R)}_{V_1 \cdot V_2, W} = \chi^{(R)}_{V_1, W} \cdot \chi^{(R)}_{V_2, W}, \tag{7.4}$$

$$\chi^{(R)}_{V, W_1 \cdot W_2} = \chi^{(R)}_{V, W_1} \cdot \chi^{(R)}_{V, W_2} \tag{7.5}$$

The proof follows immediately from (5.2).

The general expression (5.39) for the factor R_0 of the universal R-matrix allows us to find out the character of R for arbitrary finite-dimensonal representations of $Y(g)$. Indeed, the action of R on the tensor product $v \otimes w$ of highest-weight vectors reduces to the action of its factor R_0. If $\mathbf{P}(u)$ is the highest-weight vector polynomial of v and $\mathbf{Q}(u)$ is the highest-weight vector polynomial of w, then the action of the fields $(\mathrm{d}/\mathrm{d}u)\varphi_i^+(u)$ on v is given by the expression

$$\frac{\mathrm{d}}{\mathrm{d}u}\varphi_i^+(u)v = (\mathrm{d}\log P_i(u + \tfrac{1}{2}(\alpha_i, \alpha_i)) - \mathrm{d}\log P_i(u))v, \quad |u| \gg 1;$$

$\varphi_j^-(v)$ act on w as

$$\varphi_j^-(v)w = \log \frac{Q_j(v + \frac{1}{2}(\alpha_i, \alpha_j))}{Q_j(v)} w, \quad |v| \ll 1.$$

The rest is a technical application of (5.39). Due to Lemma 7.1, it is sufficient to compute the characters corresponding to the tensor product of fundamental representations.

Let us recall the notations of the previous section. Now, again, B is a symmetrized Cartan matrix of g with the matrix elements being integers without common divisor, $B_{ij} = (\alpha_i, \alpha_j)$ $i, j - 1, \ldots, r$ and $B(q)$ is a q-analog of B; $D(q)$ is an inverse matrix to $B(q)$ and $C(q)$ is an $r \times r$ matrix with coefficients from $\mathbb{Z}[q, q^{-1}]$ defined by the condition $D(q) = 1/([l(g)]_q)\ C(q)$, where $l(g)$ is proportional to the dual Coxeter number $h^\vee(g)$ (see (5.35)). A presentation (5.34), (5.35) of the inverse to q-analogue of the symmetrized Cartan matrix follows from calculation of det $B(q)$:

g	det $B(q)$	$h^\vee(g)$	$l(g)$
A_l	$[l+1]_q$	$l+1$	$l+1$
B_l	$\dfrac{[2(2l-1)]_q}{[2]_q[2l-1]_q}$	$2l-1$	$2(2l-1)$
C_l	$\dfrac{[2]_q[2(l+1)]_q}{[l+1]_q}$	$l+1$	$4(l+1)$
D_l	$\dfrac{[2]_q[2l-2)]_q}{[l-1]_q}$	$2l-2$	$2(2l-2)$
E_6	$\dfrac{[2]_q[3]_q[12]_q}{[4]_q[6]_q}$	12	12

(7.6)

E_7	$\dfrac{[2]_q[3]_q[18]_q}{[6]_q[9]_q}$	18	18
E_8	$\dfrac{[2]_q[3]_q[5]_q[30]_q}{[6]_q[10]_q[15]_q}$	30	30
F_4	$\dfrac{[2]_q[3]_q[18]_q}{[6]_q[9]_q}$	9	18
G_2	$\dfrac{[2]_q[3]_q[12]_q}{[4]_q[6]_q}$	4	12

The calculations with R_0 gives the following theorem.

THEOREM 7.2. *Let* $C_{ij}(q) = \Sigma_k C_{ij}^k q^k$, $C_{ij}^k \in \mathbb{Z}_+$. *The character* $\chi^{(R)}_{\omega_i(a),\omega_j(b)}$ *of the universal R-matrix corresponding to fundamental representations* $\omega_i(a)$ and $\omega_j(b)$, *is equal to*

$$\chi^{(R)}_{\omega_i(a),\omega_j(b)}$$

$$= \prod_k \left(\frac{\Gamma\left(\frac{a-b}{l(g)} + \frac{l(g)-k-(\alpha_i,\alpha_i)}{2l(g)}\right)\Gamma\left(\frac{a-b}{l(g)} + \frac{l(g)-k+(\alpha_j,\alpha_j)}{2l(g)}\right)}{\Gamma\left(\frac{a-b}{l(g)} + \frac{l(g)-k}{2l(g)}\right)\Gamma\left(\frac{a-b}{l(g)} + \frac{l(g)-k+(\alpha_j,\alpha_j)-(\alpha_i,\alpha_i)}{2l(g)}\right)} \right)^{C_{ij}^k}. \tag{7.7}$$

For instance, for $Y(sl_2)$ the pairing (7.7) looks like

$$\chi^{(R)}_{\omega(a),\omega(b)} = \frac{\Gamma(\frac{a-b}{2})\Gamma(\frac{a-b}{2}+1)}{\Gamma(\frac{a-b}{2}+\frac{1}{2})^2} \tag{7.8}$$

and, more generally,

$$\chi^{(R)}_{\Pi_i\omega(a_i),\Pi_j\omega(b_j)} = \prod_{i,j} \frac{\Gamma(\frac{a_i-b_j}{2})\Gamma(\frac{a_i-b_j}{2}+1)}{\Gamma(\frac{a_i-b_j}{2}+\frac{1}{2})^2}, \tag{7.9}$$

which agrees with (6.14) since

$$V_n(a) = \omega\left(a - \frac{n-1}{2}\right)\omega\left(a - \frac{n-3}{2}\right)\cdots\omega\left(a + \frac{n-1}{2}\right)$$

in $E(sl_2)$.

Quasiclassically, $R(u) = 1 + (\Omega/u)$ and the quasiclassical limit of the form (7.3) on highest weights of evaluation representation $V_\lambda(a)$ and $V_\mu(b)$ of Lie algebra $g[t]$ should be $(\langle\lambda,\mu\rangle)/(a-b)$ where $\langle,\rangle$ is an invariant scalar product in h^* (h is a Cartan subalgebra of g).

It will be interesting to obtain combinatorial and geometric interpretations of the form (7.3).

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