Applied Mathematics and Mechanics
(English Edition, Vol. 18, No. 12, Dec. 1997)

Published by SU,
Shanghai, China

KÁRMÁN-TYPE EQUATIONS FOR A HIGHER-ORDER SHEAR DEFORMATION PLATE THEORY AND ITS USE IN THE THERMAL POSTBUCKLING ANALYSIS

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(Received March 11, 1996; Revised June 6, 1997)

Abstract

Kármán-type nonlinear large deflection equations are derived according to the Reddy's higher-order shear deformation plate theory and used in the thermal postbuckling analysis. The effects of initial geometric imperfections of the plate are included in the present study which also includes the thermal effects. Simply supported, symmetric cross-ply laminated plates subjected to uniform or nonuniform parabolic temperature distribution are considered. The analysis uses a mixed Galerkin-perturbation technique to determine thermal buckling loads and postbuckling equilibrium paths. The effects played by transverse shear deformation, plate aspect ratio, total number of plies, thermal load ratio and initial geometric imperfections are also studied.

Key words composite laminated plate, higher-order shear deformation plate theory, thermal postbuckling, Galerkin-perturbation technique

I. Introduction

Composite laminated plates are widely used in the nuclear, petrochemical and aerospace industries. These plates may have significant and unavoidable initial geometric imperfections. Due to boundary constraints, varying temperature environments typically induce stresses, with ensuing buckling. Therefore, there is a need to understand the thermal buckling and postbuckling behavior of imperfect composite laminated plates.

Following the assumptions of von Kármán, Stavsky (1963)[1] derived a pair of simultaneous 4th-order differential equations for the deflection $\bar{W}$ and the stress function $\bar{F}$, governing the large deformation behavior of thin composite laminated plates. The formulation was based on the classical laminated plate theory and including thermal effects. This work was extended to study the thermal postbuckling of perfect and imperfect, thin composite laminated plates subjected to a uniform or nonuniform temperature distribution by Shen and Lin (1995)[2], using a mixed Galerkin-perturbation technique.

Recent developments in the analysis of composite laminated plates point out that plate thickness has more pronounced effects on the behavior of composite plates than on the isotropic laminates. Also, due to low transverse shear moduli relative to the in-plane Young's

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moduli, transverse shear deformations are even more pronounced in composite laminates. Thus the analysis of laminated plates requires the use of shear deformation plate theory.

In the present work the Kármán-type nonlinear large deflection equations are derived according to the Reddy's higher-order shear deformation plate theory and including thermal effects. To illustrate the accuracy of the present theory, a thermal postbuckling analysis is presented for simply supported, perfect and imperfect, symmetric cross-ply laminated plates subjected to a uniform or nonuniform parabolic temperature distribution. The analysis uses a mixed Galerkin-perturbation technique to determine the required thermal buckling loads and postbuckling equilibrium paths. The material properties are assumed to be independent of temperature. The initial geometric imperfection of the plate is taken into account but, for simplicity, its form is assumed to be identical to the buckling mode of the plate.

II. Kármán-Type Equations

Consider a rectangular plate of length a, width b and thickness t which consists of N plies of any kind, is subjected to mechanical or thermal loads. Let $\bar{U}$, $\bar{V}$ and $\bar{W}$ be the plate displacements parallel to a right-hand set of axes (X, Y, Z), where X is longitudinal and Z is perpendicular to the plate. $\bar{\Psi}_x$ and $\bar{\Psi}_y$ are the mid-plane rotations of the normals about the Y and X axes, respectively. Denoting the initial deflection by $\bar{W}^*(X, Y)$, let $\bar{W}(X, Y)$ be the additional deflection and $\bar{F}(X, Y)$ be the stress function for the stress resultants, so that

$$\bar{N}_1=\bar{F}_{,yy},\quad \bar{N}_2=\bar{F}_{,xx},\quad \bar{N}_6=-\bar{F}_{,xy} \tag{2.1}$$

Following the Reddy's higher-order shear deformation plate theory (see Reddy (1984[3]), we have the displacement field

$$\left.\begin{aligned}
&u_1=\bar{U}+Z[\bar{\Psi}_x-4(Z/t)^2(\bar{\Psi}_x+\partial\bar{W}/\partial X)/3]\\
&u_2=\bar{V}+Z[\bar{\Psi}_y-4(Z/t)^2(\bar{\Psi}_y+\partial\bar{W}/\partial Y)/3],\quad u_3=\bar{W}
\end{aligned}\right\} \tag{2.2}$$

The von Kármán strains associated with the displacement field in Eq. (2.2) are

$$\left.\begin{aligned}
&\epsilon_1=\epsilon_1^0+Z(\kappa_1^0+Z^2\kappa_1^2),\quad \epsilon_2=\epsilon_2^0+Z(\kappa_2^0+Z^2\kappa_2^2),\quad \epsilon_3=0\\
&\epsilon_4=\epsilon_4^0+Z^2\kappa_4^2,\quad \epsilon_5=\epsilon_5^0+Z^2\kappa_5^2,\quad \epsilon_6=\epsilon_6^0+Z(\kappa_6^0+Z^2\kappa_6^2)
\end{aligned}\right\} \tag{2.3}$$

where

$$\left.\begin{aligned}
&\epsilon_1^0=\partial\bar{U}/\partial X+(\partial\bar{W}/\partial X)^2/2+(\partial\bar{W}/\partial X)(\partial\bar{W}^*/\partial X)\\
&\kappa_1^0=\partial\bar{\Psi}_x/\partial X,\quad \kappa_1^2=-(4/3t^2)(\partial\bar{\Psi}_x/\partial X+\partial^2\bar{W}/\partial X^2)\\
&\epsilon_2^0=\partial\bar{V}/\partial Y+(\partial\bar{W}/\partial Y)^2/2+(\partial\bar{W}/\partial Y)(\partial\bar{W}^*/\partial Y)\\
&\kappa_2^0=\partial\bar{\Psi}_y/\partial Y,\quad \kappa_2^2=-(4/3t^2)(\partial\bar{\Psi}_y/\partial Y+\partial^2\bar{W}/\partial Y^2)\\
&\epsilon_4^0=\bar{\Psi}_y+\partial\bar{W}/\partial Y,\quad \kappa_4^2=-(4/t^2)(\bar{\Psi}_y+\partial\bar{W}/\partial Y)\\
&\epsilon_5^0=\bar{\Psi}_x+\partial\bar{W}/\partial X,\quad \kappa_5^2=-(4/t^2)(\bar{\Psi}_x+\partial\bar{W}/\partial X)\\
&\epsilon_6^0=\partial\bar{U}/\partial Y+\partial\bar{V}/\partial X+(\partial\bar{W}/\partial X)(\partial\bar{W}/\partial Y)\\
&\qquad+(\partial\bar{W}/\partial X)(\partial\bar{W}^*/\partial Y)+(\partial\bar{W}/\partial Y)(\partial\bar{W}^*/\partial X)\\
&\kappa_6^0=\partial\bar{\Psi}_x/\partial Y+\partial\bar{\Psi}_y/\partial X\\
&\kappa_6^2=-(4/3t^2)(\partial\bar{\Psi}_x/\partial Y+\partial\bar{\Psi}_y/\partial X+2\partial^2\bar{W}/\partial X\partial Y)
\end{aligned}\right\} \tag{2.4}$$

The constitutive relationships between the stress resultants $\bar{N}_i$, $\bar{M}_i$, $\bar{P}_i$, Q_i and R_i and the middle surface strains and curvatures are

$$\begin{Bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{M}} \\ \bar{\mathbf{P}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} \\ \mathbf{B} & \mathbf{D} & \mathbf{F} \\ \mathbf{E} & \mathbf{F} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\epsilon}^0 \\ \kappa^0 \\ \kappa^2 \end{Bmatrix} + \begin{Bmatrix} \bar{\mathbf{N}}^T \\ \bar{\mathbf{M}}^T \\ \bar{\mathbf{P}}^T \end{Bmatrix}, \quad \begin{Bmatrix} \mathbf{Q} \\ \mathbf{R} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{D} & \mathbf{F} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\epsilon}^0 \\ \kappa^2 \end{Bmatrix} \tag{2.5a,b}$$

where the stress resultants $\bar{N}_i$, $\bar{M}_i$, $\bar{P}_i$, Q_i and R_i are defined by

$$\left.\begin{aligned} &(\bar{N}_i,\ \bar{M}_i,\ \bar{P}_i) = \int_{-t/2}^{t/2} \sigma_i(1,\ Z,\ Z^3)dZ \qquad (i=1,\ 2,\ 6) \\ &(Q_2,\ R_2) = \int_{-t/2}^{t/2} \sigma_4(1, Z^2)dZ,\ \ (Q_1, R_1) = \int_{-t/2}^{t/2} \sigma_5(1, Z^2)dZ \end{aligned}\right\} \tag{2.6a∼c}$$

and A_{ij}, B_{ij} etc., are the plate stiffnesses, defined by

$$\left.\begin{aligned} &(A_{ij},\ B_{ij},\ D_{ij},\ E_{ij},\ F_{ij},\ H_{ij}) \\ &\quad = \sum_{k=1} \int_{t_{k-1}}^{t_k} (\bar{Q}_{ij})_k (1,\ Z,\ Z^2,\ Z^3,\ Z^4,\ Z^6)dZ \qquad (i,\ j=1,\ 2,\ 6) \\ &(A_{ij},\ D_{ij},\ F_{ij}) = \sum_{k=1} \int_{t_{k-1}}^{t_k} (\bar{Q}_{ij})_k (1,\ Z^2,\ Z^4)dZ \qquad (i, j=4,5) \end{aligned}\right\} \tag{2.7a,b}$$

where $\bar{Q}_{ij}$ are the transformed elastic constants, defined by

$$\begin{bmatrix} \bar{Q}_{11} \\ \bar{Q}_{12} \\ \bar{Q}_{22} \\ \bar{Q}_{16} \\ \bar{Q}_{26} \\ \bar{Q}_{66} \end{bmatrix} = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2 \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ c^3s & cs^3-c^3s & -cs^3 & -2cs(c^2-s^2) \\ cs^3 & c^3s-cs^3 & -c^3s & 2cs(c^2-s^2) \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2 \end{bmatrix} \begin{bmatrix} Q_{11} \\ Q_{12} \\ Q_{22} \\ Q_{66} \end{bmatrix} \tag{2.8a}$$

and

$$\begin{Bmatrix} \bar{Q}_{44} \\ \bar{Q}_{45} \\ \bar{Q}_{55} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 \\ -cs & cs \\ s^2 & c^2 \end{bmatrix} \begin{Bmatrix} Q_{44} \\ Q_{55} \end{Bmatrix} \tag{2.8b}$$

where

$$\left.\begin{aligned} &Q_{11} = \frac{E_{11}}{(1-\nu_{12}\nu_{21})},\quad Q_{22} = \frac{E_{22}}{(1-\nu_{12}\nu_{21})},\quad Q_{12} = \frac{\nu_{21}E_{11}}{(1-\nu_{12}\nu_{21})} \\ &Q_{44} = G_{23},\quad Q_{55} = G_{13},\quad Q_{66} = G_{12} \end{aligned}\right\} \tag{2.8c}$$

and

$$c = \cos\theta,\quad s = \sin\theta \tag{2.8d}$$

where $\theta =$ lamination angle with respect to the plate X-axis.

In Eq. (2.5a) the thermal forces, moments and higher-order moments are defined by

$$\begin{bmatrix} N_x^T, & M_x^T, & P_x^T \\ N_y^T, & M_y^T, & P_y^T \\ N_{xy}^T, & M_{xy}^T, & P_{xy}^T \end{bmatrix} = \sum_{k=1} \int_{t_{k-1}}^{t_k} (1, Z, Z^3) \begin{bmatrix} A_x \\ A_y \\ A_{xy} \end{bmatrix}_k T(X,Y,Z)dZ \tag{2.9a}$$

and

$$\begin{bmatrix} S_x^T \\ S_y^T \\ S_{xy}^T \end{bmatrix} = \begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} - \frac{4}{3t^2} \begin{bmatrix} P_x^T \\ P_y^T \\ P_{xy}^T \end{bmatrix} \tag{2.9b}$$

in Eq. (2.9a)

$$\begin{bmatrix} A_x \\ A_y \\ A_{xy} \end{bmatrix} = - \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 \\ s^2 & c^2 \\ 2cs & -2cs \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \end{bmatrix} \tag{2.10}$$

in which α_{11} and α_{22} are thermal expansion coefficients for a single ply.

Eq. (2.5a) can be written in the alternative form

$$\begin{bmatrix} \boldsymbol{\epsilon}^0 \\ \mathbf{M}^* \\ \mathbf{P}^* \end{bmatrix} = \begin{bmatrix} \mathbf{A}^* & \mathbf{B}^* & \mathbf{E}^* \\ -(\mathbf{B}^*)^T & \mathbf{D}^* & (\mathbf{F}^*)^T \\ -(\mathbf{E}^*)^T & \mathbf{F}^* & \mathbf{H}^* \end{bmatrix} \begin{bmatrix} \mathbf{N}^* \\ \boldsymbol{\kappa}^0 \\ \boldsymbol{\kappa}^2 \end{bmatrix} \tag{2.11}$$

where

$$\mathbf{N}^* = \bar{\mathbf{N}} - \bar{\mathbf{N}}^T, \quad \mathbf{M}^* = \bar{\mathbf{M}} - \bar{\mathbf{M}}^T, \quad \mathbf{P}^* = \bar{\mathbf{P}} - \bar{\mathbf{P}}^T \tag{2.12a}$$

and the reduced stiffness matrices are defined by

$$\left.\begin{aligned} &\mathbf{A}^* = \mathbf{A}^{-1}, \quad \mathbf{B}^* = -\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{D}^* = \mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{E}^* = -\mathbf{A}^{-1}\mathbf{E}, \\ &\mathbf{F}^* = \mathbf{F} - \mathbf{E}\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{H}^* = \mathbf{H} - \mathbf{E}\mathbf{A}^{-1}\mathbf{E} \end{aligned}\right\} \tag{2.12b}$$

It is noted that, whereas $\mathbf{A}^*$, $\mathbf{D}^*$ and $\mathbf{H}^*$ are symmetric matrices, $\mathbf{B}^*$, $\mathbf{E}^*$ and $\mathbf{F}^*$ are not necessarily so.

Substituting Eqs. (2.1) and (2.11) into equilibrium equations

$$\left.\begin{aligned} &\frac{\partial N_1}{\partial X} + \frac{\partial N_6}{\partial Y} = 0, \quad \frac{\partial N_6}{\partial X} + \frac{\partial N_2}{\partial Y} = 0 \\ &\frac{\partial Q_1}{\partial X} + \frac{\partial Q_2}{\partial Y} - \frac{4}{t^2}\left[\frac{\partial R_1}{\partial X} + \frac{\partial R_2}{\partial Y}\right] + \frac{4}{3t^2}\left[\frac{\partial^2 P_1}{\partial X^2} + 2\frac{\partial^2 P_6}{\partial X \partial Y} + \frac{\partial^2 P_2}{\partial Y^2}\right] \\ &\quad + \frac{\partial}{\partial X}\left[N_1 \frac{\partial}{\partial X}(\bar{W} + \bar{W}^*) + N_6 \frac{\partial}{\partial Y}(\bar{W} + \bar{W}^*)\right] \\ &\quad + \frac{\partial}{\partial Y}\left[N_6 \frac{\partial}{\partial X}(\bar{W} + \bar{W}^*) + N_2 \frac{\partial}{\partial Y}(\bar{W} + \bar{W}^*)\right] + q = 0 \\ &\frac{\partial M_1}{\partial X} + \frac{\partial M_6}{\partial Y} - Q_1 + \frac{4}{t^2} R_1 - \frac{4}{3t^2}\left[\frac{\partial P_1}{\partial X} + \frac{\partial P_6}{\partial Y}\right] = 0 \\ &\frac{\partial M_6}{\partial X} + \frac{\partial M_2}{\partial Y} - Q_2 + \frac{4}{t^2} R_2 - \frac{4}{3t^2}\left[\frac{\partial P_6}{\partial X} + \frac{\partial P_2}{\partial Y}\right] = 0 \end{aligned}\right\} \tag{2.13a$\sim$e}$$

In addition, use is made of the compatibility relationship

$$\frac{\partial^2\epsilon_1^0}{\partial Y^2}+\frac{\partial^2\epsilon_2^0}{\partial X^2}-\frac{\partial^2\epsilon_6^0}{\partial X\partial Y}=\left[\frac{\partial^2\overline{W}}{\partial X\partial Y}\right]^2-\frac{\partial^2\overline{W}}{\partial X^2}\frac{\partial^2\overline{W}}{\partial Y^2}+2\left[\frac{\partial^2\overline{W}^*}{\partial X\partial Y}\right]^2-\frac{\partial^2\overline{W}}{\partial X^2}\frac{\partial^2\overline{W}^*}{\partial Y^2}$$
$$-\frac{\partial^2\overline{W}^*}{\partial X^2}\frac{\partial^2\overline{W}}{\partial Y^2} \tag{2.14}$$

then Kármán-type nonlinear large deflection equations can be written as

$$L_{11}(\overline{W})-L_{12}(\overline{\Psi}_x)-L_{1\,}(\overline{\Psi}_y)+L_{14}(\overline{F})-L_{15}(N^T)-L_{16}(M^T)$$
$$=L(\overline{W}+\overline{W}^*,\ \overline{F})+q \tag{2.15}$$
$$L_{21}(\overline{F})+L_{22}(\overline{\Psi}_x)+L_{23}(\overline{\Psi}_y)-L_{24}(\overline{W})-L_{25}(N^T)=-L(\overline{W}+2\overline{W}^*,\ \overline{W})/2 \tag{2.16}$$
$$L_{31}(\overline{W})+L_{\ 2}(\overline{\Psi}_x)+L_{33}(\overline{\Psi}_y)+L_{\ 4}(\overline{F})-L_{\ 5}(N^T)-L_{\ 6}(S^T)=0 \tag{2.17}$$
$$L_{41}(\overline{W})+L_{42}(\overline{\Psi}_x)+L_{43}(\overline{\Psi}_y)+L_{44}(\overline{F})-L_{45}(N^T)-L_{46}(S^T)=0 \tag{2.18}$$

in which the operators are as follow:

$$L_{11}(\quad)=\frac{4}{3t^2}\left[F_{11}^*\frac{\partial^4}{\partial X^4}+2(F_{16}^*+F_{61}^*)\frac{\partial^4}{\partial X^3\partial Y}+(F_{12}^*+F_{21}^*+4F_{66}^*)\frac{\partial^4}{\partial X^2\partial Y^2}\right.$$
$$\left.+2(F_{26}^*+F_{62}^*)\frac{\partial^4}{\partial X\partial Y^3}+F_{22}^*\frac{\partial^4}{\partial Y^4}\right]$$

$$L_{12}(\quad)=\left[D_{11}^*-\frac{4}{3t^2}F_{11}^*\right]\frac{\partial^3}{\partial X^3}+\left[3D_{16}^*-\frac{4}{3t^2}(F_{61}^*+2F_{16}^*)\right]\frac{\partial^3}{\partial X^2\partial Y}$$
$$+\left[(D_{12}^*+2D_{66}^*)-\frac{4}{3t^2}(F_{12}^*+2F_{66}^*)\right]\frac{\partial^3}{\partial X\partial Y^2}+\left[D_{26}^*-\frac{4}{3t^2}F_{26}^*\right]\frac{\partial^3}{\partial Y^3}$$

$$L_{13}(\quad)=\left[D_{16}^*-\frac{4}{3t^2}F_{16}^*\right]\frac{\partial^3}{\partial X^3}+\left[(D_{12}^*+2D_{66}^*)-\frac{4}{3t^2}(F_{21}^*+2F_{66}^*)\right]\frac{\partial^3}{\partial X^2\partial Y}$$
$$+\left[3D_{26}^*-\frac{4}{3t^2}(F_{62}^*+2F_{26}^*)\right]\frac{\partial^3}{\partial X^2\partial Y}+\left[D_{22}^*-\frac{4}{3t^2}F_{22}^*\right]\frac{\partial^3}{\partial Y^3}$$

$$L_{14}(\quad)=B_{21}^*\frac{\partial^4}{\partial X^4}+(2B_{26}^*-B_{61}^*)\frac{\partial^4}{\partial X^3\partial Y}+(B_{11}^*+B_{22}^*-2B_{66}^*)\frac{\partial^4}{\partial X^2\partial Y^2}$$
$$+(2B_{16}^*-B_{62}^*)\frac{\partial^4}{\partial X\partial Y^3}+B_{12}^*\frac{\partial^4}{\partial Y^4}$$

$$L_{15}(N^T)=\frac{\partial^2}{\partial X^2}(B_{11}^*N_x^T+B_{21}^*N_y^T+B_{61}^*N_{xy}^T)+2\frac{\partial^2}{\partial X\partial Y}(B_{16}^*N_x^T+B_{26}^*N_y^T+B_{66}^*N_{xy}^T)$$
$$+\frac{\partial^2}{\partial Y^2}(B_{12}^*N_x^T+B_{22}^*N_y^T+B_{62}^*N_{xy}^T)$$

$$L_{16}(M^T)=\frac{\partial^2}{\partial X^2}(M_x^T)+2\frac{\partial^2}{\partial X\partial Y}(M_{xy}^T)+\frac{\partial^2}{\partial Y^2}(M_y^T)$$

$$L_{21}(\quad)=A_{22}^*\frac{\partial^4}{\partial X^4}-2A_{26}^*\frac{\partial^4}{\partial X^3\partial Y}+(2A_{12}^*+A_{66}^*)\frac{\partial^4}{\partial X^2\partial Y^2}-2A_{16}^*\frac{\partial^4}{\partial X\partial Y^3}$$
$$+A_{11}^*\frac{\partial^4}{\partial Y^4}$$

$$L_{22}(\quad)=\left[B_{21}^{*}-\frac{4}{3t^{2}}E_{21}^{*}\right]\frac{\partial^{3}}{\partial X^{3}}+\left[(B_{26}^{*}-B_{61}^{*})-\frac{4}{3t^{2}}(E_{26}^{*}-E_{61}^{*})\right]\frac{\partial^{3}}{\partial X^{2}\partial Y}$$

$$+\left[(B_{11}^{*}-B_{66}^{*})-\frac{4}{3t^{2}}(E_{11}^{*}-E_{66}^{*})\right]\frac{\partial^{3}}{\partial X\partial Y^{2}}+\left[B_{16}^{*}-\frac{4}{3t^{2}}E_{16}^{*}\right]\frac{\partial^{3}}{\partial Y^{3}}$$

$$L_{23}(\quad)=\left[B_{26}^{*}-\frac{4}{3t^{2}}E_{26}^{*}\right]\frac{\partial^{3}}{\partial X^{3}}+\left[(B_{22}^{*}-B_{66}^{*})-\frac{4}{3t^{2}}(E_{22}^{*}-E_{66}^{*})\right]\frac{\partial^{3}}{\partial X^{2}\partial Y}$$

$$+\left[(B_{16}^{*}-B_{62}^{*})-\frac{4}{3t^{2}}(E_{16}^{*}-E_{62}^{*})\right]\frac{\partial^{3}}{\partial X\partial Y^{2}}+\left[B_{12}^{*}-\frac{4}{3t^{2}}E_{12}^{*}\right]\frac{\partial^{3}}{\partial Y^{3}}$$

$$L_{24}(\quad)=\frac{4}{3t^{2}}\left[E_{21}^{*}\frac{\partial^{4}}{\partial X^{4}}+(2E_{26}^{*}-E_{61}^{*})\frac{\partial^{4}}{\partial X^{3}\partial Y}+(E_{11}^{*}+E_{22}^{*}-2E_{66}^{*})\frac{\partial^{4}}{\partial X^{2}\partial Y^{2}}\right.$$

$$\left.+(2E_{16}^{*}-E_{62}^{*})\frac{\partial^{4}}{\partial X\partial Y^{3}}+E_{12}^{*}\frac{\partial^{4}}{\partial Y^{4}}\right]$$

$$L_{25}(N^{T})=\frac{\partial^{2}}{\partial X^{2}}(A_{12}^{*}N_{x}^{T}+A_{22}^{*}N_{y}^{T}+A_{26}^{*}N_{xy}^{T})-\frac{\partial^{2}}{\partial X\partial Y}(A_{16}^{*}N_{x}^{T}+A_{26}^{*}N_{y}^{T}$$

$$+A_{66}^{*}N_{xy}^{T})+\frac{\partial^{2}}{\partial Y^{2}}(A_{11}^{*}N_{x}^{T}+A_{12}^{*}N_{y}^{T}+A_{16}^{*}N_{xy}^{T})$$

$$L_{31}(\quad)=\left[A_{55}-\frac{8}{t^{2}}D_{55}+\frac{16}{t^{4}}F_{55}\right]\frac{\partial}{\partial X}+\left[A_{45}-\frac{8}{t^{2}}D_{45}+\frac{16}{t^{4}}F_{45}\right]\frac{\partial}{\partial Y}$$

$$+\frac{4}{3t^{2}}\left[(F_{11}^{*}-\frac{4}{3t^{2}}H_{11}^{*})\frac{\partial^{3}}{\partial X^{3}}+\left[(F_{16}^{*}+2F_{61}^{*})-\frac{4}{t^{2}}H_{16}^{*}\right]\frac{\partial^{3}}{\partial X^{2}\partial Y}\right.$$

$$+\left[(F_{21}^{*}+2F_{66}^{*})-\frac{4}{3t^{2}}(H_{12}^{*}+2H_{66}^{*})\right]\frac{\partial^{3}}{\partial X\partial Y^{2}}$$

$$\left.+(F_{26}^{*}-\frac{4}{3t^{2}}H_{26}^{*})\frac{\partial^{3}}{\partial Y^{3}}\right]$$

$$L_{32}(\quad)=\left[A_{55}-\frac{8}{t^{2}}D_{55}+\frac{16}{t^{4}}F_{55}\right]-\left[D_{11}^{*}-\frac{8}{3t^{2}}F_{11}^{*}+\frac{16}{9t^{4}}H_{11}^{*}\right]\frac{\partial^{2}}{\partial X^{2}}$$

$$-2\left[D_{16}^{*}-\frac{4}{3t^{2}}(F_{16}^{*}+F_{61}^{*})+\frac{16}{9t^{4}}H_{16}^{*}\right]\frac{\partial^{2}}{\partial X\partial Y}$$

$$-\left[D_{66}^{*}-\frac{8}{3t^{2}}F_{66}^{*}+\frac{16}{9t^{4}}H_{66}^{*}\right]\frac{\partial^{2}}{\partial Y^{2}}$$

$$L_{33}(\quad)=\left[A_{45}-\frac{8}{t^{2}}D_{45}+\frac{16}{t^{4}}F_{45}\right]-\left[D_{16}^{*}-\frac{4}{3t^{2}}(F_{16}^{*}+F_{61}^{*})+\frac{16}{9t^{4}}H_{16}^{*}\right]\frac{\partial^{2}}{\partial X^{2}}$$

$$-\left[(D_{12}^{*}+D_{66}^{*})-\frac{4}{3t^{2}}(F_{12}^{*}+F_{21}^{*}+2F_{66}^{*})+\frac{16}{9t^{4}}(H_{12}^{*}+H_{66}^{*})\right]\frac{\partial^{2}}{\partial X\partial Y}$$

$$-\left[D_{26}^{*}-\frac{4}{3t^{2}}(F_{26}^{*}+F_{62}^{*})+\frac{16}{9t^{4}}H_{26}^{*}\right]\frac{\partial^{2}}{\partial Y^{2}}$$

$$L_{34}(\quad)=L_{22}(\quad)$$

$$L_{35}(N^{T})=\frac{\partial}{\partial X}\left[(B_{11}^{*}-\frac{4}{3t^{2}}E_{11}^{*})N_{x}^{T}+(B_{21}^{*}-\frac{4}{3t^{2}}E_{21}^{*})N_{y}^{T}+(B_{61}^{*}\right.$$

$$-\frac{4}{3t^2}E^*_{61})N^T_{xy}\Big]+\frac{\partial}{\partial Y}\Big[(B^*_{16}-\frac{4}{3t^2}E^*_{16})N^T_x+(B^*_{26}-\frac{4}{3t^2}E^*_{26})N^T_y$$

$$+(B^*_{66}-\frac{4}{3t^2}E^*_{66})N^T_{xy}\Big]$$

$$L_{36}(S^T)=\frac{\partial}{\partial X}(S^T_x)+\frac{\partial}{\partial Y}(S^T_{xy})$$

$$L_{41}(\ )=\Big[A_{45}-\frac{8}{t^2}D_{45}+\frac{16}{t^4}F_{45}\Big]\frac{\partial}{\partial X}+\Big[A_{44}-\frac{8}{t^2}D_{44}+\frac{16}{t^4}F_{44}\Big]\frac{\partial}{\partial Y}$$

$$+\frac{4}{3t^2}\Big[(F^*_{16}-\frac{4}{3t^2}H^*_{16})\frac{\partial^3}{\partial X^3}+\Big[(F^*_{12}+2F^*_{66})$$

$$-\frac{4}{3t^2}(H^*_{12}+2H^*_{66})\Big]\frac{\partial^3}{\partial X^2\partial Y}+\Big[(F^*_{26}+2F^*_{62})-\frac{4}{t^2}H^*_{26}\Big]\frac{\partial^3}{\partial X\partial Y^2}$$

$$+(F^*_{22}-\frac{4}{3t^2}H^*_{22})\frac{\partial^3}{\partial Y^3}\Big]$$

$$L_{42}(\ )=L_{35}(\ )$$

$$L_{43}(\ )=\Big[A_{44}-\frac{8}{t^2}D_{44}+\frac{16}{t^4}F_{44}\Big]-\Big[D^*_{66}-\frac{8}{3t^2}F^*_{66}+\frac{16}{9t^4}H^*_{66}\Big]\frac{\partial^2}{\partial X^2}$$

$$-2\Big[D^*_{26}-\frac{4}{3t^2}(F^*_{26}+F^*_{62})+\frac{16}{9t^4}H^*_{26}\Big]\frac{\partial^2}{\partial X\partial Y}$$

$$-\Big[D^*_{22}-\frac{8}{3t^2}F^*_{22}+\frac{16}{9t^4}H^*_{22}\Big]\frac{\partial^2}{\partial Y^2}$$

$$L_{44}(\ )=L_{2}(\ )$$

$$L_{45}(N^T)=\frac{\partial}{\partial X}\Big[(B^*_{16}-\frac{4}{3t^2}E^*_{16})N^T_x+(B^*_{26}-\frac{4}{3t^2}E^*_{26})N^T_y$$

$$+(B^*_{66}-\frac{4}{3t^2}E^*_{66})N^T_{xy}\Big]+\frac{\partial}{\partial Y}\Big[(B^*_{12}-\frac{4}{3t^2}E^*_{12})N^T_x$$

$$+(B^*_{22}-\frac{4}{3t^2}E^*_{22})N^T_y+(B^*_{62}-\frac{4}{3t^2}E^*_{62})N^T_{xy}\Big]$$

$$L_{46}(S^T)=\frac{\partial}{\partial X}(S^T_{xy})+\frac{\partial}{\partial Y}(S^T_y)$$

$$L(\ )=\frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial Y^2}-2\frac{\partial^2}{\partial X\partial Y}\frac{\partial^2}{\partial X\partial Y}+\frac{\partial^2}{\partial Y^2}\frac{\partial^2}{\partial X^2} \tag{2.19}$$

It is noted that these plate equations show thermal coupling as well as the interactive of stretching and bending.

III. Thermal Postbuckling of Symmetric Cross-ply Plates

Here we consider the thermal postbuckling of simply supported, symmetric cross-ply shear-deformable laminated plates subjected to uniform or nonuniform parabolic temperature loading. For such plates the following plate stiffnesses are identically zero:

$$\left.\begin{aligned}&B_{ij}=E_{ij}=0,\ A_{16}=A_{26}=D_{16}=D_{26}=F_{16}=F_{26}=H_{16}=H_{26}=0\\&A_{45}=D_{45}=F_{45}=0\end{aligned}\right\}\tag{3.1}$$

and the out-of plane loads q is taken to be zero.

The in-plane temperature variation is assumed as

$$(X,\ Y,\ Z)=T_0+T_1[1-((2Y-b)/b)^2]\tag{3.2}$$

In Eq. (3.2) when $T_1=0$, it denotes the uniform temperature field, and when $T_0=0$, it denotes the parabolic temperature distribution. From Eqs. (2.9a) and (3.2) it is noted that the thermal force N^T_{xy}, the thermal moments M^T_x, M^T_y, M^T_{xy} and the higher-order moments P^T_x, P^T_y and P^T_{xy} are zero.

Let the thermal expansion coefficients for each ply be

$$\alpha_{11}=a_{11}\alpha_0,\quad \alpha_{22}=a_{22}\alpha_0\tag{3.3}$$

where α_0 is an arbitrary reference value, and let

$$\begin{bmatrix}A^T_x\\A^T_y\end{bmatrix}=-\sum_{k=1}\int_{t_{k-1}}^{t_k}\begin{bmatrix}A_x\\A_y\end{bmatrix}_k dZ\tag{3.4}$$

Let $\lambda_T=\alpha_0T_i$, where $i=0$ for a uniform temperature distribution and $i=1$ otherwise, i. e. $T_1\neq0$. Then introducing the dimensionless quantities

$$\left.\begin{aligned}
&x=\pi X/a,\ Y=\pi Y/b,\ \beta=a/b,\ (W^*,\ W)=(\overline{W}^*,\ \overline{W})/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&F=\overline{F}/[D^*_{11}D^*_{22}]^{1/2},\ (\Psi_x,\ \Psi_y)=(\overline{\Psi}_x,\ \overline{\Psi}_y)a/\pi[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&\gamma_{14}=[D^*_{22}/D^*_{11}]^{1/2},\ \gamma_{24}=[A^*_{11}/A^*_{22}]^{1/2},\ \gamma_5=-A^*_{12}/A^*_{22}\\
&(\gamma_{110},\ \gamma_{112},\ \gamma_{114})=(4/3t^2)[F^*_{11},\ (F^*_{12}+F^*_{21}+4F^*_{66})/2,\ F^*_{22}]/D^*_{11}\\
&(\gamma_{120},\ \gamma_{122})=[D^*_{11}-4F^*_{11}/3t^2,\ (D^*_{12}+2D^*_{66})-4(F^*_{12}+2F^*_{66})/3t^2]/D^*_{11}\\
&(\gamma_{131},\ \gamma_{133})=[(D^*_{12}+2D^*_{66})-4(F^*_{21}+2F^*_{66})/3t^2,\ D^*_{22}-4F^*_{22}/3t^2]/D^*_{11}\\
&(\gamma_{212},\ \gamma_{214})=(A^*_{12}+A^*_{66}/2,\ A^*_{11})/A^*_{22}\\
&(\gamma_{31},\ \gamma_{41})=(a^2/\pi^2)(A_{55}-8D_{55}/t^2+16F_{55}/t^4,\ A_{44}-8D_{44}/t^2+16F_{44}/t^4)/D^*_{11}\\
&(\gamma_{310},\ \gamma_{312})=(4/3t^2)[F^*_{11}-4H^*_{11}/3t^2,\ (F^*_{21}+2F^*_{66})-4(H^*_{12}+2H^*_{66})/3t^2]/D^*_{11}\\
&(\gamma_{320},\ \gamma_{322})=(D^*_{11}-8F^*_{11}/3t^2+16H^*_{11}/9t^4,\ D^*_{66}-8F^*_{66}/3t^2+16H^*_{66}/9t^4)/D^*_{11}\\
&\gamma_{331}=[(D^*_{12}+D^*_{66})-4(F^*_{12}+F^*_{21}+2F^*_{66})/3t^2+16(H^*_{12}+H^*_{66})/9t^4]/D^*_{11}\\
&(\gamma_{411},\ \gamma_{413})=(4/3t^2)[(F^*_{12}+2F^*_{66})-4(H^*_{12}+2H^*_{66})/3t^2,\ F^*_{22}-4H^*_{22}/3t^2]/D^*_{11}\\
&(\gamma_{430},\ \gamma_{432})=(D^*_{66}-8F^*_{66}/3t^2+16H^*_{66}/9t^4,\ D^*_{22}-8F^*_{22}/3t^2+16H^*_{22}/9t^4)/D^*_{11}\\
&(\gamma_{T1},\ \gamma_{T2})=(A^T_x,\ A^T_y)a^2/\alpha_0\pi^2[D^*_{11}D^*_{22}]^{1/2}\\
&(M_x,\ M_y,\ P_x,\ P_y)=(\overline{M}_x,\ \overline{M}_y,\ 4\overline{P}_x/3t^2,\ 4\overline{P}_y/3t^2)a^2/\pi^2D^*_{11}[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&(\delta_x,\ \delta_y)=(\Delta_x/a,\ \Delta_y/b)b^2/4\pi^2[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/2}
\end{aligned}\right\}\tag{3.5}$$

enables Eqs. (2.15)~(2.18) to be written in dimensionless form as

$$L_{11}(W)-L_{12}(\Psi_x)-L_{13}(\Psi_y)=\gamma_{14}\beta^2L(W+W^*,\ F)\tag{3.6}$$

$$L_{21}(F)-C_1=-\gamma_{24}\beta^2L(W+2W^*,\ W)/2\tag{3.7}$$

$$L_{31}(W)+L_{32}(\Psi_x)-L_{33}(\Psi_y)=0 \tag{3.8}$$

$$L_{41}(W)-L_{42}(\Psi_x)+L_{43}(\Psi_y)=0 \tag{3.9}$$

where

$$L_{11}(\ )=\gamma_{110}\frac{\partial^4}{\partial x^4}+2\gamma_{112}\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\gamma_{114}\beta^4\frac{\partial^4}{\partial y^4}$$

$$L_{12}(\ )=\gamma_{120}\frac{\partial^3}{\partial x^3}+\gamma_{122}\beta^2\frac{\partial^3}{\partial x\partial y^2}$$

$$L_{13}(\ )=\gamma_{131}\beta\frac{\partial^3}{\partial x^2\partial y}+\gamma_{133}\beta^3\frac{\partial^3}{\partial y^3}$$

$$L_{21}(\ )=\frac{\partial^4}{\partial x^4}+2\gamma_{212}\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\gamma_{214}\beta^4\frac{\partial^4}{\partial y^4}$$

$$L_{31}(\ )=\gamma_{31}\frac{\partial}{\partial x}+\gamma_{310}\frac{\partial^3}{\partial x^3}+\gamma_{312}\beta^2\frac{\partial^3}{\partial x\partial y^2}$$

$$L_{32}(\ )=\gamma_{31}-\gamma_{320}\frac{\partial^2}{\partial x^2}-\gamma_{322}\beta^2\frac{\partial^2}{\partial y^2},\quad L_{33}(\ )=\gamma_{331}\beta\frac{\partial^2}{\partial x\partial y}$$

$$L_{41}(\ )=\gamma_{41}\beta\frac{\partial}{\partial y}+\gamma_{411}\beta\frac{\partial^3}{\partial x^2\partial y}+\gamma_{413}\beta^3\frac{\partial^3}{\partial y^3},\quad L_{42}(\ )=L_{33}(\ )$$

$$L_{43}(\ )=\gamma_{41}-\gamma_{430}\frac{\partial^2}{\partial x^2}-\gamma_{432}\beta^2\frac{\partial^2}{\partial y^2}$$

$$L(\ )=\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}-2\frac{\partial^2}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}$$

All the edges are assumed to be simply supported and to be restrained against expansion in the in-plane directions, so the boundary conditions are

$x=0,\ \pi$:

$$\delta_x=W=\Psi_y=0,\quad F_{,xy}=M_x=P_x=0 \tag{3.10a,b}$$

$y=0,\ \pi$:

$$\delta_y=W=\Psi_x=0,\quad F_{,xy}=M_y=P_y=0 \tag{3.10c,d}$$

the unit end-shortening relationships are

$$\delta_x=-\frac{1}{4\pi^2\beta^2\gamma_{24}}\int_0^\pi\int_0^\pi\left\{\left[\gamma_{24}^2\beta^2\frac{\partial^2 F}{\partial y^2}-\gamma_5\frac{\partial^2 F}{\partial x^2}\right]-\frac{1}{2}\gamma_{24}\left[\frac{\partial W}{\partial x}\right]^2-\gamma_{24}\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\right.$$
$$\left.+(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})\lambda_T C_2\right\}dxdy \tag{3.11a}$$

$$\delta_y=-\frac{1}{4\pi^2\beta^2\gamma_{24}}\int_0^\pi\int_0^\pi\left\{\left[\frac{\partial^2 F}{\partial x^2}-\gamma_5\beta^2\frac{\partial^2 F}{\partial y^2}\right]-\frac{1}{2}\gamma_{24}\beta^2\left[\frac{\partial W}{\partial y}\right]^2-\gamma_{24}\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\right.$$
$$\left.+(\gamma_{T2}-\gamma_5\gamma_{T1})\lambda_T C_2\right\}dxdy \tag{3.11b}$$

Note that in Eqs. (3.7) and (3.11), for the uniform thermal loading case, $C_1=0.0$, $C_2=1.0$ and $\lambda_T=\alpha_0 T_0$, whereas for the nonuniform parabolic thermal loading case, $C_1=8\beta^2\lambda_T(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})/\pi^2$, $C_2=[T_0/T_1+4(y/\pi-y^2/\pi^2)]$ and $\lambda_T=\alpha_0 T_1$.

Applying Eqs. (3.6)~(3.11), the thermal postbuckling behavior of a simply supported, symmetric cross-ply laminated plate is now determined by a mixed Galerkin-perturbation technique. The essence of this procedure, in the present case, is to assume that

$$\left.\begin{aligned}&W(x,\ y,\ \varepsilon)=\sum_{j=1}\varepsilon^j w_j(x,\ y),\ F(x,\ y,\ \varepsilon)=\sum_{j=0}\varepsilon^j f_j(x,\ y)\\&\Psi_x(x,\ y,\ \varepsilon)=\sum_{j=1}\varepsilon^j\psi_{xj}(x,\ y),\ \Psi_y(x,\ y,\ \varepsilon)=\sum_{j=1}\varepsilon^j\psi_{yj}(x,\ y)\end{aligned}\right\}\qquad(3.12\text{a}\sim\text{d})$$

where ε is a small perturbation parameter and tne first term of $w_j(x,\ y)$ is assumed to have the form

$$w_1(x,\ y)=A_{11}^{(1)}\sin mx\sin ny\qquad(3.13)$$

The initial geometric imperfection is assumed to have a similar form

$$W^*(x,\ y,\ \varepsilon)=\varepsilon a_{11}^*\sin mx\sin ny=\varepsilon\mu A_{11}^{(1)}\sin mx\sin ny\qquad(3.14)$$

where $\mu=a_{11}^*/A_{11}^{(1)}$ is the imperfection parameter.

Substituting Eq. (3.12) into Eqs. (3.6)~(3.9) gives a system of perturbation equations which can be solved step by step. At each step the amplitudes of the terms $w_j(x,\ y)$, $f_j(x,\ y)$, $\psi_{xj}(x,\ y)$ and $\psi_{yj}(x,\ y)$ can be determined by the Galerkin procedure. As a result, the asymptotic solutions can be obtained as

$$W=\varepsilon[A_{11}^{(1)}\sin mx\sin ny]+\varepsilon^3[A_{13}^{(3)}\sin mx\sin 3ny+A_{31}^{(3)}\sin 3mx\sin ny]+O(\varepsilon^5)\qquad(3.15)$$

$$\begin{aligned}F=&-B_{00}^{(0)}\left(\frac{y^2}{2}-C_4\frac{y^4}{24}\right)-b_{00}^{(0)}\frac{x^2}{2}+\varepsilon^2\left[-B_{00}^{(2)}\left(\frac{y^2}{2}-C_4\frac{y^4}{24}\right)-b_{00}^{(2)}\frac{x^2}{2}+B_{20}^{(2)}\cos 2mx\right.\\&\left.+B_{02}^{(2)}\cos 2ny\right]+\varepsilon^4\left[-B_{00}^{(4)}\left(\frac{y^2}{2}-C_4\frac{y^4}{24}\right)-b_{00}^{(4)}\frac{x^2}{2}+B_{20}^{(4)}\cos 2mx+B_{02}^{(4)}\cos 2ny\right.\\&+B_{22}^{(4)}\cos 2mx\cos 2ny+B_{40}^{(4)}\cos 4mx+B_{04}^{(4)}\cos 4ny+B_{24}^{(4)}\cos 2mx\cos 4ny\\&\left.+B_{42}^{(4)}\cos 4mx\cos 2ny\right]+O(\varepsilon^5)\end{aligned}\qquad(3.16)$$

$$\Psi_x=\varepsilon[C_{11}^{(1)}\cos mx\sin ny]+\varepsilon^3[C_{13}^{(3)}\cos mx\sin 3ny+C_{31}^{(3)}\cos 3mx\sin ny]+O(\varepsilon^5)\qquad(3.17)$$

$$\Psi_y=\varepsilon[D_{11}^{(1)}\sin mx\cos ny]+\varepsilon^3[D_{13}^{(3)}\sin mx\cos 3ny+D_{31}^{(3)}\sin 3mx\cos ny]+O(\varepsilon^5)\qquad(3.18)$$

Note that, for the uniform thermal loading case it is just necessary to take $C_4=0$ in Eq. (3.16), so that the asymptotic solutions have a similar form.

All the coefficients in Eqs. (3.15)~(3.18) are related and can be written as functions of $A_{11}^{(1)}$ as has being illustrated in the Appendix.

Next, substituting Eqs. (3.15) and (3.16) into boundary conditions $\delta_x=0$ and $\delta_y=0$, the thermal postbuckling equilibrium path can be written as

$$\lambda_T=\lambda_T^{(0)}+\lambda_T^{(2)}W_m^2+\lambda_T^{(4)}W_m^4+\cdots\qquad(3.19)$$

in which W_m is the dimensionless form of the maximum deflection of the plate, which is assumed to be at the point $(x,\ y)=(\pi/2m,\ \pi/2n)$ and $\lambda_T^{(0)}$, $\lambda_T^{(2)}$ and $\lambda_T^{(4)}$ are given in detail in the Appendix.

Eq. (3.19) can be employed to obtain numerical results for the thermal postbuckling load-deflection curves of symmetric cross-ply laminated plates under a uniform or nonuniform parabolic temperature distribution, from which results for isotropic and orthotropic thick plates follow as a limiting case, From the Appendix, the solution for the initial thermal buckling load for a perfect plate is exact and can readily be obtained numerically, by setting $\mu=0$ (or $\bar{W}^*/t=0$), while taking $W_m=0$ (or $\bar{W}/t=0$). In all the cases, the minimum initial buckling load is determined by applying Eq. (3.19) for values of various the buckling mode (m, n), i. e. for different numbers of half-waves in the X- and Y-directions, respectively.

IV. Numerical Examples and Discussion

Thermal postbuckling induced by uniform and nonuniform parabolic temperature distributions has been studied by a mixed Galerkin-perturbation method presented. A number of examples were solved to illustrate the performance of perfect and imperfect, symmetric cross-ply laminated plates. Typical results are presented in dimensionless graphical form in which $\lambda_T^*=12(\alpha_{11}+\nu_{21}\alpha_{22})b^2\lambda_T/\alpha_0\pi^2t^2$. For all of the examples all plies were of equal thickness and the material properties used were (except Table 1 and Fig. 1): $E_{11}/E_{22}=13.8957$, $G_{12}/E_{22}=G_{13}/E_{22}=0.4801$, $G_{23}/E_{22}=0.1838$, $\nu_{12}=0.33$, $\alpha_{11}/\alpha_0=0.139$ and $\alpha_{22}/\alpha_0=9.0$.

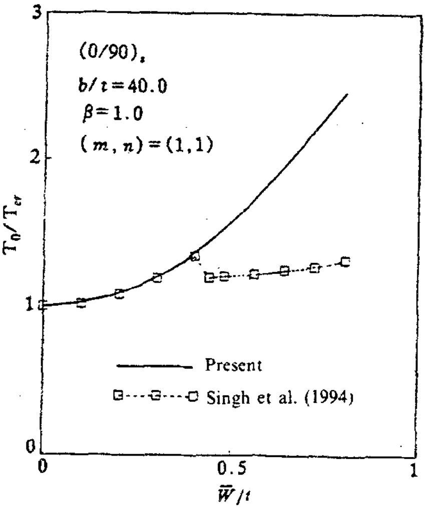


Fig. 1 Comparisons of thermal postbuckling load-deflection curves of (0/90)$_s$ square plate under a uniform temperature rise

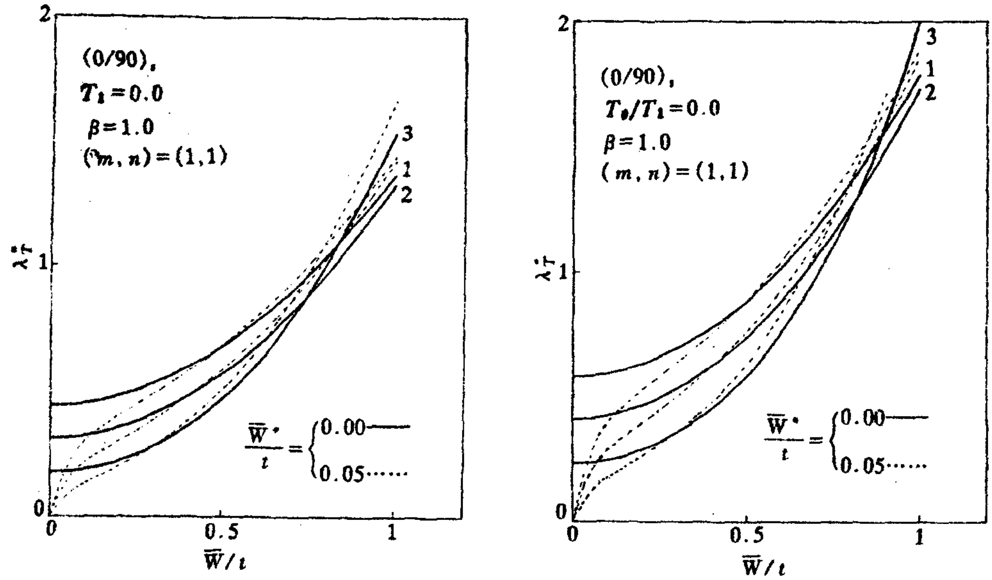
As part of the validation of the present method, the thermal buckling loads, $\bar{\lambda}_T=\alpha_0T_0\times10^3$ for perfect, simply supported, isotropic and 10-layer $(0)_{s}$ composite laminated square plate subjected to a uniform temperature rise with different thickness ratio are compared in Table 1 with results of 3-dimensional solutions given by Noor and Burton (1992)[4], using their material properties, i. e. $E_{11}/E_{22}=15$, $G_{12}/E_{22}=G_{13}/E_{22}=0.5$, $G_{23}/E_{22}=0.3356$, $\nu_{12}=0.3$, $\alpha_{11}/\alpha_0=0.015$ and $\alpha_{22}/\alpha_0=1.0$; and of classical plate theory (CPT) obtained from [2]. Clearly, the results obtained from the present method and the 3-dimensional elasticity theory are in good agreement, but CPT gives higher buckling temperature for moderately thick and thick plates. In addition, the thermal postbuckling load-deflection curves for perfect 4-ply $(0/90)_s$ symmetric cross-ply laminated square plates with $b/t=40$ subjected to uniform temperature rise are compared in Fig. 1 with results of Singh et al. (1994)[5], using their material properties, i. e. ($E_{11}/E_{22}=25$, $G_{12}/E_2=G_{13}/E_{22}=0.5$, $G_{23}/E_{22}=0.2$, $\nu_{12}=0.25$, and $\alpha_{22}/\alpha_{11}=10$. They show that the results from the method presented agree well with the comparator solutions when $\bar{W}/t\leqslant0.4$, whereas further into the postbuckling region change of buckling mode is shown in Ref. [5].

Fig. 2 gives the thermal postbuckling load-deflection curves of 4-ply $(0/90)_s$ symmetric cross-ply laminated square plates under uniform or nonuniform parabolic temperature loading with different thickness ratio $b/t(=10.0, 5.0)$ and are compared with their classical counterpatrs. The results calculated show that the thermal buckling load of $(0/90)_s$ plate with $b/t=10.0$ is about 29% lower than that of CPT in both uniform and nonuniform temperature loading

Table 1 Comparisons of various theories on the thermal buckling load $\bar{\lambda}_T=\alpha_0 T_0\times 10^3$ for perfect square plates

Layer-up	b/t	Noor & Burton[4]	Present	CPT[a]
isotropic	100	0.1264	0.1265	0.1265
	20	3.109	3.1194	3.1633
	10	11.83	11.9782	12.6533
	5	39.90	41.3175	50.6134
$(0)_s$	100	0.7463	0.7466	0.7486
	20	17.39	17.5202	18.7160
	10	57.82	59.1271	74.8639
	5	143.6	149.9049	299.4555

[a]Calculated using classical plate theory given in [2].



1: CPT
2: HSDPT ($b/t=10.0$)
3: HSDPT ($b/t=5.0$)

(a) uniform temperature rise

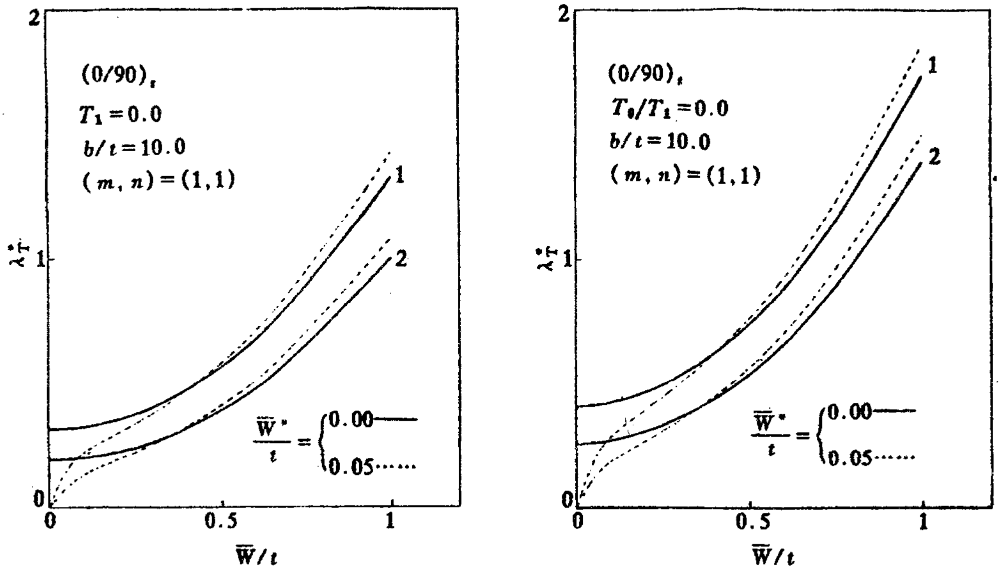
1: CPT
2: HSDPT ($b/t=10.0$)
3: HSDPT ($b/t=5.0$)

(b) nonuniform temperature rise

Fig. 2 Effect of plate thickness ratio b/t on thermal postbuckling of $(0/90)_s$ plates under uniform or nonuniform parabolic temperature distribution

cases. It is found that the thermal buckling loads are decreased by decreasing the plate thickness ratio b/t, but in the deep postbuckling range the thick plate will has a higher postbuckling load than does a thin plate. It can also be seen that the laminated plate under nonuniform parabolic temperature loading has a higher initial buckling load and a higher postbuckling load than does a plate under uniform thermal loading.

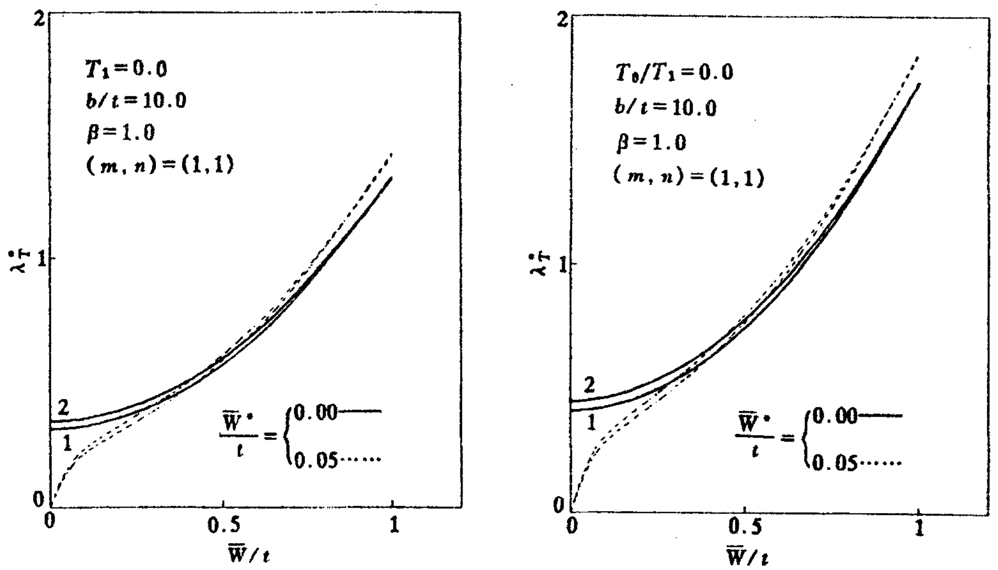
Fig. 3 shows the effect of the plate aspect ratio $\beta(=1.0, 1.5)$ on the thermal postbuckling response of the same laminated plates under uniform or nonuniform parabolic temperature loading. Then Fig. 4 shows the effect of the total unmber of plies $N(=4, 8)$ on the thermal postbuckling response of symmetric cross-ply laminated square plates under uniform or nonuniform parabolic temperature loading. As expected, these results show that the thermal



1: $\beta=1.0$ 2: $\beta=1.5$
(a) uniform temperature rise

1: $\beta=1.0$ 2: $\beta=1.5$
(b) nonuniform temperature rise

Fig. 3 Effect of plate aspect ratio β on thermal postbuckling of (0/90)$_s$ plates under uniform or nonuniform parabolic temperature distribution



1: (0/90)$_s$, $N=4$ 2: (0/90)$_{2s}$, $N=8$
(a) uniform temperature rise

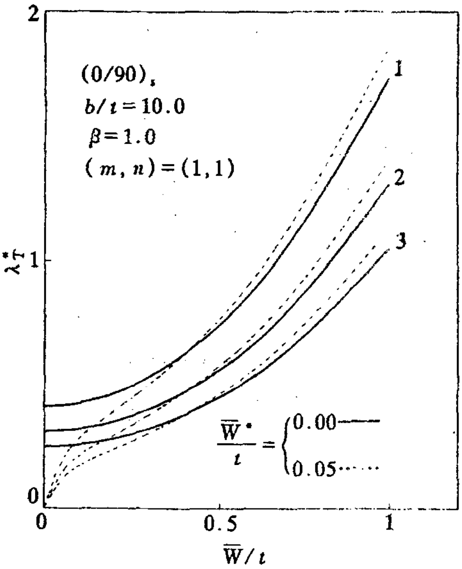
1: (0/90)$_s$, $N=4$ 2: (0/90)$_{2s}$, $N=8$
(b) nonuniform temperature rise

Fig. 4 Effect of total number of plies N on thermal postbuckling of symmetric laminated plates under uniform or nonuniform parabolic temperature loading

buckling load and postbuckling strength are increased by decreasing the plate aspect ratio β or by increasing the total number of plies N with N having rather less effect.

Fig. 5 shows the effect of thermal load ratio T_0/T_1 ($=0.0$, 0.25, 0.5) on the post-buckling responses of the same laminated plates under nonuniform parabolic temperature loading. It can be seen that the thermal buckling load is decreased by increasing the thermal load ratio T_0/T_1 and that the thermal postbuckling equilibrium path becomes significantly lower as the thermal load ratio T_0/T_1 increases. In Figs. 3~5 the plate thickness ratio is $b/t=10.0$.

Thermal postbuckling load-deflection curves for imperfect as well as perfect plates are plotted in each of Figs. 2~5. The imperfect curves show that the effect of an initial geometric imperfection on the thermal postbuckling response is substantial. This conclusion is valid for both classical and shear-deformable composite laminated plates.



1: $T_0/T_1=0.0$ 2: $T_0/T_1=0.25$ 3: $T_0/T_1=0.50$

Fig. 5 Effect of thermal load ratio T_0/T_1 on the postbuckling of (0/90)$_s$ plates under nonuniform parabolic temperature distribution

V. Concluding Remarks

The essence of this paper lies in the development of the Reddy's higher-order shear deformation theory to study the thermal postbuckling of imperfect shear-deformable laminated plates under a uniform or nonuniform parabolic temperature distribution. The numerical examples presented relate to the performance of perfect and imperfect, symmetric cross-ply laminated plates. They show that the characteristics of thermal postbuckling are significantly influenced by the transverse shear deformation, plate aspect ratio, thermal load ratio and initial geometric imperfections, whereas the total number of plies has rather less effect. The results presented do not cover all possible cases, however we believe some interesting and noteworthy effects can be studied with the present equations.

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Appendix

In Eqs. (3.15)~(3.18), the coefficients are related. Illustrative examples are

$$C_{11}^{(1)}=-m\frac{g_{04}}{g_{00}}A_{11}^{(1)},\quad D_{11}^{(1)}=-n\beta\frac{g_{03}}{g_{00}}A_{11}^{(1)}$$

$$B_{20}^{(2)}=\frac{1}{32}\frac{\gamma_{24}n^2\beta^2}{m^2}(1+2\mu)(A_{11}^{(1)})^2,\quad B_{02}^{(2)}=\frac{1}{32}\frac{m^2}{\gamma_{24}n^2\beta^2}(1+2\mu)(A_{11}^{(1)})^2$$

$$A_{13}^{(3)}=\frac{1}{16}\frac{\gamma_{14}}{\gamma_{24}}\frac{m^4}{J_{13}}(1+\mu)^2(1+2\mu)(A_{11}^{(1)})^3$$

$$A_{31}^{(3)}=\frac{1}{16}\frac{\gamma_{14}}{\gamma_{24}}\frac{n^4\beta^4}{J_{31}}(1+\mu)^2(1+2\mu)(A_{11}^{(1)})^3$$

$$C_{13}^{(3)}=-m\frac{g_{134}}{g_{130}}A_{13}^{(3)},\quad C_{31}^{(3)}=-3m\frac{g_{314}}{g_{310}}A_{31}^{(3)}$$

$$D_{13}^{(3)}=-3n\beta\frac{g_{133}}{g_{130}}A_{13}^{(3)},\quad D_{31}^{(3)}=-n\beta\frac{g_{313}}{g_{310}}A_{31}^{(3)}$$

$$B_{20}^{(4)}=-\frac{1}{16}\frac{\gamma_{24}n^2\beta^2}{m^2}(1+\mu)A_{11}^{(1)}A_{31}^{(3)},\quad B_{02}^{(4)}=-\frac{1}{16}\frac{m^2}{\gamma_{24}n^2\beta^2}(1+\mu)A_{11}^{(1)}A_{13}^{(3)}$$

Finally, in Eq. (3.19)

$$(\lambda_T^{(0)},\lambda_T^{(2)},\lambda_T^{(4)})=(S_0,\ S_2,\ S_4)/\gamma_{14}C_{11}$$

where

$$S_0=\frac{\Theta_{11}}{(1+\mu)},\quad S_2=\frac{1}{16}\frac{\gamma_{14}}{\gamma_{24}}\Theta_2(1+2\mu),\quad S_4=\frac{1}{256}\frac{\gamma_{14}^2}{\gamma_{24}^2}C_{11}(C_{24}-C_{44})+C_{33}$$

$$\Theta_{11}=(\gamma_{110}m^4+2\gamma_{112}m^2n^2\beta^2+\gamma_{114}n^4\beta^4)$$
$$+[m^2(\gamma_{120}m^2+\gamma_{122}n^2\beta^2)g_{04}+n^2\beta^2(\gamma_{131}m^2+\gamma_{133}n^2\beta^2)g_{03}]/g_{00}$$

$$\Theta_2=[(3\gamma_{24}^2-\gamma_5^2)(m^4+\gamma_{24}^2n^4\beta^4)+4\gamma_{24}^2\gamma_5m^2n^2\beta^2]/(\gamma_{24}^2-\gamma_5^2)$$

$$\Theta_{13}=(\gamma_{110}m^4+18\gamma_{112}m^2n^2\beta^2+\gamma_{114}81n^4\beta^4)$$
$$+[m^2(\gamma_{120}m^2+\gamma_{122}9n^2\beta^2)g_{134}+9n^2\beta^2(\gamma_{131}m^2+\gamma_{133}9n^2\beta^2)g_{133}]/g_{130}$$

$$\Theta_{31}=(81\gamma_{110}m^4+18\gamma_{112}m^2n^2\beta^2+\gamma_{114}n^4\beta^4)$$
$$+[9m^2(\gamma_{120}9m^2+\gamma_{122}n^2\beta^2)g_{314}+n^2\beta^2(\gamma_{131}9m^2+\gamma_{133}n^2\beta^2)g_{313}]/g_{310}$$

$$C_{24}=2(1+\mu)^2(1+2\mu)^2\Theta_2\left[\frac{m^4}{J_{13}}+\frac{\gamma_{24}^2n^4\beta^4}{J_{31}}\right]$$

$$g_{00}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}n^2\beta^2)-\gamma_{331}^2m^2n^2\beta^2$$

$$g_{03}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}n^2\beta^2)$$
$$-\gamma_{331}m^2(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}n^2\beta^2)$$

$$g_{04}=(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}n^2\beta^2)(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}n^2\beta^2)$$
$$-\gamma_{331}n^2\beta^2(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}n^2\beta^2)$$

$$g_{130}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}9n^2\beta^2)(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}9n^2\beta^2)-\gamma_{331}^2 9m^2n^2\beta^2$$

$$g_{133}=(\gamma_{31}+\gamma_{320}m^2+\gamma_{322}9n^2\beta^2)(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}9n^2\beta^2)$$

$$-\gamma_{331}m^2(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}9n^2\beta^2)$$

$$g_{134}=(\gamma_{41}+\gamma_{430}m^2+\gamma_{432}9n^2\beta^2)(\gamma_{31}-\gamma_{310}m^2-\gamma_{312}9n^2\beta^2)$$

$$-\gamma_{331}9n^2\beta^2(\gamma_{41}-\gamma_{411}m^2-\gamma_{413}9n^2\beta^2)$$

$$g_{310}=(\gamma_{31}+\gamma_{320}9m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}+\gamma_{430}9m^2+\gamma_{432}n^2\beta^2)-\gamma_{331}^2 9m^2n^2\beta^2$$

$$g_{313}=(\gamma_{31}+\gamma_{320}9m^2+\gamma_{322}n^2\beta^2)(\gamma_{41}-\gamma_{411}9m^2-\gamma_{413}n^2\beta^2)$$

$$-\gamma_{331}9m^2(\gamma_{31}-\gamma_{310}9m^2-\gamma_{312}n^2\beta^2)$$

$$g_{314}=(\gamma_{41}+\gamma_{430}9m^2+\gamma_{432}n^2\beta^2)(\gamma_{31}-\gamma_{310}9m^2-\gamma_{312}n^2\beta^2)$$

$$-\gamma_{331}n^2\beta^2(\gamma_{41}-\gamma_{411}9m^2-\gamma_{413}n^2\beta^2)\text{。}$$

$$J_{13}=\mathfrak{S}_{13}C_{11}(1+\mu)-\mathfrak{S}_{11}C_{13},\quad J_{31}=\mathfrak{S}_{31}C_{11}(1+\mu)-\mathfrak{S}_{11}C_{31}$$

in above equations, for uniform thermal loading case

$$C_{11}=(\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2),\quad C_{13}=(\gamma_{T1}m^2+9\gamma_{T2}n^2\beta^2)$$

$$C_{31}=(9\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2),\quad C_{33}=0$$

and for nonuniform parabolic thermal loading case

$$C_{11}=(\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2)\left(\frac{T_0}{T_1}+\frac{3}{2}\right)+2\gamma_8\frac{m^2}{\pi^2n^2}$$

$$C_{13}=(\gamma_{T1}m^2+9\gamma_{T2}n^2\beta^2)\left(\frac{T_0}{T_1}+\frac{3}{2}\right)+\frac{2}{9}\gamma_8\frac{m^2}{\pi^2n^2}$$

$$C_{31}=(9\gamma_{T1}m^2+\gamma_{T2}n^2\beta^2)\left(\frac{T_0}{T_1}+\frac{3}{2}\right)+18\gamma_8\frac{m^2}{\pi^2n^2}$$

$$C_{33}=\frac{1}{256}\ \frac{\gamma_{14}^2}{\gamma_{24}^2}(1+\mu)(1+2\mu)^2\mathfrak{S}_2\left[\frac{3}{2}\gamma_8\frac{m^2}{\pi^2n^2}\ \frac{m^4}{J_{13}}\right]$$

$$\gamma_8=(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})/\gamma_{24}^2$$