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The Undecidability of the First-order Theory of Diagonalizable Algebras

Abstract. The undecidability of the first-order theory of diagonalizable algebras is shown here.

Introduction

A *diagonalizable algebra* is a pair $\langle \mathscr{A}, \tau\rangle$, where $\mathscr{A}$ is a Boolean algebra, and τ is a unary operation on $\mathscr{A}$ satisfying the following properties:

$$\tau 1 = 1, \quad \tau(x\cdot y) = \tau x\cdot \tau y, \quad \tau(\tau x \to x) = \tau x.$$ [1]

The diagonalizable algebras constitute an equational class, which we call DA. An important aspect of the diagonalizable algebras is the following:

Let $\varphi \equiv \{p_n\}_{n\in\omega}$ be a sequence of sentences of Peano Arithmetic PA; let us define a mapping φ^* from the set of the polynomials of DA into the set of PA sentences in the following inductive manner:

$$\varphi^*0 = \ulcorner 0 = 1\urcorner; \quad \varphi^*1 = \ulcorner 0 = 0\urcorner; \quad \varphi^* x_i = p_i; \quad \varphi^*(f+g) = \varphi^* f \vee \varphi^* g;$$
$$\varphi^*(\nu f) = \neg \varphi^* f; \quad \varphi^*\tau f = Theor\ \overline{\varphi^* f}.$$

Then, if $f = 1$ is valid in DA, for every φ as above, $\varphi^* f$ is a theorem of PA. (See R. MAGARI, [3]).

In [6], R. SOLOVAY shows that, conversely, if, for every φ, $\varphi^* f$ is a theorem of PA, then $f = 1$ is valid in DA.

In [5], F. MONTAGNA extends MAGARI's result to the first-order theory $\mathscr{T}$ of the diagonalizable algebras. More precisely, the author defines, for every φ as above, a mapping φ° from the set of the formulas of $\mathscr{T}$ into the set of the sentences of PA, and shows that, if A is a theorem of $\mathscr{T}$, then, for every φ, $\varphi^\circ A$ is a theorem of PA.

It is known (see C. BERNARDI, [1]) that DA (that, is, the set of the equations which are valid in all diagonalizable algebras) is decidable, therefore, the set of the polynomials f such that, for every φ, $\varphi^* f$ is a theorem of DA, is decidable.

In this paper, we show that, on the contrary, the first-order theory $\mathscr{T}$ of diagonalizable algebras is undecidable. The decision problem remains

[1] In the following, $+$, $\cdot$, ν denote the operations of join, meet, and complementation; $x \to y$ is an abbreviation for $\nu x + y$.

open for the set of the formulas A of $\mathscr{T}$ such that, for every φ, $\varphi^\circ A$ is a theorem of PA.

By [7], in order to show the claim it suffices to find a model $\mathscr{A} = \langle A, \tau\rangle$ of DA, and formulas $F(x)$, $\psi(x, y)$, $\vartheta_0(x, y, z)$, $\vartheta_1(x, y, z)$ of the signature of $\mathscr{A}$, so that, putting $\mathscr{F} = \{x \in A: \models_{\mathscr{A}} F(x)\}$, $\approx = \{\langle x, y\rangle \in A^2: \models_{\mathscr{A}} \psi(x, y)\}$ $\boldsymbol{+} = \{\langle x, y, z\rangle \in A^3: \models_{\mathscr{A}} \vartheta_0(x, y, z)\}$ and $\boldsymbol{\times} = \{\langle x, y, z\rangle \in A^3: \models_{\mathscr{A}} \vartheta_1(x, y, z)\}$, the following conditions hold:

1) $\approx$ is a congruence relation in the structure $\mathfrak{F} = \langle \mathscr{F}, \boldsymbol{+}, \boldsymbol{\times}\rangle$.

2) The quotient $\mathfrak{F}/\approx$ is isomorphic to the standard model $\mathfrak{N} = \langle N, +, .\rangle$ of the natural numbers.

Now, let us define $\mathscr{A}$ as follows:
Let X be the set $\omega \cup \{\langle\{i, j\}, n\rangle: i, j, n \in \omega,\ i \neq j, i, j < n\}$. Let R be the binary relation on X defined by: xRy iff $\exists i, j, n: i \neq j; i, j < n, x = \langle\{i, j\}, n\rangle$ and $y = i$ or $y = j$ or $y > n$. Let us define the mapping τ from $\mathscr{P}(X)$ into $\mathscr{P}(X)$ by $\tau A = \{x \in X: \forall\ z \in X$, if xRz, then $z \in A\}$.

Now, let $\mathscr{A}$ be the algebra $\langle \mathscr{P}(X), \cup, \cap, \mathscr{C}, X, \emptyset, \tau\rangle$. Since R is a transitive and reverse well-founded relation on X, $\mathscr{A}$ is a diagonalizable algebra (see R. Magari, [4]). It is easily seen that $\tau\emptyset = \omega$, $\tau^2\emptyset = X$. Now, we define

$$
\begin{aligned}
Ix &= \text{“}x\cdot\tau x\text{”}\\
Opx &= \text{“}Ix = x\text{”}\\
Cnx &= \text{“}\forall u \forall v[Opu \wedge Opv \wedge u+v \geqslant x \wedge u\cdot v\cdot x = 0 \rightarrow u\cdot x = 0 \vee v\cdot x\\
&\quad = 0]\text{”}\\
Atx &= \text{“}\neg x = 0 \wedge \forall u(u < x \rightarrow u = 0)\text{”}\\
Cmp(x, y) &= \text{“}Cnx \wedge x \leqslant y \wedge \forall u[Cnu \wedge u \leqslant y \rightarrow \neg x < u].\text{”}
\end{aligned}
$$

It is easily seen that the set $\{B \subseteq X: \models_{\mathscr{A}} Op\, B\}$ constitutes a topology on X. We denote this topology by T. Of course, for every $B, C \subseteq X$, $\models_{\mathscr{A}} CnB$ iff B is connected, $\models Cmp(B, C)$ iff B is a connected component of C; finally $B \subseteq X$ is open iff for each $x, y \in X$, if $x \in B$, and xRy, then $y \in B$.

Definition 1. Let B be a non-empty subset of X. We define $T_{1B} = \{C \subseteq B: \exists D \subseteq X[Op\, D \wedge C = X \cap D]\}$.

Of course, T_{1B} is a topology on B; moreover, it is easily seen that for every $C \subseteq B$, C is open in T_{1B} iff, for every $x, y \in B$, if $x \in C$ and xRy, then $y \in C$.

Definition 2. Let Y be a subset of X, and let a, b be elements of Y; we define $a\, R_Y b$ iff there is a finite sequence $x_0 \ldots x_n$ of elements of Y such that $x_0 \equiv a$, $x_n \equiv b$, and, for every i, either $x_i R x_{i+1}$ or $x_{i+1} R x_i$.

R_y is an equivalence relation on Y.

Lemma 1. *Let Y be a subset of X. Then, Y is connected iff for every $a, b \in Y$ $aR_y b$.*

PROOF. Suppose that Y is connected; let x be an arbitrary element of Y, and let $\overline{Y}$ be the set $\{y \in Y: xR_y y\}$. Then, $\overline{Y}$ is open in the topology T_{1Y}. Moreover, $Y - \overline{Y}$ is also open in T_{1Y}; in fact, if $a \in Y - \overline{Y}$, $b \in Y$ and aRb, then $b \in Y - \overline{Y}$. Otherwise, we would have $xR_Y b$, and then $xR_Y a$, $a \in \overline{Y}$, a contradiction. Since $(Y - \overline{Y}) \cup \overline{Y} = Y$ and $\overline{Y}$ is non-empty, we can conclude that $Y - \overline{Y}$ is empty and that $Y = \overline{Y}$. Therefore, for every $y \in \overline{Y}$, $xR_Y y$.

On the other hand, suppose that, for every $x, y \in Y$ $xR_Y y$. Let A, B be non-empty subsets of Y such that $A \cup B = Y$; suppose that both A and B are open in T_{1Y}. Let $a \in A$, $b \in B$; since $a\, R_{\overline{Y}} b$, there is a sequence $x_0 \ldots x_n$ of elements of Y such that $x_0 = a$, $x_n = b$, and, for every i, either $x_i R x_{i+1}$ or $x_{i+1} R x_i$. Let x_{i_0+1} be the first element of the sequence, which belongs to B; then $x_{i_0} \in A$; since B is closed under R, if $x_{i_0+1} R x_{i_0}$ $x_{i_0} \in A \cap B$; since A is also closed under R, if $x_{i_0} R x_{i_0+1}$, $x_{i_0+1} \in A \cap B$; in both cases, $A \cap B$ is non-empty. Then, Y is connected. Q.E.D.

By Lemma 1, we can easily deduce that, if $Y \subseteq X$, the connected components of Y are exactly the equivalence classes determined by the relation R_Y.

DEFINITION 3. We define $\sigma x = \nu\tau\nu x$; $Fx = x \leqslant \tau 0 \wedge \neg \sigma x = \sigma 1$;

$$\mathscr{F} = \{x \in A : \models_{\mathscr{A}} Fx\}.$$

LEMMA 2. *Let Y be a subset of X; then $\mathscr{F}Y$ is true iff Y is a finite subset of ω.*

PROOF. We have $\sigma Y = \{x \in X: \exists y \in Y \quad xRy\}$, $\sigma X = \mathscr{C}\tau\emptyset = \{\langle\{i, j\}, n\rangle: i, j, n \in \omega,\ i \neq j,\ i, j < n\}$. Suppose that Y is a finite subset of ω; then $Y \subseteq \omega = \tau\emptyset$; moreover, $\sigma Y \neq \sigma 1$; in fact, let $n = \max Y$, and let $z = \langle\{n+1, n+2\}, n+3\rangle$; by the definition of R, zRx iff $x \geqslant n+1$; therefore, for every $x \in Y$, zRy; so $z \in \sigma 1 - \sigma Y$, and $\mathscr{F}Y$ is true.

On the other hand, if $\mathscr{F}Y$ is true, then $Y \subseteq \tau 0 = \omega$; moreover Y is a finite set; indeed suppose, by contradiction, that Y is infinite; let $\langle\{i, j\}, n\rangle$ be an arbitrary element of $\sigma 1$. Since Y is infinite, there is an $m \in Y$, such that $n < m$; then $\langle\{i, j\}, n\rangle Rm$ therefore $\langle\{i, j\}, n\rangle \in \sigma Y$. So, we would conclude $\sigma Y = \sigma 1$, which contradicts our hypothesis. Q.E.D.

DEFINITION 4. We define $\psi(u, v)$ to be the formula $Fu \wedge Fv \wedge [u = v \vee \exists z \forall w (Cmp(w, z) \to At(u \cdot w) \wedge At(v \cdot w)]$; moreover, we put $\approx\, = \{\langle x, y\rangle \in \mathscr{A}^2 : \models_{\mathscr{A}} \psi(x, y)\}$.

LEMMA 3. *For every $x, y \in \mathscr{A}$, we have $x \approx y$ iff x, y are finite subsets of ω, and $\overline{\overline{x}} = \overline{\overline{y}}$.*

PROOF. If $A \approx B$, then both A and B are finite subsets of ω: moreover, either $A = B$, or there is a $Z \subseteq X$ such that, for every connected component W of Z, both $W\cap(A-B)$ and $W\cap(B-A)$ have exactly one element; if $A=B$, then $\overline{\overline{A}}=\overline{\overline{B}}$; otherwise, let us consider an arbitrary $a\in A$; there is a unique connected component W_a of Z such that $W_a\cap(A-B)=\{a\}$; furthermore $W_a\cap(B-A)$ has exactly one element, say b_a; now let us define for every $a\in A-B$, $fa=b_a$; it is easily seen that f is a one-to-one function from $A-B$ onto $B-A$; then $\overline{\overline{A-B}}=\overline{\overline{B-A}}$, therefore $\overline{\overline{A}}=\overline{\overline{B}}$.

Conversely, let A and B be finite subsets of ω such that $\overline{\overline{A}}=\overline{\overline{B}}$; if $A=B$, then the claim is obvious; otherwise, let $A-B=\{a_1\dots a_n\}$, $B-A=\{b_1\dots b_n\}$, and let $h=\max\{a_1\dots a_n, b_1\dots b_n\}$. Let Z be the set $\{a_1\dots a_n, b_1\dots b_n, (\{a_1,b_1\},h+1)\dots(\{a_n,b_n\},h+1)\}$. By Lemma 1, it is easily seen that the connected component of Z are $W_1=\{a_1,b_1,(\{a_1,b_1\},h+1)\},\dots,W_n=\{a_n,b_n,(\{a_n,b_n\},h+1)\}$, therefore for every W_i, $W_i\cap(A-B)$ and $W_i\cap(B-A)$ have exactly one element. Therefore $A\approx B$. Q.E.D.

DEFINITION 5. We define $\vartheta_0(x,y,z)=$ "$Fx\wedge Fy\wedge Fz\wedge\exists u\big(Fu\wedge\psi(u,y)\wedge u\cdot x=0\wedge\psi(z,x+u)\big)$" and $\boldsymbol{+}=\{\langle x,y,z\rangle\in\mathscr{A}^3\colon\models_{\mathscr{A}}\vartheta_0(x,y,z)\}$.

It is easily seen that, if x,y,z are finite subsets of ω, then we have $\boldsymbol{+}(x,y,z)$ iff $\overline{\overline{x}}+\overline{\overline{y}}=\overline{\overline{z}}$.

DEFINITION 6. We define $\vartheta_1(u,v,w)=$ "$Fu\wedge Fv\wedge Fw\wedge\exists x\exists y[\psi(x,u)\wedge\psi(y\cdot\tau 0,w)\wedge y\geqslant x\wedge\forall z\big(Cmp(z,y)\to At(z\cdot x)\wedge\psi(\tau 0\cdot z,v)$" and $\boldsymbol{\times}=\{\langle x,y,z\rangle\in\mathscr{A}^3\colon\models_{\mathscr{A}}\vartheta_1(x,y,z)\}$.

LEMMA 4. *For every $A,B,C\subseteq X$, we have $\boldsymbol{\times}(A,B,C)$ iff A,B,C are finite subsets of ω, and $\overline{\overline{A}}\cdot\overline{\overline{B}}=\overline{\overline{C}}$.*

PROOF. If $\boldsymbol{\times}(A,B,C)$, then, by Lemma 2, A,B,C are finite subsets of ω; moreover, there are $D,E\subseteq X$ such that $\overline{\overline{D}}=\overline{\overline{A}}$, $E\supseteq D$ and for every connected component F of E, $\overline{\overline{F\cap\tau 0}}=\overline{\overline{B}}$ and $F\cap D$ has exactly one element; therefore, E has exactly $\overline{\overline{D}}=\overline{\overline{A}}$ components; now, let $\overline{\overline{A}}=n$, and let $F_1\dots F_n$ be the components of E; then $\overline{\overline{C}}=\overline{\overline{E\cap\tau\emptyset}}=\sum_{i=1}^{n}\overline{\overline{F\cap\tau\emptyset}}=\sum_{i=1}^{n}\overline{\overline{B}}=\overline{\overline{A}}\cdot B$.

On the other hand, suppose that A,B,C are finite subsets of ω, and $\overline{\overline{A}}\cdot\overline{\overline{B}}=\overline{\overline{C}}$. Let $\overline{\overline{A}}=n$, $\overline{\overline{B}}=m$. Let $D=\{0,m,2m,\dots,(n-1)m\}$, $\overline{F}_i=\{im,im+1,\dots,(i+1)m-1\}$ $(i=0\dots n-1)$, $\overline{\overline{F}}{}_i^1=\{(\{h,k\}),nm)\colon h,k\in\overline{F}_i\}$, $F_i=\overline{F}_i\cup F_i^1$, $E=\bigcup_{i=0}^{n-1}F_i$. By Lemma 1, it is easily seen that the connected components of E are $F_0\dots F_{n-1}$; moreover, $\overline{\overline{E\cap\tau\emptyset}}=mn=\overline{\overline{C}}$, $E\supseteq D$, and for every i, $F_i\cap D=\{i\}$, $\overline{\overline{F_i\cap\tau 0}}=m=\overline{\overline{B}}$, therefore we can conclude $\boldsymbol{\times}(A,B,C)$. Q.E.D.

Now, we are able to show the following theorem:

THEOREM 1. *$\mathscr{T}$ is undecidable.*

PROOF. From Lemmas 1–4, it follows that, putting $\mathfrak{F} = \langle \mathscr{F}, +, \times \rangle$, $\mathscr{F}/\approx$ is isomorphic to the standard model $N = \{N, +, \cdot\}$ of natural numbers, therefore the claim follows. Q.E.D.

REMARK. Let $\mathscr{T}^n$ be $\mathscr{T} + \{\tau^n 0 = 1\}$. Since for $n \geqslant 2$ $\mathscr{A}$ is also a model of $\mathscr{T}^n$, the above argument shows that, if $n \geqslant 2$, $\mathscr{T}^n$ is undecidable.

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