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On Classical and Quantum Relativistic Dynamics[1]

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A canonical formalism for the relativistic classical mechanics of many particles is proposed. The evolution equations for a charged particle in an electromagnetic field are obtained and the relativistic two-body problem with an invariant interaction is treated. Along the same line a quantum formalism for the spinless relativistic particle is obtained by means of imprimitivity systems according to Mackey theory. A quantum formalism for the spin-$\frac{1}{2}$ particle is constructed and a new definition of spin $\frac{1}{2}$ in relativity is proposed. An evolution equation for the spin-$\frac{1}{2}$ particle in an external electromagnetic field is given. The Bargmann Michel, and Telegdi equation follows from this formalism as a quasiclassical approximation. Finally, a new relativistic model for hydrogenlike atoms is proposed. The spectrum predicted is in agreement with Dirac's when radiative corrections have been added.

1. CLASSICAL RELATIVISTIC MECHANICS

The difficulties encountered in relativity in elaborating a canonical dynamics which is covariant and nontrivial are well known. Let us simply recall the no-go theorems of Currie.[(1)] Under such conditions it is only possible to overcome these difficulties by accepting the need to radically change the point of view of Einstein's theory.

The relativistic dynamics that we propose (previously introduced in Ref. 2) does not encounter these difficulties, and the present paper is devoted to a survey of several of its developments.

The essential differences between the usual point of view and the one adopted in Ref. 2 are the following: In the usual Einstein theory each (classical) particle is identified with a trajectory in spacetime and the dynamics of the system is simply reduced to a description of these trajectories. Accord-

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ing to that point of view, nothing changes, since nothing moves over these trajectories, and we cannot properly speak of a system's evolution. Further, the concept of probability, one of the characteristics of quantum physics, is meaningless in such a scheme.

The point of view presented here and adopted in Ref. 2 is simultaneously based on both Einsteinian and Newtonian ideas of mechanics, admitting the existence of a spacetime supplied with the geometry given by the Poincaré group and identifying each particle with a single point of that spacetime (or an event according to Einstein). Moreover, to describe the true evolution of the system, we postulate the existence of a new time (described by the parameter τ) which passes "uniformly and inexorably" as Newton imagined. That time τ is called "historical time" because it corresponds to the ordering relation determined by successive measuring processes in quantum theory or given by the laws of thermodynamics. Such a parameter τ has been introduced by other authors, but (except for Aghassi *et al.*[3]) in each case as a mathematical convenience without physical interpretation.

In such a scheme the state of each particle is described by eight independent numbers

$$q^\mu = (q^1, q^2, q^3, q^4) = (\mathbf{q}, t)$$
$$p_\mu = (p_1, p_2, p_3, p_4) = (\mathbf{p}, -E)$$

where q^μ is identified with the position in spacetime and p_μ with the state of motion, i.e., the momentum–energy. The metric tensor is $g_{\mu\nu} = (1, 1, 1, -c^2)$ (where c is the velocity of light in vacuum).

Then it is obvious that the "mass"

$$m \equiv (-g^{\mu\nu} p_\mu p_\nu / c^2)^{1/2}$$

depends on the state of the particle. A well-known example is the bound proton in the nucleus whose mass is smaller than that of the free proton, the mass deficiency corresponding to the binding energy. Actually the description of an interacting system entails the consideration of states which are not confined to a "mass shell" condition.

The dynamical principle that we invoke to obtain the evolution is a generalization of the Hamilton principle. Consider the differential 1-form

$$\omega = \sum_{i=1}^{N} p_{i\mu}\, dq_i^\mu - K(p_i, q_i)\, d\tau \qquad (1)$$

where K denotes a function of the state (p_i, q_i) of each particle $i = 1,..., N$.

The dynamical principle is expressed as follows[3]: Given a closed curve C in $\Gamma = \{p_i, q_i, \tau\}$, the integral $\int_C \omega$ is invariant for any (continuous) deformations of C generated by arbitrary displacements of its points along the trajectories corresponding to the evolution of the system. This principle is equivalent to the canonical equations

$$\dot{q}_i^{\mu} \equiv dq_i^{\mu}/d\tau = \partial K/\partial p_{i\mu} \qquad \text{and} \qquad \dot{p}_{i\mu} \equiv dp_{i\mu}/d\tau = -\,\partial K/\partial q_i^{\mu} \tag{2}$$

These equations are manifestly covariant when the differential 1-form ω is invariant (which is the case when K transforms like a scalar field under the Lorentz group), the historical time τ being invariant.

Let us now illustrate the previous considerations by some examples.

1.1. Particle in External Electromagnetic Field

In the absence of radiation phenomena, the evolution of a charged particle in an external electromagnetic field described by the 4-vector potential $A_\mu(x) = (\mathbf{A}(x), -V(x))$ is governed by canonical equations, where K is given by

$$K = (1/2M)g^{\mu\nu}[p_\mu - eA_\mu(q)][p_\nu - eA_\nu(q)] \tag{3}$$

In this expression e denotes the charge of the particle and M its mass. Here M is a dynamical constant which is characteristic of the particle.

In this particular case the canonical equations (2) imply

$$\begin{aligned} \dot{q}^{\mu} &= (1/M)[p^{\mu} - eA^{\mu}(q)] \\ \dot{p}_\mu &= (e/M)[p^{\nu} - eA^{\nu}(q)]\,\partial_\mu A_\nu(q) \end{aligned} \tag{4}$$

where $\partial_\mu A_\nu$ denotes the partial derivative of A_ν with respect to x^μ. From these equations follows the well-known relation

$$M\ddot{q}_\mu = e[\partial_\mu A_\nu(q) - \partial_\nu A_\mu(q)]\dot{q}^{\nu} \tag{5}$$

Moreover,

$$g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = 2K/M$$

is conserved, since K is a constant of motion. For

$$K = -\tfrac{1}{2}Mc^2 \qquad \text{we have} \qquad g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = -c^2$$

[3] For the canonical formalism see Ref. 4.

and then the proper time takes the same values as the historical time. Moreover,

$$g^{\mu\nu}[p_\mu - eA_\mu(q)][p_\nu - eA_\nu(q)] = -M^2c^2$$

for any τ.

1.2. Two-Body Problem with Potential Interaction[5]

In this example the interaction between both particles is described by a scalar potential $\Phi(|q_1 - q_2|)$, where $|q| = (g_{\mu\nu}q^\mu q^\nu)^{1/2}$.

We have

$$K = \frac{1}{2M_1} g^{\mu\nu} p_{1\mu} p_{1\nu} + \frac{1}{2M_2} g^{\mu\nu} p_{2\mu} p_{2\nu} + \Phi(|\, q_1 - q_2\,|) \tag{6}$$

where M_1 and M_2 are the masses of particles 1 and 2.

In that case, as in the corresponding Galilean case, it is possible to define some center-of-mass coordinates which are more convenient for the discussion. They are defined by

$$\begin{aligned} P_\mu &= p_{1\mu} + p_{2\mu}, \qquad & Q^\mu &= \frac{M_1 q_1{}^\mu + M_2 q_2{}^\mu}{M_1 + M_2} \\ p_\mu &= \frac{M_1 p_{2\mu} - M_2 p_{1\mu}}{M_1 + M_2}, \qquad & q^\mu &= q_2{}^\mu - q_1{}^\mu \end{aligned} \tag{7}$$

This change of coordinates characterizes a canonical tıansformation which leaves K invariant. In terms of the new variables we have

$$K = K_0 + k \tag{8}$$

where

$$K_0 = (1/2M) g^{\mu\nu} P_\mu P_\nu, \qquad M = M_1 + M_2$$

and

$$k = \frac{1}{2\mu} g^{\mu\nu} p_\mu p_\nu + \Phi(|\, q\,|), \qquad \mu = \frac{M_1 M_2}{M_1 + M_2}$$

The corresponding canonical equations then imply that

$$\begin{aligned} \dot{Q}^\mu &= P^\mu/M, \qquad & \dot{P}_\mu &= 0 \\ \dot{q}^\mu &= p^\mu/\mu, \qquad & \dot{p}_\mu &= -\partial_\mu \Phi(|\, q\,|) \end{aligned} \tag{9}$$

where ∂_μ denotes the partial derivative with respect to q^μ.

Thus, the total momentum–energy $P_\mu = (\mathbf{P}, -E)$ is a constant of motion and $Q^\mu = (\mathbf{Q}, T)$ defines a spacetime point called the center of mass. The time

$$T = (E/Mc^2)\tau \tag{10}$$

passes uniformly, and so Møller's condition on the time, used by Cook,[6] is justified by our model.

Now it is important to note from (9) that the antisymmetric tensor

$$j^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu \tag{11}$$

which is a generalization of the relative angular momentum, also is a constant of motion. As a consequence, the relative motion (in q^μ coordinates) takes place in a plane $\mathscr{T}$ containing the origin $q = 0$. Moreover, this plane $\mathscr{T}$ is spacelike (i.e., $g_{\mu\nu} q^\mu q^\nu > 0$ for any $q \neq 0$ in $\mathscr{T}$) if and only if $j^{\mu\nu} j_{\mu\nu} > 0$. In that case (which is the most important one for applications) there exists a family of reference frames for which $\mathscr{T}$ is a spaceplane, in other words, for which $q^\mu = (\mathbf{q}, t = 0)$ for any τ. Namely, in that case

$$t_1 = t_2 = T = (E/Mc^2)\tau \tag{12}$$

and from (9), $p_\mu = (\mathbf{p}, 0)$ since $\dot{t} = 0$. Consequently, the relative motion is governed by the "Newtonian" equations

$$\dot{\mathbf{q}} = \mathbf{p}/\mu, \qquad \dot{\mathbf{p}} = -\nabla\Phi(|\mathbf{q}|) \tag{13}$$

which follow from (9) in this particular situation.

Thus, in spite of the apparent difficulties due to the spacetime metric, the previous model describes in a satisfactory way what we expect for a relativistic two-body system, and, as we have seen, it is possible, for each Newtonian system with a potential $\Phi(|\mathbf{q}|)$, to construct a completely covariant relativistic model describing the same type of motion. Nevertheless, it is important to note that $m_1{}^2 = -g^{\mu\nu} p_{1\mu} p_{1\nu}/c^2$ and $m_2{}^2 = -g^{\mu\nu} p_{2\mu} p_{2\nu}/c^2$ are generally not constants of motion, nor even conserved quantities in a scattering process.

2. QUANTUM RELATIVISTIC MECHANICS

Most attempts to construct a relativistic quantum mechanics have led only to the case of the free particle. It was initially claimed that the reason for this impasse is the difficulty of controlling all the representations of the

Poincaré group. However, when the same group-theoretical arguments were applied to the Galilean case, exactly the same result was found. The reason for the impasse is clear in this case. It is that the Galilean group has been interpreted as the group of motion from the active point of view, and one is therefore naturally led to a representation for a free particle which is given in the Heisenberg picture.[7]

For the Galilean case, an effort has therefore recently been made to develop a group-theoretical argument which avoids this difficulty. An effective procedure was found to be the following: First construct the set of observables which characterize the system, and build the dynamics only afterwards. With this procedure, one naturally obtains the observables in the Schrödinger picture, which is specifically not dependent on the dynamics. This result is found by studying the action of the Galilean group on the measuring apparatus from the passive point of view, by defining each observable by an imprimitivity system according to Mackey's theory.[8,9] In the Galilean case, this procedure leads to the most general solution compatible with the Galilei principle.

The point of view adopted here in relativistic mechanics naturally leads us to generalize the previous procedure to the relativistic case. As an illustrative example, we construct a model of a relativistic, quantum spinless particle. Later we shall consider the spin-$\frac{1}{2}$ particle and briefly present some applications.

2.1. Quantum Spin-zero Particle

The states of the (spinless) quantum particle are assumed to be described by the vectors (more precisely by the rays) of a Hilbert space H. According to our point of view, a relativistic particle is supposed to have:

(a) A spacetime position observable q characterized by a spectral family $q: B(\mathbb{R}_x{}^4) \to P(H)$ from $B(\mathbb{R}_x{}^4)$, the Borel sets of $\mathbb{R}_x{}^4$, to $P(H)$, the set of orthogonal projections in H.

(b) A momentum–energy observable p characterized by a spectral family $p: B(\mathbb{R}_k{}^4) \to P(H)$ from $B(\mathbb{R}_k{}^4)$, the Borel sets of $\mathbb{R}_k{}^4$, to $P(H)$.

Moreover, it is assumed that there is no other observable not depending on p and q.

These observables p and q are completely characterized by the following group G acting on the corresponding measuring apparatus. The group G is generated by:

(i) The spacetime translations a^μ

$$x^\mu \mapsto x^\mu + a^\mu, \qquad k_\mu \mapsto k_\mu$$

(ii) The momentum–energy translations w_μ

$$x^\mu \mapsto x^\mu, \qquad k_\mu \mapsto k_\mu + w_\mu$$

(iii) The Lorentz transformations $\Lambda^\mu{}_\nu$

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu, \qquad k_\mu \mapsto \Lambda_\mu{}^\nu k_\nu$$

(for any $x \in \mathbb{R}_x{}^4$ and $k \in \mathbb{R}_k{}^4$).

Considered from the passive point of view, the action of G allows us to write the following relations, called "imprimitivity systems," which characterize q and p, respectively. The imprimitivity system based on q imposes that for any $g \in G$ and any $\Delta \in B(\mathbb{R}_x{}^4)$

$$q(g \cdot \Delta) = U(g)q(\Delta)U^{-1}(g) \tag{14}$$

where U denotes a unitary ray representation of G in H and $g \cdot \Delta$ the action of g on $\Delta \in B(\mathbb{R}_x{}^4)$ induced by the action of G on $\mathbb{R}_x{}^4$.

Similarly, the imprimitivity system based on p imposes that for any $g \in G$ and any $\Delta \in B(\mathbb{R}_k{}^4)$

$$p(g \cdot \Delta) = U(g)p(\Delta)U^{-1}(g) \tag{15}$$

where U denotes the same ray representation as in (14) and $g \cdot \Delta$ the action of g on $\Delta \in B(\mathbb{R}_k{}^4)$ induced by the action of G on $\mathbb{R}_k{}^4$.

The solution for the imprimitivity systems (14) and (15) is obtained by taking for the Hilbert space H

$$H = L^2(\mathbb{R}_x{}^4, d^4x)$$

and for U the unitary ray representation in H defined by

$$\begin{aligned} (U(a)\psi)(x) &= \psi(x - a) \\ (U(w)\psi)(x) &= \exp(iw_\mu x^\mu/\hbar)\ \psi(x) \\ (U(\Lambda)\psi)(x) &= \psi(\Lambda^{-1}x) \end{aligned} \tag{16}$$

for any $\psi \in H$. The solution depends on a parameter which turns out to be identified with the Planck constant $\hbar$. The previous representation is a nontrivial ray representation, since it is easy to verify that we have the following generalized Weyl commutation rules:

$$U(w)U(a) = \exp(iw_\mu a^\mu/\hbar)\ U(a)U(w) \tag{17}$$

The observable q is defined by the spectral family $q\colon B(\mathbb{R}_x^4) \to P(H)$ whose projections are given by

$$(q(\Delta)\psi)(x) = \chi_\Delta(x)\psi(x), \qquad \psi \in H \tag{18}$$

where χ_Δ denotes the characteristic function associated to the Borel set Δ.

The spectral family $p\colon B(\mathbb{R}_k^4) \to P(H)$ associated to p is given by

$$\widehat{(p(\Delta)\psi)(k)} = \chi_\Delta(k)\hat{\psi}(k) \tag{19}$$

where $\hat{\psi}(k)$ is obtained by the unitary Fourier transformation, defined on a dense subset by

$$\phi(x) \mapsto \hat{\phi}(k) = (2\pi\hbar)^{-2} \int_{\mathbb{R}_x^4} d^4x \exp(-ik_\mu x^\mu/\hbar)\, \phi(x) \tag{20}$$

Thus, the self-adjoint operators p_μ and q^μ corresponding to p and q are respectively given by

$$(p_\mu\psi)(x) = -i\hbar\, \partial_\mu\psi(x), \qquad (q^\mu\psi)(x) = x^\mu\psi(x) \tag{21}$$

and these operators satisfy the Heisenberg commutation relations

$$i[p_\mu, q^\nu] = \hbar\, \delta_\mu^\nu\, \mathbb{1} \tag{22}$$

The evolution of such a relativistic particle is parametrized by the historical time τ and governed by a Schrödinger equation

$$i\hbar\, d_\tau\psi_\tau = K\psi_\tau \tag{23}$$

where K is a self-adjoint operator, which is, moreover, of scalar type to guarantee covariance.

We have thus obtained a description of the particle in the Schrödinger picture (relative to τ), and dynamical considerations are related to the choice of K.

As an illustrative example, we consider a charged particle in an external electromagnetic field described by the 4-vector potential $A_\mu(x)$. By neglecting radiation phenomena we have

$$K = (1/2M)g^{\mu\nu}[p_\mu - eA_\mu(q)][p_\nu - eA_\nu(q)] \tag{24}$$

where e and M denote the charge and the mass of the particle. We easily verify that

$$\begin{aligned} \dot{q}^\mu &\equiv (i/\hbar)[K, q^\mu] = (1/M)[p^\mu - eA^\mu(q)] \\ \dot{p}_\mu &\equiv (i/\hbar)[K, p_\mu] \\ &= (e/2M)[\partial_\mu A_\nu(q), p^\mu - eA^\mu(q)]_+ \end{aligned} \tag{25}$$

In the case of a free particle where

$$K = (1/2M)g^{\mu\nu}p_\mu p_\nu = (-\hbar^2/2M)\,\Box \tag{26}$$

more comments can be made. First, the usual relativistic theory for spinless particles, more precisely the Klein–Gordon equation, is interpreted here as an eigenstate equation

$$K\psi(x) = -\tfrac{1}{2}Mc^2\psi(x) \tag{27}$$

This fact suggests a change of representation: the four-dimensional Fourier transform (20) followed by a change of variables given by

$$\hat{\psi}(k) \mapsto f(\mathbf{k}, m) = \frac{|\,m\,|^{1/2}c^{3/2}}{(\mathbf{k}^2 + m^2c^2)^{1/4}}\,\hat{\psi}(\mathbf{k}, -c\,\mathrm{sign}(m)\,(\mathbf{k}^2 + m^2c^2)^{1/2}) \tag{28}$$

the corresponding scalar product being

$$\langle g, f\rangle = \int_{-\infty}^{+\infty} dm \int_{\mathbb{R}^3} d^3k\; g^*(\mathbf{k}, m)\, f(\mathbf{k}, m) = \langle\phi, \psi\rangle \tag{29}$$

Let us now consider the observable

$$\mathbf{q}_r = \mathbf{q} - \frac{1}{2}\left(\frac{\mathbf{p}}{E}\,t + t\,\frac{\mathbf{p}}{E}\right) \tag{30}$$

It is easy to verify the following commutation rules:

$$i[q_r^{\,i}, q_r^{\,k}] = 0, \qquad i[p^i, q_r^{\,k}] = \hbar\,\delta^{ik}\,\mathbb{1} \tag{31}$$

Moreover, $\mathbf{q}_r$ is a constant of motion

$$[K, \mathbf{q}_r] = 0 \tag{32}$$

Decomposed in the previous new representation, $\mathbf{q}_r$ turns out to be just $i\hbar\partial/\partial\mathbf{k}$, i.e., the Newton–Wigner position operator.

2.2. Quantum Spin-$\frac{1}{2}$ Particle[11]

As previously for the spinless particle, a particle with spin is supposed to have a spacetime position observable q and a momentum–energy observable p. Moreover, it is assumed to possess some new observables compatible (commuting) with p and q.

According to the Mackey theory, this situation is obtained by describing the states of the particle with N-component wave functions $\psi \in \mathbb{C}^N \otimes L^2(\mathbb{R}_x^4, d^4x)$, the scalar product being written

$$\langle \psi, \phi \rangle = \int_{\mathbb{R}_x^4} d^4x \sum_{i=1}^{N} \psi_i^*(x)\phi_i(x) = \int_{\mathbb{R}_x^4} d^4x \; \psi^+(x)\phi(x) \tag{33}$$

The Lorentz group acts on the states by unitary operators $U(\Lambda)$ of the following form:

$$(U(\Lambda)\psi)(x) = D(\Lambda)\psi(\Lambda^{-1}x) \tag{34}$$

where D is a nontrivial, irreducible, unitary ray representation of the Lorentz group (acting in the space components of ψ). Such a representation is necessarily infinite-dimensional and consequently implies the existence of an infinity of compatible internal states for the particle (infinite spin). This fact is in disagreement with experiments and the well-established double degeneracy for electrons in atoms or metals.

This difficulty disappears when one introduces a (continuous) superselection rule for the spin-$\frac{1}{2}$ particle, i.e., a family of Hilbert spaces.[12]

Let us consider the Stern–Gerlach apparatus defining the spin observable and more precisely the symmetry of the magnetic field of this apparatus. Such a magnetic field is characterized by a strong gradient. It defines not only the space direction of the spin, but also a unique timelike direction, the direction of the time given by the frame where the field is purely magnetic. Then the spin state of the particle is characterized by a direction in space (the spin) and a timelike 4-vector n^μ such that $n^4 > 0$ and $g_{\mu\nu}n^\mu n^\nu = -c^2$.

We postulate that this 4-vector n^μ is a superselection rule and we introduce a family of Hilbert spaces H_n indexed by n. To every timelike unit 4-vector n^μ we associate a Hilbert space H_n of two-component wave functions which is identical to $\mathbb{C}^2 \times L^2(\mathbb{R}_x^4, d^4x)$. In other words, each spin-$\frac{1}{2}$ particle state is characterized by a given n and a given $\psi_n \in H_n$.

The Lorentz group acts on the states in the following way[13]:

$$(U(\Lambda)\psi)_n(x) = D(L^{-1}(n)\Lambda L(\Lambda^{-1}n))\psi_{\Lambda^{-1}n}(\Lambda^{-1}x) \tag{35}$$

where $L(n)$ are boosts such that $L(n)^\mu{}_\nu n_0^\nu = n^\mu$ with $n_0^\mu = (0, 0, 0, 1)$ and D is the usual 2×2 unitary ray representation of the rotation group corresponding to a spin-$\frac{1}{2}$. Thus a Lorentz transformation Λ maps every H_n onto $H_{\Lambda n}$ by (35).

As in the spinless case, the self-adjoint operators corresponding to p and q are respectively given by

$$(p_\mu\psi)(x) = -i\hbar\, \partial_\mu\psi(x), \qquad (q^\mu\psi)(x) = x^\mu\psi(x) \tag{36}$$

for any ψ in any H_n. Obviously the corresponding spectral families verify the imprimitivity relations (14) and (15) (extended to the case of a superselection rule n) for the representation U of G given by (35) and

$$\begin{aligned}(U(a)\psi)_n(x) &= \psi_n(x-a)\\ (U(w)\psi)_n(x) &= \exp(iw_\mu x^\mu/\hbar)\ \psi_n(x)\end{aligned} \tag{37}$$

The spin observable, for $n = n_0$, is given by the four matrices

$$W^i_{n_0} = \tfrac{1}{2}\sigma^i, \quad i = 1, 2, 3 \qquad \text{and} \qquad W^4_{n_0} = 0 \tag{38}$$

where σ^i are the Pauli matrices. For any n we can define

$$W^\mu_n = L(n)^\mu{}_\nu W^\nu_{n_0} \tag{39}$$

and consequently $n_\mu W^\mu_n = 0$. This definition is justified by the following physical considerations: Let us consider a unit spacelike 4-vector s^μ,

$$g_{\mu\nu}s^\mu s^\nu = 1$$

such that $g_{\mu\nu}s^\mu n^\nu = 0$ and let us consider the observable defined by the operator

$$s_\mu W^\mu_n$$

The corresponding two eigenstates are associated with a measurement of the spin with a Stern–Gerlach apparatus whose (i) time direction is given by n^μ and (ii) direction of the magnetic field gradient is given by s^μ, i.e., $(\mathbf{s}_0, 0) = L^{-1}(n)s$ in the reference frame of the apparatus. This follows immediately from the relation

$$s_\mu W^\mu_n = s_\mu L(n)^\mu{}_\nu W^\nu_{n_0} = \tfrac{1}{2}\mathbf{s}_0\boldsymbol{\sigma}$$

As regards the covariance, it is easy to verify from (35) and from the definition (39) that

$$U^{-1}(\Lambda)W^\mu_{\Lambda n}U(\Lambda) = \Lambda^\mu{}_\nu W^\nu_n \tag{40}$$

as expected. Finally, the W^μ_n satisfy the following commutation rules:

$$[W^\mu_n, W^\nu_n] = i\epsilon^{\mu\nu}{}_{\rho\lambda}W^\rho_n n^\lambda \tag{41}$$

where $\epsilon_{\mu\nu\rho\lambda} = \pm 1$ for $\mu\nu\rho\lambda$ an even or odd permutation of 1, 2, 3, 4, respectively, and $\epsilon_{\mu\nu\rho\lambda} = 0$ otherwise.

Because of the existence of a superselection rule n, the evolution in τ is governed by a Schrödinger equation

$$i\hbar\, d_\tau \psi_\tau = K_{n(\tau)} \psi_\tau \tag{42}$$

connected with an equation of the form

$$\dot{n}_\tau^\mu = f^\mu(n_\tau, \psi_\tau) \tag{43}$$

where the f^μ fulfil the condition $n_\mu f^\mu(n, \psi) = 0$.

Due to the dependence on ψ_τ, Eq. (43) involves irreversible processes. In other respects, physically, the comparison of this model with the Dirac theory[11] suggests that the evolution is such that n_τ^μ tends to be parallel to $\langle p^\mu \rangle_{\psi_\tau}$, the mean value of the momentum–energy. In many cases and various applications, both directions can be assumed to be effectively parallel during the evolution.

2.3. Spin-$\frac{1}{2}$ Particle Interacting with an External Electromagnetic Field

Let us consider the case of the spin-$\frac{1}{2}$ particle in an external electromagnetic field $A_\mu(x)$, neglecting radiation phenomena. The operator K_n is given by the corresponding one for the spin-0 case, modified by the terms due to the interaction of the spin with the electromagnetic field. Especially for the electron (or positron) we propose

$$\begin{aligned} K_n = {} & (1/2M) g^{\mu\nu}[p_\mu - eA_\mu(q)][p_\nu - eA_\nu(q)] \\ & - (g_1 \mu_0/Mc^2)[p^\mu - eA^\mu(q)]\tilde{F}_{\mu\nu}(q) W_n^\nu \\ & + (g_2{}^2 \mu_0{}^2/8Mc^4) F_{\mu\nu}(q) n^\nu F^\mu{}_\rho(q) n^\rho - (g_3 \mu_0/c^2) n^\mu \tilde{F}_{\mu\nu}(q) W_n^\nu \end{aligned} \tag{44}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}^{\mu\nu} = -\frac{1}{2}c^2 \epsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}$. The charge and the mass of the particle are e and M; $\mu_0 = e\hbar/2M$ denotes the Bohr magneton; and g_1, g_2, and g_3 are dimensionless phenomenological constants.

As a first application we consider the evolution of the spin in an electromagnetic field that is not necessarily homogeneous. More precisely, from the expression (44) for K_n and from the definition of W_n^μ, we can write an expression for the derivative $\dot{W}_n^\mu$ (for more details see Ref. 11). We have

$$\dot{W}_n^\mu = (i/\hbar)[K_n, W_n^\mu] + n^\mu(\dot{n}_\mu W_n^\mu)/c^2 \tag{45}$$

where the last term corresponds to the contribution of the evolution of n. A straightforward calculation of the commutator in (45) leads to the following expression:

$$\begin{aligned}\dot{W}_n^\mu = {} & (g_1\mu_0/\hbar Mc^2)\{n^\rho F_{\rho\lambda}(q)W_n^\lambda[p^\mu - eA^\mu(q)] \\ & - F^\mu{}_\lambda(q)W_n^\lambda n_\rho[p^\rho - eA^\rho(q)] \\ & + F^\mu{}_\lambda(q)n^\lambda W_{n\rho}[p^\rho - eA^\rho(q)]\} \\ & + (g_3\mu_0/\hbar c^2)\{n^\rho F_{\rho\lambda}(q)W_n^\lambda n^\mu + c^2F^\mu{}_\lambda(q)W_n^\lambda\} + n^\mu(\dot{n}_\nu W_n^\nu)/c^2 \end{aligned} \quad (46)$$

From this general expression of $\dot{W}_n^\mu$, we justify the BMT equation[14] as a semiclassical approximation.

Let us consider a state corresponding to a wave packet around the mass shell, sharply defined in spacetime and momentum–energy and a given direction of spin. For such a state we can approximate $\langle F_{\mu\nu}(q)\rangle$ by $F_{\mu\nu} = F_{\mu\nu}(\langle q\rangle)$. For the time derivative of q^μ we have

$$\dot{q}_n^\mu = \frac{i}{\hbar}[K_n, q^\mu] = \frac{1}{M}\left[p^\mu - eA^\mu(q) - \frac{g_1\mu_0}{c^2}\tilde{F}^\mu{}_\nu(q)W_n^\nu\right]$$

and we can suppose that

$$d_\tau\langle q^\mu\rangle = \langle\dot{q}_n^\mu\rangle \cong (1/M)\langle p^\mu - eA^\mu(q)\rangle \cong n^\mu \quad (47)$$

for any τ, according to the previous considerations about the evolution of n^μ. Moreover,

$$\dot{n}^\mu \cong d_\tau\langle\dot{q}_n^\mu\rangle \cong \left\langle\frac{i}{\hbar}\left[K_n, \frac{p^\mu - eA^\mu(q)}{M}\right]\right\rangle \cong \frac{e}{M}F^\mu{}_\nu n^\nu \quad (48)$$

Performing these approximations in (46) and reordering terms, we finally find the following equation, which is the BMT equation for the precession of the polarization of a particle moving in an electromagnetic field:

$$\dot{W}_n^\mu = (\mu_0/\hbar)\{(g_1 + g_3)F^\mu{}_\nu W_n^\nu + (g_1 + g_3 - 2)n^\mu n^\rho F_{\rho\nu}W_n^\nu\} \quad (49)$$

In view of this result, $g_1 + g_3 = g$ is interpreted as the g-factor including the anomalous magnetic moment.

As a second application, we consider the hydrogen atom (or hydrogenlike atom) considered as an electron interacting with the electromagnetic field of a nucleus, given by[15]

$$A_\mu(x) = \left(0, 0, 0, -\frac{Z(-e)}{4\pi\epsilon_0}\frac{1}{r}\right), \qquad r = |\mathbf{x}| \quad (50)$$

where ϵ_0 denotes the vacuum dielectric constant. $Z(-e)$ denotes the charge of the nucleus, e being the electron charge.

Roughly speaking, the states of the electron corresponding to the spectrum are those states described by wave packets moving in spacetime with the nucleus: i.e., moving along the time axis and staying close to it. Then, according to the previous considerations in Section 2.2 about the evolution, we must consider states such that $n = n_0$. More precisely, the spectrum of the atom is all the values of $E - Mc^2$ corresponding to solutions of the eigenvalue equations

$$K_{n_0}\psi(x) = -\tfrac{1}{2}Mc^2\psi(x) \qquad \text{and} \qquad n_0^{\mu} p_{\mu}\psi(x) = -E\psi(x) \tag{51}$$

which are "localized" around the time axis. For the given electromagnetic field in (50) we have

$$\begin{aligned} K_{n_0} = \frac{1}{2M}\left\{\mathbf{p}^2 - c^2\left[p^4 - \frac{e}{c^2}V(q)\right]^2\right\} \\ + \frac{g_1\mu_0}{2Mc^2}[\mathbf{p} \wedge \mathbf{E}(q)]\boldsymbol{\sigma} + \frac{g_2{}^2\mu_0{}^2}{8Mc^4}\mathbf{E}^2(q) \end{aligned} \tag{52}$$

In this expression $\mathbf{E}$ denotes the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{Z(-e)}{4\pi\epsilon_0}\frac{\mathbf{x}}{r^3}$$

Obviously K_{n_0} does not depend on q^4 and thus it commutes with $n_0^{\mu}p_{\mu} = p_4 = -i\hbar\,\partial_t$. Consequently, the solutions of (51) are of the form

$$\psi(x) = \exp(-iEt/\hbar)\,\varphi(\mathbf{x}) \tag{53}$$

where $\varphi(\mathbf{x})$ is a solution of

$$K_{n_0}^{(E)}\varphi(\mathbf{x}) = -\tfrac{1}{2}Mc^2\varphi(\mathbf{x}) \tag{54}$$

the restriction of (51) to the spectral subspace of p_4 corresponding to E. Taking into account the explicit form of $V(x)$ and $E(x)$ we have

$$\begin{aligned} K_{n_0}^{(E)} = \frac{1}{2Mc^2}\left[c^2\mathbf{p}^2 - \left(E + \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}\right)^2\right] \\ - \frac{g_1\mu_0}{2Mc^2}\frac{Ze}{4\pi\epsilon_0}\left(\mathbf{p} \wedge \frac{\mathbf{x}}{r^3}\right)\boldsymbol{\sigma} + \frac{g_2{}^2\mu_0{}^2}{8Mc^4}\frac{Z^2e^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4} \end{aligned} \tag{55}$$

Finally the spectrum is identified with the values of $E - Mc^2$ such that there exist solutions of (53) in $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3, d^3x)$. Since K_{n_0} and p_4 are invariant under rotations, they commute with the total angular momentum operators $\mathbf{J} = \mathbf{L} + \hbar\boldsymbol{\sigma}/2$, where $\mathbf{L} = \mathbf{q} \wedge \mathbf{p}$ are the orbital angular momentum operators. Moreover, the operator $\mathbf{L}^2$ also commutes with K_{n_0} and p_4. Consequently we can determine solutions of (53) of the form (in spherical coordinates r, θ, ϕ)

$$\varphi(r, \theta, \phi) = R(r) Y^m_{j,l}(\theta, \phi) \tag{56}$$

where $Y^m_{j,l}(\theta, \phi)$ denotes the usual angular eigenfunctions of $\mathbf{J}^2$, $\mathbf{L}^2$, J_z, and $R(r) \in L^2(\mathbb{R}_+, r^2\, dr)$. Thus, in contradistinction to the Dirac model, the orbital angular momentum l is here a good quantum number. We also point out that according to the symmetries of the solutions we have

$$\int_{\mathbb{R}^3} d^3x\; \varphi^+(\mathbf{x})\mathbf{p}\varphi(\mathbf{x}) = 0$$

This means that the corresponding wave packet around the mass shell, built up from the solutions of (51), represents a particle moving in spacetime along the time axis, as expected.

The corresponding radial equation for $R(r)$ in (56) is obtained from (54) and (55) to be

$$\Big[\frac{-\hbar^2}{2M}\frac{1}{r^2}d_r r^2 d_r + \frac{\hbar^2}{2M}\frac{l(l+1)}{r^2} - \alpha^2 E\frac{a_0}{r} - \alpha^4\frac{Mc^2}{2}\frac{a_0^2}{r^2} + \alpha^4\frac{g_1 Mc^2}{4}c(j,l)\frac{a_0^3}{r^3} + \alpha^6\frac{g_2^2 Mc^2}{32}\frac{a_0^4}{r^4} - \frac{E^2 - M^2c^4}{2Mc^2}\Big]R(r) = 0 \tag{57}$$

Here $\alpha = e^2/4\pi\epsilon_0\hbar c$ denotes the fine structure constant, $a_0 = 4\pi\epsilon_0\hbar^2/Me^2$ the Bohr radius, and $c(j, l) = l - 1$ or l according to whether $l = j + \frac{1}{2}$ or $j - \frac{1}{2}$. Formally, this radial equation looks like the corresponding one in the nonrelativistic case where the particle interacts with the singular potential

$$-\alpha^2 E\frac{a_0}{r} - \alpha^4\frac{Mc^2}{2}\frac{a_0^2}{r^2} + \alpha^4\frac{g_1 Mc^2}{4}c(j,l)\frac{a_0^3}{r^3} + \alpha^6\frac{g_2^2 Mc^2}{32}\frac{a_0^4}{r^4}$$

and one can show that the exact solutions of (57) in $L^2(\mathbb{R}_+, r^2\, dr)$ behave near the origin as

$$R(r) = \exp(-\alpha^2 g_2 a_0/4r)\; O(r^{(g_1/g_2)c(j,l)}) \tag{58}$$

for $r \to 0$.

The spectrum, i.e., the values of $E - Mc^2$ such that there exist solutions of (57) in $L^2(\mathbb{R}_+, r^2\, dr)$, has been calculated in terms of the power expansion in α^2 up to terms in α^4. The following results have been obtained.

For $l = 0$

$$E - Mc^2 = -Mc^2 \left\{ \frac{Z^2\alpha^2}{2n^2} + \frac{Z^4\alpha^4}{2n^4} \left[n - \frac{3}{4} - n(g_2 - 1) \right] + O(\alpha^6) \right\} \quad (59)$$

where n, the principal quantum number, takes positive integer values.

For $l = j \pm \frac{1}{2} \neq 0$ we respectively have

$$E - Mc^2 = -Mc^2 \left\{ \frac{Z^2\alpha^2}{2n^2} + \frac{Z^4\alpha^4}{2n^4} \times \left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \pm \frac{n(g_1 - 1)}{(2j + 1)(j + \frac{1}{2} \pm \frac{1}{2})} \right) + O(\alpha^6) \right\} \quad (60)$$

where n takes integer values strictly greater than l.

The previous results need some comments. First, for $g_1 = g_2 = 1$ the spectrum coincides (at least up to terms in α^4) with the corresponding spectrum of the Dirac model and exhibits the well-known degeneracy in l.

For g_1 and g_2 not equal to one, expansions (59) and (60) differ (up to terms in α^4) from the Dirac spectrum by energy shifts of the form

$$\frac{Mc^2Z^4\alpha^4}{2n^3}(g_2 - 1) \qquad \text{for} \quad l = 0 \quad (61)$$

and

$$\mp \frac{Mc^2Z^4\alpha^4}{2n^3} \frac{g_1 - 1}{(2j + 1)(j + \frac{1}{2} \pm \frac{1}{2})} \qquad \text{for} \quad l = j \pm \tfrac{1}{2} \neq 0 \quad (62)$$

Such terms remove the degeneracy relative to l and then contribute to the Lamb shifts. Usually the Lamb shifts are obtained as radiative corrections to the Dirac results. The correcting energy shift terms obtained by such a procedure[16] may be compared with the above energy shifts (61) and (62). Expressions of both types are rather similar, particularly in what concerns their n and Z dependence.

Actually for $g_1 = g - 1 = 1.0023$ and $g_2 = 1.048$, our spectrum exhibits Lamb shifts with good numerical agreement.

For the ${}^nS_{\frac{1}{2}} - {}^nP_{\frac{1}{2}}$ energy separations we have the expression

$$\frac{\alpha^4Mc^2}{2} \frac{Z^4}{n^3} \left(g_2 - 1 + \frac{g_1 - 1}{3} \right), \qquad n \geqslant 2$$

which corresponds to the frequency $8.464Z^4/n^3$ [GHz]. The predicted numerical values for different n and Z are compared with the experimental results in Table I.

Table I[a]

	Theory			Experiment		
n	$Z = 1$	$Z = 2$	$Z = 3$	^1H	$^4He^+$	$^6Li^{2+}$
2	1.058	16.92	85.7	1.058[17]	14.04[17]	63.0[17]
3	0.314	5.04	—	0.315[18]	4.18[19]	—
4	0.132	2.11	—	0.133[18]	1.76[20]	—
5	0.068	1.08	—	0.065[18]	0.90[21]	—

[a] Results in GHz.

For the above cases ($j = \frac{1}{2}$) the linewidth is always smaller than the Lamb shift, but for $j \geqslant 3/2$ it is larger than the energy splitting in our spectrum. Nevertheless, we can compare our results for $j > \frac{1}{2}$ with very accurate experimental results for the ${}^nP_{\frac{3}{2}} - {}^nP_{\frac{1}{2}}$ energy separations in the hydrogen atom.

From (60) it is given by

$$(\alpha^4 Mc^2/4n^3)g_1$$

which corresponds to the frequency $87.786/n^3$ [GHz]. We have in [GHz]

n	Theory	Experiment
2	10.973	10.969[22]
3	3.251	3.248[22]

Finally it is important to note that the experimental determination of the Lamb shift is sometimes extrapolated from the Zeeman splitting of the energy levels for different magnetic fields. In our model, the dominating terms for the Zeeman effect are the usual relativistic ones (with $g_1 + g_3 = g$). Hence, we are led to the same interpretation of that experiment as the usual one.

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