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FOUNDATIONS OF CONDITIONAL LOGIC*

1. FOUNDATIONAL ISSUES

Conditional statements occupy a central place in reasoning, and hence their proper analysis is a principal task of logic. Now, ever since material implication was proposed, and found wanting as an explication, new analyses of conditionality have been put forward by logicians and philosophers. The resulting variety of formal explications itself raises several background questions, and it is to this foundational theme that the present paper is devoted.

General questions concerning existing semantic accounts of the meaning of conditionals are exemplified by the following. About the language of conditional statements, should conditionality be treated as an operation upon propositions, or rather as a relation between these? As for the semantic apparatus, how can one judge the need for, or the relative merits of the various types of model and truth definition proposed in the literature? Finally, with respect to the 'logical evidence', what is the status of the intuitions of validity, often invoked as a touch-stone for the conditional logic resulting from some particular analysis?

These are issues which may give rise to lively, but also inconclusive philosophical debate. For instance, operational and relational views of conditionals both have their adherents, and some people even entertain both, to the point of confusing object-language and meta-language of their formalization. (This is the familiar criticism of C. I. Lewis' account of entailment; mentioned, e.g., in Scott (1971).) To mention another example, the validity of a principle such as Conditional Excluded Middle ('if X, then Y, or, if X, then not Y') has strong intuitive support, but also provokes grave doubts . . . sometimes within the same observer.

What we need, then, is a general unifying perspective, enabling us to arrive at more definite issues and results. Definite, not in the sense of a universal settling of old scores, but of establishing the true logical relations between various options. For instance, argument about the validity of some

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specific inference will become pointless on the view to be developed here: the truly fruitful enterprise being a dispassionate exploration of the ranges of conditionals validating proposed patterns of deduction.

In this paper, conditional statements 'if X, then Y' will be analyzed using the generalized quantifier perspective of contemporary formal semantics. (Cf. Barwise and Cooper (1981) for a general exposition – though one not concerned with conditionals.) More specifically, the conditional particle *if* will denote a generalized quantifier *relation between sets of antecedent and consequent 'occasions'* (denoted by 'X', 'Y', respectively; cf. Section 2). Evidently, not all such relations qualify as relations of conditionality, and hence various intuitive constraints will be explored (Section 3); both more general properties of logical constants and more specific ones concerning conditionality. The range of conditionals remaining within these constraints is then investigated, for the most natural case of finite universes of occasions (Section 4). Only three candidates remain: consensus ('all X are Y'), democracy ('half or more X are Y'), and anarchy ('some X is Y'). Three major escape routes out of this extremely restricted area are charted (Section 5), viz. admission of infinite sets of occasions (explored in Section 6), introduction of probability measures on sets of occasions (Section 7), or postulating 'hidden variables', in the form of some relevant hierarchy among possible occasions (Section 8) – the latter being the preferred route in current conditional theories. Even so, many constraints remain, and the new range of admissible truth definitions for conditionals may be explored (Section 9). Finally, we turn from this perspective of *a priori* semantic options toward the logical evidence. Some major 'conditional logics' are analyzed in the above light (Section 10), an analysis which may be extended to arbitrary patterns of conditional inference (Section 11).

One striking feature of the present perspective is its integrative power. As this research developed, apparently unrelated contributions by such diverse authors as De Finetti, Scott and D. Lewis all found their natural place in this scheme. Many further illustrations had to be omitted, for reasons of exposition – notably the existence of unmistakeable connections with traditional, pre-Fregean views of conditionality. Even so, it is not a claim of this paper that the generalized quantifier perspective is the uniquely correct one for the study of conditionals. One important limitation ought to be mentioned at the outset. Broadly speaking, there are two major directions in logical studies of conditionality; one 'vertical', having to do with iterations of conditionals and the resulting implicative relations, another 'horizontal', concerned with the interaction between single conditionals and the Boolean connectives *and*, *or*, *not*. The former direction is most prominent in the modal 'entailment' tradition (cf. Hughes and Creswell, 1968), the latter in the study of counterfactuals and related topics (cf. D. Lewis, 1973). Our approach is partial to the horizontal direction, for reasons explained in Section 2. Extensions to the vertical case are mentioned occasionally, without being developed in depth.

In addition to the above perspective itself, this paper contains quite a few new results, mainly concentrated in Sections 4, 10 and 11. But its major intended contribution is a broadening of current logical horizons. If many of the themes in this paper look unfamiliar to the reader of the current literature in philosophical logic, then all the better. Logical semantics should generate technical issues of its own, in addition to importing traditional concerns from mainstream logic, such as the ubiquitous quest for completeness theorems. What we want eventually is a general logical understanding of possible formal semantics, and it is this foundational theme which pervades the following pages.

2. CONDITIONALS AS GENERALIZED QUANTIFIERS

Throughout the subject of logic, one finds two views of conditionals: sometimes implication is a mere connective, then again it is taken to express a relation between propositions. The tendency is of long standing. Thus for instance, Immanuel Kant listed 'hypothetical propositions' under the heading of 'Relation' in his famous Table of Categories. Again, both points of view occur intermingled in C. I. Lewis' account of his intended 'strict implication'. If this is a confusion, as the canonical textbook exposition has it, it is a remarkably tenacious one – a phenomenon which itself requires explanation.

In order to arrive at a logical perspective which does justice to both points of view, one can take a cue from ordinary language. Unlike coordinating connectives such as *and*, *or*, the conditional particle *if* functions in subordinate constructions

$$(\textit{if}\ X)\ Y;$$

where, categorially, '*if X*' operates as a sentence modifier. As is usual in such linguistic contexts, the full denotation of the expressions '*X*', '*Y*' may be involved – i.e., the ranges of occasions ('worlds', 'situations', 'models') where these are true, not just a truth value on some specific occasion. Thus, the force of a conditional particle may be compared with that of ordinary determiner expressions such as *all*, *most* (e.g., (*all X*) *Y*), which exhibit similar linguistic behaviour. (The precise nature and extent of this linguistic analogy need not be explored here, as no claims will be staked on it. To mention just one possibly fruitful parallel, the particle *then* seems to function very much like an anaphoric pronoun.)

Now, determiners are being studied in contemporary linguistic semantics through the logical notion of a 'generalized quantifier' (cf. Barwise and Cooper, 1981; van Benthem, 1983 (a, b)), with various illuminating effects. For present purposes, the relevant point in these investigations is their methodological perspective. Logical determiners, such as *all*, *most* or *some*, correspond to functions mapping sets $[\![X]\!]$ (i.e., the denotation of '*X*') to sets of sets of individuals, viz. the extensions of the predicates to which the relevant subject term ('all *X*', 'most *X*', 'some *X*') applies. Equivalently, yet more suggestively, determiners correspond to structural relations between extensions of predicates, such as *inclusion* or *overlap*. This leads to an easy visualization, in terms of the familiar Venn diagrams. Now conversely, not all relations between sets of individuals qualify as denotations for (logical) determiners; but there is a lot of interest to the study of just which ones do.

We propose to apply the same perspective to conditional statements

$$\textit{if } X, (\textit{then}) \; Y;$$

regarded as expressing some semantic relation

$$[\![\textit{if}]\!]([\![X]\!], [\![Y]\!])$$

between the sets of antecedent and consequent occasions. (Henceforth, this relation will be written without denotation brackets; an innocent abuse of notation, with heuristic virtues.) Again, the principal task of a logical investigation will be to delimit a range of suitable conditional relations between sets of occasions. Further questions will then arise in due course.

More precisely, on the structural, non-linguistic side, we shall be concerned with *generalized quantifiers*, viewed as functors *F* assigning, to each universe of discourse E, some binary relation F_E between subsets of E. Thus typically, for a universe E of 'relevant' occasions, and $A, B \subseteq E$,

$$if_E A, B$$

will mean that the conditional relation holds in E between A and B. (The letters 'A', 'B' will be used henceforth when referring to sets in models, without a specific formal language in mind.) Examples are inclusion (all A are B), majority (most A are B) or overlap (some A is B). For a context-dependent example, where the universe E is essential, consider '(relatively) *many*', meaning that the proportion of B-worlds in A exceeds that of the B-worlds in the whole universe E. In Section 3, postulates will be introduced eliminating such a contextual dependence for our conditionals; but the above scheme gives the general pattern for natural language determiners.

In the following sections, the uses of the present approach will become clear. Here, we want to point at some of its peculiarities. First, the view of conditionals as relations between sets of occasions would seem to favour *generic* conditional statements over *individual* ones. The former refer to sets of events, as in

'If (i.e., whenever) she comes, she quarrels'.

The latter are about particular events, however, as in

'If he came, he cried'.

Our view is that both statements presuppose variety of occasions. The first is about several events in one world, the second about one event in several possible worlds. The present broad concept of 'occasion' is meant to include both, as well as combinations of the two. Against this general background, specific choices of relevant universes of occasions E may account for particular kinds of conditionals. For instance, the location of some distinguished 'actual world' in E may be important in the treatment of the contrast between indicative and subjunctive conditionals. (Indeed, E itself may consist of some set of world-lines connected with that actual world.) In this paper, however, the abstract common pattern is the central concern, while such further specifications are left to specific applications.

Another notable aspect of the generalized quantifier approach is that iterated conditionals become awkward to handle. This reflects the fact that natural language has no direct means of iterating determiner expressions.

Thus, this paper is almost exclusively (though not irrevokably) concerned with simple conditionals, in the earlier-mentioned 'horizontal' tradition. In itself, this restriction need not be a defect. There is a well-attested danger of facile logical formalisms leading us into iteration that just is not there in ordinary speech. For instance, on the causal reading of conditional relations between (sets of) events, iterated conditionals do not make sense; unless the second layer is interpreted in a different spirit. And in fact, the latter meaning shift is also present in standard examples of iteration, such as

'If this glass breaks if hit, it will break'.

To stress all these points, the notation to be used here differs from the ordinary use of arrows. We study assertions of the form

$$if\,XY;$$

where 'X', 'Y' may be complex, but conditional-free Boolean terms involving $\neg$, $\wedge$, $\vee$. Most of the (in our opinion) fundamental types of conditional inference are already statable at this level, witness the discussion in Sections 10 and 11. Thus, one can study schemata of the form

$$\frac{if\,X_1 Y_1 \quad \cdots \quad if\,X_n Y_n}{if\,XY}.$$

Examples are transitivity (from *if* XY and *if* YZ to *if* XZ) or monotonicity (from *if* XY to *if* $X(Y \vee Z)$). A pattern like Conditional Excluded Middle however (*if* XY *or if* $X \neg Y$), would call for the addition of Boolean combinations of conditional assertions themselves. Essentially, this would now also allow disjunctive conclusions in patterns of inference.

From a more general logical point of view, the preceding patterns of inference may be viewed as universal statements in a first-order language with variables $X, Y, Z, \ldots$, Boolean term operations and one binary relation symbol *if*. One further direction of enquiry, not usually pursued in logical semantics, would be to study the entire range of first-order assertions about conditionality. Another might consist in adding further binary relations between sets of occasions, such as inclusion, and studying their interplay with conditionality. This paper is devoted to the earlier simple case, however.

It remains to be noted that iterations may be introduced after all, once the conditional relation is provided with an additional parameter:

$$if_{E,w}A, B;$$

i.e., *if* A, B as seen from the vantage point of some particular world w in E. Through lambda abstraction, conditional statements can then be made to correspond to sets of worlds, which may again be used as arguments A, B. Much of the following investigation can be transferred to this setting without any major changes – but the earlier more austere presentation has been preferred for its simplicity and elegance.

3. INTUITIONS OF CONDITIONALITY

What kind of generalized quantifier is a conditional? Before passing on to the usual display of paradigmatic (non-)-inferences, let us reflect more deeply. Our intuitions come in various kinds, and it is important to consider the more volatile ones first, concerning the *kind* of notion that we are after, before these are drowned in a list of very specific desiderata. Only in the light of such background intuitions, one can take a proper look at more concrete claims of validity or invalidity of conditional inferences.

The difference may also be illustrated by an example from a different field of semantics. In the logical study of Time, attention is often restricted to the choice of specific axioms for the temporal precedence order matching certain desired validities in the tense logic. But, there as well, there exist preliminary global intuitions, such as 'anisotropy' or 'homogeneity', constituting the texture of our idea of Time, constraining rather than generating specific relational conditions.

Indeed, the ill repute of the term 'intuition' may be partly due to a misapplication. It is highly unlikely that intuition would settle such specific issues as the validity of concrete inference schemata. An appeal to intuitions in discussions of the latter type often amounts to a refusal to argue about the evidence. On the other hand, the proper place for intuition would seem to be at the level of the general structure of our concepts – in the spirit of Kant's philosophy. To paraphrase this great philosopher, we have certain *a priori* intuitions concerning the basic logical notions, and no human mind is entirely without them.

Global intuitions themselves come in various kinds, having different levels of generality. The following principles will illustrate this. The first postulate is very specific for conditionals, the next holds for determiner expressions in general, and finally some constraints are imposed on logical constants as such.

CONFIRMATION

A conditional statement *if XY* claims that 'significantly many' ('enough') X-occasions are Y-occasions. As such, it is tied up closely with what might be called 'positive' or 'negative' evidence; i.e., cases where X and Y hold, or cases where X holds without Y, respectively (cf. Figure 1). Briefly, our intuition is that addition of positive evidence, or removal of negative evidence will not affect a true conditional assertion.

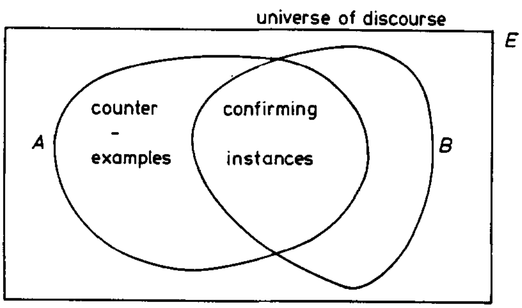

Fig. 1.

Out of the various ways to make this idea more concrete, here is one suggestive formulation, which leads to several structural constraints upon conditional relations $if_E A, B$. Suppose that one decides that *if* X, Y is true in E on the basis of partial information about the extensions $[\![X]\!]$, $[\![Y]\!]$, say $[\![X]\!] = A$, $[\![Y]\!] = B$. Now, further information may tell us that these estimates should be revised to $A' \supseteq A$, $B' \supseteq B$. Then the above intuition tells us that, if no counter-examples are added in this way, the conditional relation will continue to hold. Formally, for $A \subseteq A'$, $B \subseteq B'$,

$$\text{if } if_E A, B \text{ and } A' - A \subseteq B', \text{ then } if_E A', B'.$$

In practice, it is more convenient to split this up into two cases:

(1) fixed A, growth of B:

$$if\, A, B \text{ implies } if\, A, B \cup C, \quad \text{for any set } C;$$

(2) simultaneous growth of A, B:

$$\textit{if}\ A, B \text{ implies } \textit{if}\ A \cup C, B \cup C. \quad \text{for any set } C.$$

The first of these principles is well-known as 'weakening of the consequent', or (upward right) 'monotonicity'.

Next, consider the case where there have been errors in judgments already made about $[\![X]\!]$, $[\![Y]\!]$, and one retreats to $A' \subseteq A$, $B' \subseteq B$. At least, the above intuition tells us that mere removal of counter-examples will not affect the conditional assertion:

$$\text{if } \textit{if}_E\ A, B \text{ and } A - A' \subseteq A - B, \quad \text{then } \textit{if}_E\ A'B.$$

In a more elegant form this may be stated as the implication

(3) $\textit{if}\ A, B \cap C$ implies $\textit{if}\ A \cap B, C$.

What about a stronger principle, dual to the above (1), stating that 'strengthening the antecedent' will not do any harm?

(3)′ $\textit{if}\ A, C$ implies $\textit{if}\ A \cap B, C$.

This would mean that possible removal of confirming instances does not affect the conditional either. Except for the extreme case where A is included in C to begin with, such a principle has little to recommend itself as a general constraint.

A fourth and final aspect of Confirmation would seem to be that 'optimal' evidence should verify a conditional:

(4) $\textit{if}\ A, B$, whenever $A \subseteq B$.

This completes the exploration of what is perhaps the most distinctive feature of conditionality. The next intuition to be considered is a more general one, prescribing a special role for the left-hand argument in conditional statements – a phenomenon also observable with determiner expressions (cf. van Benthem, 1983(a)).

ANTECEDENCE

A conditional statement invites us to take a mental trip to the land of the antecedent. Thus, the assertion of the consequent is only relevant in as far as it holds among the antecedent occasions (cf. Figure 2):

$$\textit{if}_E\ A, B \text{ if and only if } \textit{if}_E\ A, B \cap A.$$

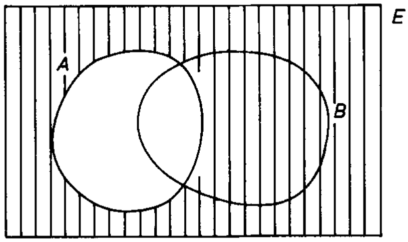

Fig. 2.

Most current accounts of conditionals obey this restriction; which also shows in equivalences such as 'every dragon is greedy $\leftrightarrow$ every dragon is a greedy dragon'.

A stronger version of this same intuition would be the claim that the whole context of E outside of the antecedent occasions is immaterial:

$$if_E A, B \text{ if and only if } if_A A, B \cap A.$$

We shall derive this stronger equivalence from a more general principle to be adopted below.

The following set of principles is of a yet more general logical flavour.

EXTENSION

Logical constants should be stable, in the sense that, once established, further growth of the universe will not affect decisions already taken. The relevant cross-contextual constraint is found in Westerståhl (1982):

$$\text{if } A, B \subseteq E \subseteq E', \quad \text{then}$$

$$if_E A, B \text{ if and only if } if_{E'} A, B.$$

The effect of this principle is 'context-neutrality': logical statements about sets involve no more than (the union of) these sets themselves. In conjunction with Antecedence, the context may even be restricted to the antecedent domain.

It is this principle that will allow us to drop the context parameter E in the remainder of this paper, whenever convenient.

But, there are also more local requirements of a logical nature.

ACTIVITY

A logical constant should do some work, showing some variety of behaviour within its proper field of action. Thus, in the light of the above, we state that

for each non-empty set A, there exist $B, B' \subseteq A$ such that

if A, B and not *if* AB'.

In conjunction with the earlier postulate of Confirmation, clause (1) (monotonicity), this is equivalent to requiring

if A, A

if $A, \emptyset$ for no A except the empty set.

The former condition already appeared as clause (4) of Confirmation. The latter is new; and indeed, it need not always be satisfied. (One may think of a probabilistic approach where A is a non-empty set of measure zero.) Nevertheless, we shall assume Activity henceforth; largely for convenience. Subsequent results can easily be modified so as to do without it; but this would bring in all kinds of border-line cases obscuring the main issues.

A deeper, but also less fathomable logical intuition says that all phenomena previously noted should occur quite 'regularly'. We are concerned with an arbitrary situation, not any specific semantic structure.

UNIFORMITY

There should be uniformity in the range of a conditional relation. Accidental features, such as the absolute size of the antecedent set, should not matter to its truth value behaviour. Typically, such an intuition invites us to make comparisons across different antecedent sets.

To make this idea more concrete, consider the following *thought-experiment*. Start from sets A, B with *if* A, B (or not, as the case may be). Now, one adds a counter-example, noting what happens. (I.e., one notes the truth value of *if* $A \cup \{a\}, B$; with a outside of $A \cap B$.) Afterwards, removing the counter-example a, one adds a confirming instance, again noting what happens (to *if* $A \cup \{a\}, B \cup \{a\}$; with a outside of A). Finally, one notes what happens when both acts are performed at the same time. The outcomes may be pictured as 'confirmation patterns' of truth values, of the form

	old situation	
add counter-example		add confirming instance.
	add both	

A priori, sixteen possible truth value patterns can occur as the outcome of our thought-experiment. Of these, Confirmation allows only the six patterns displayed in Figure 3.

	+			+			+			−			−			−	
+		+	−		+	−		+	−		+	−		+	−		−
	+			+			−			−			+			−	

Fig. 3.

Now, Uniformity will typically constrain the occurrences of such outcomes. Our thought-experiment must at least exhibit certain regularities, independent from the particular location A, B where it is performed. One obvious requirement, given the nature of the above experiment is *uniqueness of outcomes*: the combined addition should always have its truth value uniquely determined by the results of the separate experiments. Thus, the second and the third (or fourth and fifth) patterns in the above sequence cannot occur together for the same conditional.

There is room for a whole hierarchy of uniformity constraints here, depending on when the outcomes of a sequence of thought-experiments are to form stable, recurring patterns. But here, only the above minimal kind of regularity will be imposed.

The final principle to be formulated here may be too technical to come to the untutored mind as an 'intuition'. Still, it is one which, in one form or another, appears in virtually all discussions of logical constants.

QUANTITY

Logical constants should be 'topic-neutral', in the sense that there is no special role for any particular individual occasion. Formally, this is usually presented as invariance under permutations of the universe:

$$\text{for every bijection } F \text{ between } E, E', \text{ and } A, B \subseteq E,$$

$$if_E\, A, B \text{ if and only if } if_{E'}\, F[A], F[B].$$

In plainer terms: *if* A, B only depends on the *number of occasions* in the relevant sets $A \cap B, A - B, B - A, (E - A) \cap (E - B)$.

This principle of stark austerity may need some clarification. Quantity may be viewed as a form of Occam's Razor: there should be no more to conditionals than meets the eye. More specifically, no semantic constructs should be relevant but those sanctioned explicitly by the syntactic material in a conditional sentence. Now, as has been argued in Section 2, the proper denotations of the antecedent and consequent sentences are bare sets of occasions – and the particle *if* has to denote a relation between these. Violations of this principle must then always result in 'hidden variable theories', postulating additional semantic structure among occasions beyond what meets the eye. The latter procedure is quite respectable, of course: science introduces its theoretical terms often in just this way. But, in a foundational study such as the present one, we want to explore the limits of the former, more austere realm of conditionals – if only to see just where this is to be transcended, and what are the options.

4. A TRILEMMA

Even if all our general intuitions concerning conditionals are plausible by themselves, their combined effect may be surprising. After all, the problem with our intuitions is usually not their availability or vitality, but rather their *consistency*. In this section, it will be determined which generalized quantifiers are left by the combined postulates of Section 3.

Now, in a first analysis, there are good reasons for restricting attention here to *finite universes*. It is in accord with the intuitive semantics of natural language, it is the area where proposed explications for conditionals usually work most smoothly (cf. Lewis, 1981), and the restriction also curbs the mathematical temptation to embark upon exotic infinite combinatorics irrelevant to the main issue.

There exists a convenient geometric representation of our conditionals in the finite realm. In view of Antecedence and Extension, the truth of *if* A, B only depends on $A, B \cap A$ – or equivalently, on $A - B, A \cap B$. Quantity entails that only the numbers of occasions in these two sets matter. Thus, any conditional on the finite universes is representable as a subset of the *tree* of couples

$$|A - B|, |A \cap B|$$

depicted in Figure 4.

So, a conditional may be pictured as a tree pattern of truth values +, –; as used in the earlier statement of Uniformity.

|A| =							
0				0,0			
1			1,0		0,1		
2		2,0		1,1		0,2	
3	3,0		2,1		1,2		0,3
⋮				etcetera			

Fig. 4.

A simple combinatorial argument will now establish the range of conditionals left open by our postulates: essentially, two 'democratic' cases and one 'anarchistic' one.

4.1. THEOREM. On the finite sets, the only conditionals in the present sense are those defined by *all, not least* and *some*.
Here, 'not least' is short for 'half or more', and 'some' stands for 'some or all'.

Proof. That these three conditionals satisfy all earlier postulates follows by geometric inspection of their tree patterns. The key observation here is that

– Confirmation (1), (2), (3) amount to the requirement that, whenever (a, b) belongs to the conditional, then so does the area $(0, \infty)$, $(0, b)$, (a, b), (a, ∞) (cf. Figure 5).

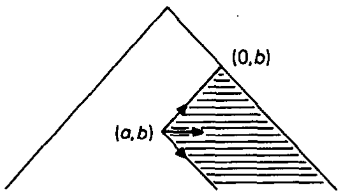

Fig. 5.

– Activity (including Confirmation (4)) says that the right edge of the tree lies within the +-region, while the left edge (minus the top) lies outside of it.

Conversely, consider the pattern for any conditional satisfying our postulates. The top position gets +, by Activity. The next row gets –, +; again by Activity. The third row leaves a choice in the middle – its boundaries being fixed by Activity. One possibility is – – +, in which case Uniformity produces a –-diagonal alongside the right edge. By Confirmation then, (in fact, by monotonicity in the rows), the tree becomes that of the conditional *all* (cf. Figure 6(i)).

(i)	(ii)	(iii)
+	+	+
− +	− +	− +
− − +	− + +	− + +
− − − +	− + + +	− − + +
⋮	− + + + +	− − + + +
	⋮	− − − + + +
		⋮

Fig. 6.

The other possibility for the third row is − + +. This fixes three positions in the fourth row, as before, while leaving the truth value at its second position open.

CASE 1. The fourth row is − + + +.

In this case, the experiment $-^{-}+$ has produced the outcome +, and hence it will continue to do so (Uniformity). By Activity and Confirmation then, the tree pattern becomes that of *some or all* (cf. Figure 6(ii)).

CASE 2. The fourth row is − − + +.

This determines four positions in the fifth row as before. The remaining one (in the middle) is fixed by Uniformity: $-^{+}+$ will invariably produce the combined truth value outcome +. By a similar observation concerning $-^{-}+$, the tree pattern becomes that of *half or more* (cf. Figure 6(iii)). □

There is a Calvinist flavour to the above results: only a few ways of life are open to a righteous conditional, none of them very attractive.

The logical escape routes will be charted in the course of this paper.

Finally, the above tree of numbers has more uses than the one just demonstrated. It is particularly useful in picturing the effect of various intuitions; say, when trying to escape from the above trilemma. For instance, it turns out that Uniformity played a decisive role in this extreme limitation. Without it, the full range of *a priori* possibilities is realized.

4.2. THEOREM. On the finite sets, there are $2^{\aleph_0}$ conditionals satisfying Antecedence, Confirmation, Extension, Activity and Quantity.

Proof. Any subset of the tree satisfying the geometrical equivalents of Confirmation and Activity will qualify. Such subsets are determined by a left-most positive boundary, as in Figure 7. Each such boundary is characterized by a 'marching order': n_1 steps south-east, n_2 steps south-west, n_3 steps south-east, etcetera. Obviously, there are $2^{\aleph_0}$ such marching orders. □

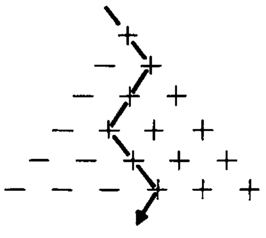

Fig. 7.

5. THREE WAYS OUT

Up till this point, our study has been concerned with a purely numerical approach to conditionals, reaching its limits in the trilemma of Section 4. Now, we will explore options for transcending this approach, thus increasing our scope to encompass more actual examples of conditionals studied in the literature. Three lines of investigation may be discerned here, perhaps not the only ones, but certainly the most important ones.

First, there is one option which violates none of the earlier intuitions, viz. lifting the restriction to finite sets. Essentially, we have been studying conditionals as binary relations on finite sets of natural numbers – and we may now pass on to the full power set $\mathscr{P}$(IN). One attraction of this *infinitary* approach is the following. In doing semantics on the finite models, one is typically concerned with an arbitrary, but usually 'large' number of occasions. The spirit of this view is sometimes better captured by infinite sets, abstracting from all peculiarities of particular finite numbers.

On the other hand, one wants to exclude irrelevant concerns about higher cardinal arithmetic, and hence Section 6 will be devoted to the numbers

$$0, 1, 2, \ldots, \infty$$

The predominant tendency in conditional logic seems to be, not to exploit possible additional resources of infinity, but to enrich the old (finite) models by imposing additional semantic structure. The intuition to go then is not a typical conditional principle such as Confirmation or Antecedence, but rather one of the general logical ones, viz. Quantity. (Quantitative violations of, say, Extension or Uniformity will be left unexplored here.)

In principle, there are many ways of proceeding here. One is to assign 'weights' to different possible occasions, by introducing some *probability measure* P on subsets of E. Conditionals then arise which essentially exploit this additional structure, such as the following principle of 'likelihood':

$$\mathit{if}\, A, B \text{ if } P(A \cap B) > P(A - B).$$

Notice that this inductive approach reduces to the earlier numerical one when P is the equiprobability measure.

The inductive approach will be considered in more detail in Section 7, for its suggestive value – and a connection is found with earlier work in the foundations of statistics. Nevertheless, the main thrust of this paper lies in a different direction. For, the usual procedure in possible worlds semantics for conditional logic has been rather to differentiate between individual occasions through accessibility and similarity patterns. For instance, Quantity is violated in the counterfactual semantics of Lewis (1973); witness the models of Figure 8. (Comparative similarity here is just relative distance, and truths are as indicated. The two situations depicted are numerically indistinguishable – yet the true conditional

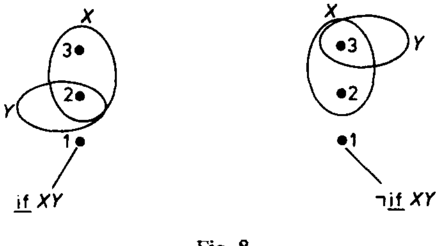

Fig. 8.

statements are affected by the interchange of worlds 2 and 3.) This hierarchical or *intensional* approach will be studied systematically in Section 8.

Further connections between these three ways out of the austerity of Section 4 will not be pursued here, nor any alternatives to them.

6. INFINITY

Adding the infinite case to the previous finite realm may be pictured as the addition of one final row to the number tree of Section 4 (cf. Figure 9). More concretely, conditionals may now be viewed as binary relations between sets of natural numbers, finite or infinite.

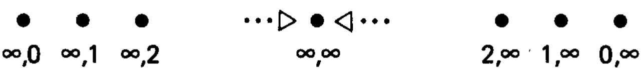

Fig. 9.

One typical example of the greater freedom in this perspective is the following idea of a conditional connection.

6.1. EXAMPLE. Let *if* A, B hold when the number of exceptions (i.e., $|A - B|$) is 'negligible' as compared to the total size of A. What this amounts to is

$$\mathit{if}\ A, B \text{ if } \begin{cases} A \subseteq B, & \text{when } A \text{ is finite} \\ A - B \text{ is finite}, & \text{when } A \text{ is infinite.} \end{cases}$$

We shall return to this example presently.

Of the earlier intuitions concerning conditionals, Antecedence, Confirmation and Activity remain equally plausible in the full power set $\mathscr{P}(\mathbb{N})$. Uniformity becomes less attractive, however; as one does not expect infinite sets to behave like finite ones in all respects. And indeed, the above conditional fails to satisfy this postulate: adding single counter-examples destroys validity for finite antecedent sets, but not for infinite ones.

A more relevant question within the present context would seem to be which view-point connects behaviour in the infinite case with, possibly uniform, behaviour in the finite realm. This will then constrain the admissible extrapolations to $\mathscr{P}(\mathbb{N})$ of those conditional relations on the finite sets that were allowed by the earlier intuitions.

Regardless of the precise reasoning employed, the only really interesting conditionals on the infinite row are those represented in Figure 10. Two of these are straightforward extrapolations of the main conditions in Section 4, the one in the middle is the above Example 6.1.

− − −	··· − ···	− − +	(*all*)
− − −	··· − ···	+ + +	(*most*)
− − −	··· + ···	+ + +	(*not least*)

Fig. 10.

An obvious question is if the new conditionals arising in this way distinguish themselves significantly from the earlier three; in particular, in terms of valid inferences. Let us consider the three most prominent cases. *All* (inclusion) has precisely the same logic as it had on the finite domains only, as is easy to verify. Our conjecture is that the same holds for *not least*. The logic of Example 6.1 is more intriguing, however. It validates the Conjunction principle (from *if* XY and *if* XZ to *if* $X(Y \wedge Z)$), which sets it apart from the *not least* logic. But, it also fails to validate transitivity; which distinguishes it from the logic of *all*. (To see this, consider an infinite A with finite $A' \subseteq A$ outside of B: *if* A, B, *if* A', A, yet not *if* A', B.) Indeed, in an earlier version of this paper, it was conjectured that this new logic coincides with the basic counterfactual logic of Burgess 1981 – which would have provided a purely quantitative modelling of the basic logic in the tradition of D. Lewis. But actually, John Burgess (personal communication) provided a counter-example. Here is a simple inference, adapted from Burgess' example, which is valid in the logic of Example 6.1, though not in the basic Lewis logic:

$$\frac{\textit{if}(Y \vee \neg Y) \neg Y \quad \textit{if}\, XY \quad \textit{if}\, YZ}{\textit{if}\, XZ}.$$

To see this, notice that the first premise forces $[\![Y]\!]$ to be finite, and the remaining one extends this to $[\![X]\!]$ and $[\![Z]\!]$. A Lewis-semantic counter-example occurs in the marked world of Figure 11, however.

The problem seems to be that the logic of Example 6.1 encodes part of the peculiarities of finite (and infinite) sets in this particular structure.

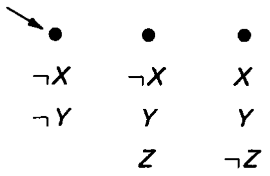

Fig. 11.

QUESTION. What is the logic of Example 6.1?

QUERY. Which of the common conditional logics in the literature may be modelled as the logic of some suitable conditional relation on subsets of IN?

Actually, one may also restrict attention here to suitable parts of the full power set of IN. Various Boolean subalgebras are relevant – e.g., that of just the finite and co-finite sets of natural numbers. •

Also, one need not stop at mere conditional inference schemata. For instance, what is the full first-order theory of the structure ⟨IN, *if*, ⊆⟩, with *if* as defined in Example 6.1? There is room for applications of ordinary logical model theory here.

Finally, the interplay between finite and infinite sets of occasions, though interesting, may be beside the point. After all, if infinite sets are to model 'arbitrary' finite ones, then one may want to consider just the former. Accordingly, one may restrict attention to the last 'infinite' row in the number tree. Then, again the question arises which logics are generated by the above three choices. Our conjecture is that the only difference occurs in the middle case; where the earlier Burgess example is now blocked – as finite sets are no longer available. Hans Kamp has conjectured that the resulting logic will now indeed become the basic counterfactual one. But again, there arises a new principle beyond the latter logic. For infinite sets A, B, *if* $A \cup B, B$ implies *if* A, B: a form of strengthening the antecedent which is invalid in counterfactual reasoning.

7. INDUCTION

One possible additional structure upon our universes is provided by a probability measure P assigning real numbers $P(A)$ to sets of occasions A, subject to the following conditions:

$$0 \leqslant P(A) \leqslant 1; \quad P(\emptyset) = 0, \quad P(E) = 1 \qquad \text{(normality)}$$

$$\text{for disjoint } A, B, P(A \cup B) = P(A) + P(B) \qquad \text{(additivity).}$$

In this section, only finite universes will be considered.

Several 'inductive' conditionals may be defined employing the above measure.

7.1. EXAMPLE. Set *if* A, B if $P(A \cap B) = P(A)$.

This conditional says that $B \cap A$ fails to cover A by a zero-set.

Despite the similarity with Example 6.1, the logic of this conditional is just the classical one of entailment. For instance, the strong principle of strengthening the antecedent (cf. Section 3) is established thus:

Suppose that *if* A, B. Then $P(A \cap B) = P(A)$, and hence $P(A - B) = 0$. For any C then, $P((A \cap C) - B) \leqslant P(A - B) = 0$; i.e, $P(A \cap C \cap B) = P(A \cap C)$.

When P is the equiprobability measure on E, this conditional reduces to the earlier *all*. At the other extreme lies the merest tip of the balance:

7.2. EXAMPLE. Set *if* A, B if $P(A \cap B) \geqslant P(A - B)$.

This stipulation is closely related to an idea in Lenzen (1980), reading A-conditional belief of B in terms of 'more likely than not, within the A-realm'.

In fact, the present section is an illustration, inspired by Lenzen's book, of how existing ideas in the foundations of statistics and epistemic logic fit in quite naturally with the generalized quantifier framework – thus providing some independent confirmation of the latter's value.

Again, when P is the equiprobability measure on E, the conditional of Example 7.2 collapses to an earlier one, viz. *not least*. As was observed in Section 5, the earlier numerical approach is a limiting case of the inductive one.

In the spirit of the preceding investigation, what is at issue are not so much particular examples as general constraints upon admissible probability measures, and conditionals based upon them – within and across various universes. Such a probabilistic picture suggests intuitions of its own that would not come to the fore in the purely numerical setting of Section 3.

For instance, conditionals $if_{E,P}$, now viewed as functions from $\mathscr{P}(E) \times \mathscr{P}(E)$ to $\{0, 1\}$, may be required to satisfy various *smoothness* properties involving P. (E.g., conditionals should be in 'equilibrium', in the sense that truth values assigned ought to be stable under small shifts in arguments; with 'small' as measured by P.) This idea even invites a generalization to 'fuzzy conditionals' assuming truth values in the real interval $[0, 1]$ (a suggestion due to Lotfi Zadeh); where *if* then becomes *continuous* in the usual sense.

These themes are not developed here; the main point of the present section being to draw attention to a parallel between the present generalized quantifier prespective and an earlier historical one.

In his foundational studies of probability, Bruno de Finetti introduced a notion of 'relative probability' between sets of outcomes:

$$A \leqslant B$$

meaning that 'A is at most as plausible as B', (Cf. Lenzen, 1980, Chapter 4, p. 85.) He then produced a list of intuitive desiderata (in the spirit of Section 3), including the following:

(1) $\emptyset \leqslant A \leqslant E$

(2) $\emptyset \not\geqslant E$

(3) if $A \leqslant B \leqslant C$, then $A \leqslant C$ (transitivity)

(4) $A \leqslant B$ or $B \leqslant A$ (connectedness)

(5) if $A \cap B \leqslant A \cap C$, then $A - C \leqslant A - B$ (contraposition).

The guiding hope was that these would provide necessary and sufficient conditions for this primitive relation to be represented through some probability measure P on E:

$$A \leqslant B \text{ if and only if } P(A) \leqslant P(B).$$

Later investigations have revealed that further, less intuitive combinatorial postulates are required for this purpose (cf. Lenzen, o.c., p. 87).

There is an intimate connection between de Finetti's notion and the earlier inductive conditional *if* of Example 7.2.

7.3. THEOREM. *If* A, B iff $A - B \leqslant A \cap B$,

$$A \leqslant B \text{ iff } if\ A \Delta B, B.$$

Here 'Δ' denotes symmetric difference.

Proof. The first equivalence is obvious. The second equivalence follows from additivity:

$$\text{If } P(A) \leqslant P(B), \text{ then } P(A - B) + P(A \cap B) \leqslant P(B - A) + P(B \cap A); \text{ whence } P(A - B) \leqslant P(B - A):$$
$$\text{i.e., } P((A \,\Delta\, B) \cap B) \geqslant P((A \,\Delta\, B) - B).$$

A check upon these equivalences is found in the following calculations:

(i) *If* A, B (iff $A - B \leqslant A \cap B$) iff *if* $(A - B) \,\Delta\, (A \cap B), A \cap B$ iff *if* $A, A \cap B$. (Antecedence)

(ii) $A \leqslant B$ (iff *if* $A \,\Delta\, B, B$) iff $(A \,\Delta\, B) - B \leqslant (A \,\Delta\, B) \cap B$ iff $A - B \leqslant B - A$.

Through the above observation, the De Finetti axioms (1), . . . , (5) generate a conditional logic. Amongst others, one finds that contraposition becomes universally valid, connectedness is Conditional Excluded Middle, while transitivity becomes the following 'Δ-principle':

if $A \,\Delta\, B, B$, *if* $B \,\Delta\, C, C$ imply *if* $A \,\Delta\, C, C$.

(Cf. also Example 10.3 below.)

QUERY. To axiomatize the De Finetti logic.

All principles of this logic discovered up till now also hold for the earlier conditional *not least*. This is obvious for Conditional Excluded Middle; but there is also a less obvious check.

7.4. EXAMPLE. The Δ-principle holds for *not least*.

For, consider the Venn diagram of Figure 12. '*Not least* $A \,\Delta\, B$ are B' means that $1 + 2 \leqslant 3 + 4$, '*not least* $B \,\Delta\, C$ are C' means that $3 + 5 \leqslant 2 + 6$. It follows that $1 + 2 + 3 + 5 \leqslant 3 + 4 + 2 + 6$, and hence that $1 + 5 \leqslant 4 + 6$: i.e., *not least* $A \,\Delta\, C$ are C. □

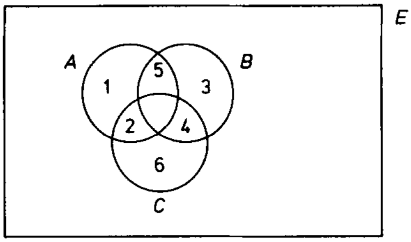

Fig. 12.

We conclude with a

QUESTION. Do the De Finetti logic and the *not least* logic coincide?

8. INTENSION

The usual approach in conditional semantics has been the hierarchical one. Possible worlds can be more or less close to some vantage world, and the conditional is only concerned with 'closest' antecedent worlds. In the perspective of this paper, where the universe itself may already be derived from some vantage point, a hierarchy is just some binary relation. Thus, one now considers *finite* structured universes $\langle E, R \rangle$, say; in which the generalized quantifier *if* assigns a binary relation between subsets of E. The earlier numerical perspective was a democratic one, so to speak, with an empty (or universal) relation R – but in the general case, certain individual occasions may possess greater influence than others.

8.1. EXAMPLE. Top-ranking decisions.

One typical hierarchical conditional considers top-ranking occasions only:

all R-maximal occasions in A are in B.

We shall review the intuitions of Section 3 for this example. First, obviously, Quantity has been given up – but there remains a related principle.

QUALITY

Conditional relations are invariant under the action of *R-isomorphisms* between universes $\langle E, R \rangle$.

For, such R-isomorphisms preserve the relevant hierarchical structure. (For instance, if F is an R-isomorphism of $\langle E, R\rangle$, and w was R-maximal in A, then $F(w)$ will be R-maximal in $F[A]$; and vice-versa.)

Within a single universe, the force of this postulate depends on the hierarchy. If R is empty (or universal), i.e., all individuals are equal, then every permutation of E is an R-automorphism, and Quality reduces to Quantity. Thus, the present approach subsumes the earlier numerical one. If, on the other hand, every individual is uniquely distinguished by its position in the hierarchy, then the only R-automorphism is the identity map, and the constraint becomes empty.

We may view the situation as one of postulating 'hidden variables'. The conditional relations need no longer be invariants of the full permutation group of E, but only of some subgroup – and we postulate additional structure of E in order to characterize the latter.

Now, continuing with the other intuitive constraints, Antecedence remains equally plausible, in the form

$$if_{\langle E, R\rangle}A, B \text{ iff } if_{\langle E, R\rangle}A, B \cap A.$$

The case of Confirmation is more interesting. Evidently, this principle should remain valid – but this may impose certain (mild) conditions upon the hierarchy, notably *transitivity and irreflexivity* of R.

8.2. EXAMPLE. The conditional of Example 8.1 satisfies clauses (1), (2) and (4) of Confirmation, but not necessarily clause (3).

Proof of (2). Suppose that $if\, A, B$. Consider any top-ranking w in $A \cup C$. Either $w \in C$, and hence $w \in B \cup C$; or $w \in A$ – whence it is R-maximal in A, and therefore, by the assumption, $w \in B$, $w \in B \cup C$.

The case of (3). Suppose again that $if\, A, B$. Now remove a counter-example from $A - B$. R-maximal worlds in the remaining set A^- ought to have been R-maximal in A (and hence belong to B): otherwise, the desired principle may break down. But, in general, this will require the above two conditions, as well as finiteness of the universe:

Suppose that w_1 is not R-maximal in A. Then, for some $w_2 \in A$, $w_1 R w_2$. Continuing, by finiteness, transitivity and irreflexivity, we must arrive at some $w \in A$, $w_1 R w$ which is R-maximal in A, and hence belongs to B as well. So, this w has not been removed – and w_1 is still not R-maximal in A^-. □

The example shows an interplay of three 'degrees of freedom' concerning a particular conditional: general constraints, a particular choice of truth definition, and requirements upon hidden variables occurring in the latter. This theme will be discussed at greater length in Section 9 below.

Activity remains as plausible as before. Notice that its validity for our paradigm conditional depends on the existence of R-maximal occasions in non-empty sets A: which again depends on the assumption of finiteness.

Indeed, in infinite universes, the conditional of Example 8.1 would not necessarily satisfy either Activity of Confirmation. This reflects a familiar problem with the well-known approach of D. Lewis. On finite models, the preferred truth definition reads as in Example 8.1, but the infinite case forces one to consider the less intuitive clause (in our terms):

'some $A \cap B$-world is R-closer than every $(A - B)$-world'.

To mention just one problem, it is not obvious that this clause is equivalent with that of Example 8.1 on finite universes. And in fact, it is stronger; unless one assumes yet another condition on the hierarchy, viz. some form of *connectedness* (cf. Lewis, 1973). The latter requirement was dropped in Burgess (1981), who axiomatizes the resulting basic conditional logic. By way of illustration, here is the relevant connection.

8.3. THEOREM. The conditional of Example 8.1 validates precisely the Burgess logic.

Proof. It is easy to verify that all Burgess axioms (presented in Section 10 below) are valid for the top-ranking conditional. Conversely, a Burgess counter-example, which can always be assumed to be finite, may be regarded as a hierarchy in the above sense. (Actually, there is a small problem here, as Burgess' truth definition differs from the above. But, modulo finiteness, transitivity and reflexivity, the two formulations turn out to coincide.) □

After this digression, we consider the remaining intuitions of Section 3. Extension remains valid, in the form appropriate to the present context:

If E is an *R-submodel* of E', then, for all $A, B \subseteq E$,

$$if_{\langle E, R\rangle} A, B \text{ iff } if_{\langle E', R'\rangle} A, B.$$

Finally, there was the logical idea of Uniformity. Again, this principle acquires a new flavour in the present perspective.

The characteristic thought-experiment of Section 3 now consists in adding individual worlds to a hierarchy. More than before, the uniformity principles postulating regularity of outcomes will be influenced by the *form of description* chosen for the latter. There is a plethora of possibilities here, in terms of preserving or destroying certain desired R-patterns.

For purposes of illustration, here is a very limited view of the matter. By Confirmation, confirming instances can always be added, at each position in the hierarchy. For counter-examples, let us distinguish three possible actions:

(1) insert in top position (without $A \cap B$-superiors),

(2) insert below some $A \cap B$-world (not necessarily *immediately* below),

(3) insert below some $(A - B)$-world (not necessarily immediately below).

Uniformity now says that allowing such an action once means allowing it always. The strength of this principle will be gauged in the proof of Theorem 8.4 below.

A richer perspective such as the hierarchical one suggests *new intuitions* as well. One simple example is the following.

RELEVANCE

At each individual occasion, the hierarchy is only relevant in as far as it is 'accessible'. Formally, call a substructure of $\langle E, R \rangle$ a 'sub-hierarchy' if each of its occasions retains all its R-superiors and R-inferiors from E. (In tense logic, a sub-hierarchy would be called a 'generated substructure'.) Then the principle is this:

Conditional statements are preserved in passing from a hierarchy to its sub-hierarchies. Or, formally,

$$\text{if } \langle E', R' \rangle \text{ is a sub-hierarchy of } \langle E, R \rangle, \text{ then}$$
$$if_{\langle E, R \rangle} A, B \text{ implies } if_{\langle E', R' \rangle} A \cap E', B \cap E'.$$

As in the classification theorem of Section 3, the effects of the combined hierarchical intuitions may be investigated for the finite transitive irreflexive hierarchies of this section.

8.4. THEOREM. The only two conditionals satisfying Quality, Antecedence, Confirmation, Activity, Extension, as well as Uniformity, Relevance are *all X are Y, all top-ranking X are Y.*

Proof. A simple calculation establishes that these two conditionals satisfy all mentioned principles.

Conversely, consider any conditional *if* subject to these constraints.

CLAIM 1. Relevance rules out action (1) of Uniformity.

By Confirmation, one single $A \cap B$-occasion verifies *if* A, B. Action (1) would allow the addition of a single R-isolated $(A - B)$-occasion, while *if* A, B remains true. But then, by Relevance, the latter alone would verify the conditional: which contradicts Activity.

CLAIM 2. Either *if* is inclusion, or it allows both action (2) and (3) of Uniformity.

For, if there exists any situation $if_{\langle E, R\rangle} A, B$ with A not included in B, then that hierarchy contains some $(A - B)$-occasion. Now, this occasion w_1 cannot occur in top position. For, otherwise, removing this occasion leaves the conditional true (by Confirmation); and hence, in retrospect, action (1) was allowed after all. So, by an earlier argument, there must be some other R-maximal occasion w_2 above w_1 – and, evidently, it must be in $A \cap B$. But then, the same reasoning of removal/reversal (applied to w_1, w_2) shows that action (2) is admissible. Thus, $(A - B)$-occasions may be inserted below w_2, in particular also below w_1. And that again means that action (3) is admissible as well.

Finally, the third observation completes the argument.

CLAIM 3. When actions (2), (3) of Uniformity are admitted, the conditional must be that of top-ranking occasions.

Here, in one direction, each situation where all top-ranking A and B can be created from single $A \cap B$-occasions (where the conditional holds, by Confirmation) through judicious addition of confirming instances intermingled with (2), (3)-insertions.

Conversely, suppose that at least one situation is admitted with some top-ranking $(A - B)$-occasion. Omit this occasion (by Confirmation): in reverse, action (1) has been allowed, in contradiction to the first claim. □

Thus, upon one particular analysis of our broad intuitions, the hierarchical perspective allows just ordinary modal entailment, as in the purely numerical case, while adding one new basic possibility ('top-ranking'), which has turned out to generate precisely the basic subjunctive logic in the Lewis–Stalnaker tradition.

9. THE RANGE OF CONDITIONAL TRUTH DEFINITIONS

Even though conditionals were treated as abstract generalized quantifiers in the above, the presentation of specific examples, or the statement of classification theorems usually proceeds by definition in some standard logical language. Indeed, such descriptions may be viewed as possible *truth definitions* for abstract conditionals, satisfying certain intuitive constraints, with respect to some background class of universes of occasions.

First, let us consider the logically simplest case. Call a conditional *if first-order definable* if there exists some formula $\varphi = \varphi(X, Y)$ in the monadic first-order language with identity and unary X, Y such that,

$$\text{for all } E \text{ and } A, B \subseteq E,$$

$$if_E A, B \text{ if and only if } \langle E, A, B\rangle \models \varphi.$$

For instance, two of the conditionals in Section 3 were first-order definable:

$$\forall x(Xx \rightarrow Yx) \qquad \text{(all)}$$

$$\exists x(Xx \wedge Yx) \vee \forall x(Xx \rightarrow Yx) \qquad \text{(some or all).}$$

Other examples are: in *at least five* (*at most six, all but at most seven*) occasions.

Because of the preservation of first-order statements under isomorphism, all these conditionals satisfy Quantity.

Definitely outside of this class is the third conditional in the trilemma of Section 3. *Not least* is not even definable on the finite sets alone, even if an infinite defining set of first-order formulas were allowed. A simple model-theoretic proof of this fact employs compactness of the first-order language, in combination with the observation that any two models $\langle E, A, B\rangle$, $\langle E', A', B'\rangle$ with $E = A$, $E' = A'$ and $A \cap B$, $A - B$, $A' \cap B'$, $A' - B'$ all *infinite*, validate the same first-order sentences in the above language.

A useful semantic characterization of first-order definability may be derived from the well-known general result of Fraïssé, characterizing first-order formulas in terms of invariance under back-and-forth morphisms up to some finite threshold. More precisely, in the present simple case, set

$$U \sim_n V \text{ if } |U| = |V| < n \text{ or } |U| \geqslant n, |V| \geqslant n.$$

By extension, set $\langle E, A, B\rangle \sim_n \langle E', A', B'\rangle$ if all four relevant 'slots' $A \cap B, A - B, B - A, (E - A) \cap (E - B)$ in $\langle E, A, B\rangle$ stand in the $\sim_n$-relation to their primed counterparts.

9.1. THEOREM. A conditional *if* is first-order definable if and only if, for some fixed natural number n,

$$\langle E, A, B\rangle \sim_n \langle E', A', B\rangle \text{ implies } if_E A, B \text{ iff } if_{E'} A', B,$$

for all models $\langle E, A, B\rangle, \langle E', A', B'\rangle$.

As an immediate application, it is seen that the earlier restriction to finite models is immaterial for first-order definable conditionals. For, if a principle of inference is refuted for such a conditional on an infinite model, then it can already be refuted in some suitably large finite model.

The behaviour of these conditionals is easily pictured in the earlier tree of numbers. After an initial period of 'childhood diseases', a first-order definable conditional reaches the level $|A| = 2n$ (n as in Theorem 9.1), where the following happens (cf. Figure 13). The truth value at (n, n) is repeated in the whole downward generated triangle $\{(k, 1) | k \geqslant n, 1 \geqslant n\}$. The truth value of $(n + k, n - k)$ is repeated along the south-west diagonal $\{(n + k, 1) | 1 \geqslant n - k\}$, and likewise that of $(n - k, n + k)$ along its south-east diagonal.

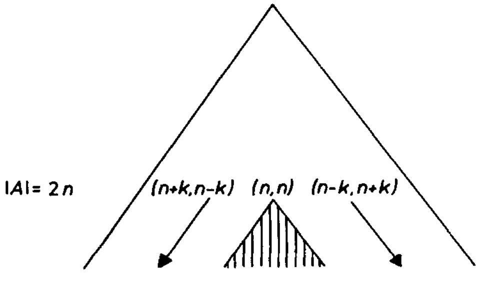

Fig. 13.

Given this pattern, it is immediate why *not least* could not be first-order definable: it lacks a homogeneous triangle (of + or –) in its tree.

The limitations of the present first-order language also become apparent in the following illustration, drawn from the Introduction to Suppe (1974) (Sections IIA, IIC, especially pp. 22, 37).

INTERMEZZO: THE LOGIC OF DISPOSITIONS

The only well-known alternative to material implication known in the thirties was modal entailment: *all X are Y* ('*X* is a sufficient condition for *Y*'). Now, when philosophers of science started considering conditional statements in scientific contexts, they ran into the problem that entailment does not work for dispositional statements. E.g., the sentence

> 'This lump of sugar is soluble',

which means presumably

> '*If* this lump of sugar is put into water, it dissolves',

canoot be transcribed as

> 'All watering occasions for this lump are dissolution occasions'.

One kind of problem is that continuously dry objects would have to be called soluble then, for trivial reasons. This could be remedied by enlarging the setting to all possible occasions (whether actualized in this world or not). But even so, there remains another problem, viz. that the conditional is too strong in another sense. One is only referring to all watering occasions 'under normal circumstances'. Typically, this means that dispositional conditionals will not admit of strengthening their antecedents, as this may bring in non-normal circumstances. (*If* this lump of sugar is put into water *and* withdrawn at once, then . . .)

These problems led Carnap to formulate an amendment to the 'Received View' of scientific theories. In addition to ordinary first-order predicate logic, one would have to allow intensional (notably, counterfactual) logic, even in the observational base of the theory. (Cf. Suppe, o.c., p. 42, referring to Carnap, 1956.)

There is a curious weakness to the argument for such a move. One considers a certain kind of natural language statement (dispositional, in this case), one tries a simple-minded predicate-logical transcription: this then turns out to fail – and one concludes that *no* predicate-logical transcription *whatsoever* will be adequate. This pattern of reasoning is also quite current in defense of the thesis that 'predicate logic is insufficient for semantics'.

Can we settle the problem in a more definite manner? To do that, some more clarity is needed as to what logic of dispositionals is to be explicated. One obvious candidate here is the basic subjunctive logic for counterfactuals mentioned in Sections 8 and 10. Assuming our earlier broad constraint of Activity, we can now obtain a definite answer.

9.2. THEOREM. No first-order definable conditional $\varphi(X, Y, =)$ generates precisely the basic subjunctive conditional logic.

Proof. Let φ be any first-order sentence in X, Y, = validating at least the axioms of Burgess (1981). We show that φ defines *inclusion*, and hence that it validates undesirable principles beyond subjunctive logic, such as strengthening of the antecedent.

First, consider $\langle E, A, B\rangle$ with $A \subseteq B$. As the Burgess logic has *if* XX as an axiom and *if* $XY \leftrightarrow$ *if* $X(Y \wedge X)$ for a derived principle, if follows that φ holds for (A, A and hence for) A, B.

Conversely, suppose, for the sake of reductio ad absurdum, that in some model, φ holds for A, B without $A \subseteq B$. I.e., the model verifies the monadic sentence $\varphi \wedge \neg\forall x(Xx \rightarrow Yx)$. As was observed earlier, this sentence must then already be true in some finite model. But then, since φ satisfies Quantity and Antecedence, the proof of Theorem 11.4 below (concerning the principle of Conjunction, which also belongs to the Burgess logic), shows that φ holds for $A, \emptyset$. By Activity then, A equals $\emptyset$, whence $A \subseteq B$ after all: a contradiction. □

This result also provides the justification for the break with monadic first-order definability in current counterfactual semantics.

It might be objected that, perhaps, dispositional statements have a logic different from the above subjective one. But, our impossibility result can be generalized, using earlier insights. The dispositional logic (or logics) will lie somewhere in the spectrum between the minimal constraints of Section 3 and the full power of modal entailment. In any case, it will lack monotonicity for antecedents, whether downward or upward. But then, Theorem 9.1 already provides a refutation. For, any first-order definable conditional satisfying both Antecedence and Confirmation will eventually settle down to either a left-upward monotone type or to a left-downward monotone type.

9.3. THEOREM (Antecedence, Confirmation): A conditional is first-order definable if and only if, above a certain threshold value n for $|A|$, it is expressed by one of the following types: *at least k X are Y, all but at most k X are Y* $(k = 0, 1, 2, \ldots)$.

Proof. By the earlier geometrical observations on the tree of numbers, the existence of a homogeneous 'characteristic triangle' induces the first type (if +) or the second (if −). □

One curious feeling remains. As with many other instances of definability questions, laborious formal analysis has eventually confirmed the earlier heuristics: 'it cannot be done *simply*. So, it cannot be done *at all*'.
Could it be that a profound Principle of Perfection governs our world:
The truth is always simple?

END OF INTERMEZZO

From the present austere monadic first-order language, one may ascend in at least two directions. One is to increase logical power, passing on to *higher-order* notions; the other is to increase power of perception, *enriching the vocabulary*. Actually, there is an argument for a preference here. A truth definition ought to be as simple as possible, not presupposing any higher-order entities in the semantic models. If the latter are thought important, they should be incorporated into these models explicitly. Thus, e.g., the probabilistic approach of Section 7 shifted the higher-order complexity of the truth definition *not least* to the models, which now contained probability measures P: whence the definition for the conditional could become first-order again in terms of P. We will return to this issue of general constraints on truth definitions below, at the end of this section.

Now, it may be ascertained, for the specific case of conditionals, which constraints on possible first-order truth definitions (with any number of 'hidden variables') are induced by the intuitive postulates of Section 3. Essentially these amount to 'preservation properties' – and the latter have been studied extensively in logical model theory.

First, Quantity will give way to a suitable version of *Quality*, as in Section 8.

Next, Antecedence requires that each quantifier be *restricted to A*, or, in our linguistic formulation, to the antecedent predicate X. As a consequence, subformulas Xz not accompanying quantifiers receive the constant truth value 'true', and may therefore be eliminated.

Confirmation requires at least monotonicity (clause (1)), and hence each occurrence of an atomic subformula Yz in the truth definition may be taken to be *positive* in the usual syntactic sense. The other parts of Confirmation constrain this even further.

QUERY. To find a preservation theorem (in the usual model-theoretic sense) for those first-order sentences that satisfy Antecedence and Confirmation.

On the basis of these constraints, further restrictions may decrease the possible range of truth definitions. For instance, strengthening of the antecedent essentially removes occurrences of the existential quantifier. It is shown in van Benthem (1983b) that, for the earlier X, Y,=-language, this leaves only the type

there are at most k X, or all but at most n X are Y.

To conclude, here is a more general look at our enterprise. Here, and in logical semantics in general, there are several 'degrees of freedom' (cf. the discussion following Example 8.2). Once a language has been fixed, there are still choices to be made of *semantic models*, a *truth definition* and also a (conditional) *logic* to be arrived at. Our intuitions place certain broad constraints upon these choices, as we have seen. But basically, one can study all kinds of variation in this scheme. Thus, even within the constraints of Section 3 (or those of Section 8), possibly combined with others, such as the requirement of first-order definability, or 'non-creativity' (to be introduced below), there remain whole ranges of 'triple choices' that fit.

Here are the main two possibilities of variation. First, fixing some desired conditional logic, there may be various mixtures of truth definition and model class that generate precisely this logic. In fact, even fixing both the logic and the truth definition may still leave widely diverging model classes that 'fit' – a phenomenon well-documented in the area of modal logic (cf. Hughes and Creswell, 1968). Less orthodox, but equally feasible, is the fixing of both logic and model class, determining the range of fitting truth definitions – essentially, the topic of the following two sections. (The remaining third case, fixing model class and truth definition already determines the logic uniquely.)

The existence of a variety of solutions does not imply that some solutions are not better than others. Even in the general logical case, there are global constraints already. Notably, one would wish the truth definition to be as simple as possible, implying nothing whatsoever about the semantic models. First-order definability was one aspect of this matter; another might be called *non-creativity*: the truth of $if_{\langle E, R\rangle}A, B$ (to take the setting of Section 8) ought to imply no particular conditions upon the relational structure of the hierarchy $\langle E, R\rangle$. This requirement explains the preference, in Section 8, for implementing Confirmation through additional constraints on the hierarchy, rather than through some complex truth definition. (In this light, the above preservation question concerning Confirmation, though interesting as such, is not absolutely vital.) The issue of various possible 'trade-offs' between truth definition and model conditions remains outside the scope of this paper (cf. van Benthem, 1983(c)).

As this discussion will have indicated, our present study of conditionals eventually raises some very general issues as to the structure of semantic theory, and the fitting locations for the burden of explanation.

10. CONDITIONAL LOGICS

In addition to the abstract view of possible conditionals pursued in the preceding sections, there is the more familiar and concrete topic of specific conditional inferences, and privileged conditional logics.

Besides global intuitions, there exist also convictions concerning the validity, or desirability of particular inference patterns for conditionals. Now, proposals for 'conditional logics' have varied widely. Moreover, their background motivation is sometimes unclear – especially in those cases where merely some suspect 'classical' laws are removed from the usual corpus. Therefore, let us take stock of the natural conditional logics that have appeared in the course of the preceding study.

Recall the restricted language of Section 2, with statements *if XY*, where X, Y may be Boolean compounds of unary predicate expressions.

Several privileged logics have occurred in the earlier sections. First and foremost, the basic intuitions of Section 3 give rise to a

minimal conditional logic M,

whose principles may be read off as follows.

Antecedence:

(1) $$\frac{\text{if}\, XY}{\text{if}\, X(Y \wedge X)}$$

(2) $$\frac{\text{if}\, X(Y \wedge X)}{\text{if}\, XY}$$

Confirmation:

(3) $$\frac{\text{if}\, XY}{\text{if}\, X(Y \vee Z)}$$

(4) $$\frac{\text{if}\, XY}{\text{if}\, (X \vee Z)(Y \vee Z)}$$

(5) $$\frac{\text{if}\, X(Y \wedge Z)}{\text{if}\, (X \wedge Y)Z}$$

(6) $\text{if}\, XX$

Actually, axiom (3) is redundant here (although it will be retained in what follows):

$\text{if}\, XY$, $\text{if}\, X(Y \wedge X)$ (by (1)),

$\text{if}\, (X \vee (Z \wedge X))((Y \wedge X) \vee (Z \wedge X))$ (by (4)),

$\text{if}\, X((Y \vee Z) \wedge X)$ (by Boolean identities),

$\text{if}\, X(Y \wedge Z)$ (by (2)).

Alternatively, axiom (2) already follows from (3).

The next important logics arise in connection with the trilemma of Section 4. First, there was (S5-) modal entailment, axiomatizable as the

classical conditional logic C,

which consists of *M* together with the following additions:

(7) $$\frac{if\,XY \qquad if\,YZ}{if\,XZ}$$ (Transitivity)

(8) $$\frac{if\,XY}{if(X \wedge Z)Y}$$ (Left-monotonicity)

(9) $$\frac{if\,XY \qquad if\,XZ}{if\,X(Y \wedge Z)}$$ (Conjunction)

(10) $$\frac{if\,XY \qquad if\,ZY}{if(X \vee Z)Y}$$ (Disjunction).

There is some redundancy here, and in fact C can also be axiomatized by the familiar set of principles

(i) reflexivity and transitivity

(ii) the infimum laws for $\wedge$ and the supremum laws for $\vee$.

In all these cases, it is assumed that Boolean identities may be used freely inside the antecedent and consequent expressions. □

The second major conditional to come out of Section 4 was that of *not least*. Here the precept is to call Y a consequence of X if the number of confirming instances is no less than that of the counter-examples. For obvious reasons then, let us call this the

preferential conditional logic P.

This logic lacks classical laws such as transitivity or left-monotonicity. (But then, such non-validities have been reported by many students of conditionals.) On the other hand, although this observation falls outside of the present language, the *not least* conditional validates Conditional Excluded Middle, an inference normally associated with the case of just one single relevant world. One final reason for interest in this logic consists in the possible coincidence with the De Finetti logic of Section 7.

Our conjecture is that P can be recursively axiomatized. But in practice, its principles are difficult to locate. Recall that P did contain the Δ-principle of Example 7.4, an axiom also valid for *all* (and hence in C).

In fact, the relations between the three logics introduced up till now are as follows.

10.1. THEOREM. $M \subseteq P \subseteq C$; and all inclusions are proper.

Proof. The only two assertions which are not immediate follow here.

– $M \neq P$: the Δ-principle of Example 7.4 does not hold for all conditionals satisfying the minimal logic. A counter-example is provided by *all but at most one*.

– $P \subseteq C$: Let the inference from *if* $X_1 Y_1, \ldots,$ *if* $X_n Y_n$ to *if* XY be refuted by inclusion in some model. Notice that *not least* X_i are Y_i $(1 \leqslant i \leqslant n)$, because *all* are. On the other hand, there exists at least one object in $X - Y$. Now, add a number of copies of this object, behaving exactly the same as far as (non-) membership of the relevant sets is concerned, such that the cardinality of $X - Y$ exceeds that of $X \cap Y$. Such an addition changes none of the previous relationships *if* $X_i Y_i$. But then, we now have a counter-example for the same inference with respect to *not least*. □

The final logic to arise from Section 4 reflects mere overlap (*some or all*). This *exemplary conditional logic E* validates the two inference patterns of reflexivity and *symmetry*:

$$(11) \qquad \frac{if\,XY}{if\,YX}.$$

In van Benthem (1983b) it is shown how this implies upward monotonicity in both arguments, and all possibilities are classified.

Evidently, this is not a serious competitor. As a matter of some interest, we mention a

CONJECTURE. $M = C \cap E$.

Of greater interest are some intermediate logics immediately suggested by the earlier ones. For instance, notice how the classical logic C adds *two* kinds of principle to the minimal M. There are straightforward axioms of 'transmission', such as transitivity and left-monotonicity; but there is also 'combination', as in Conjunction and Disjunction. The latter principles are of interest by themselves, and we define the

subjunctive conditional logic S

as the result of adding Conjunction and Disjunction to M. The motivation for this name lies in the following result.

10.2. THEOREM. S is precisely the basic subjunctive conditional logic of Burgess 1981.

Proof. The principles of Burgess' presentation are

(i) *if* XX

(ii) *if* XY, *if* XZ imply *if* $X(Y \wedge Z)$

(iii) *if* $X(Y \wedge Z)$ implies *if* XY

(iv) *if* XY, *if* XZ imply *if* $(X \wedge Y)Z$

(v) *if* XY, *if* ZY imply *if* $(X \vee Z)Y$.

All derivations involved are straightforward, both ways. □

As a result of this proof, the Burgess principle (iv), appearing rather ad hoc, receives a natural motivation through clause (3) of Confirmation.

Thus, the remaining basic logic of this paper, that of top-ranking (Section 8) has been introduced as well. It remains to establish the natural place of S in the above scheme.

First, obviously, $M \subseteq S \subseteq C$; where all inclusions are proper. There is a deeper connection between M and S, however. Notice that M contained only one-premise inferences, while C, P, S all added two-premise ones.

10.3. THEOREM. M coincides with the one-premise fragment of S.

The following argument will make it clear that the above axiomatization for M (and S) is rather perspicuous and useful.

Proof. Consider any invalid inference from *if* XY to *if* ZU in M. We shall find a Lewis-model which is an S-counter-example. First, some transformations are useful, into M-equivalent assertions:

– *if* XY to *if* $X(Y \wedge X)$ to *if* $(X_1 \vee X_2)X_2$, where X_1, X_2 are disjoint disjunctions of complete state descriptions (composed out of the proposition letters occurring in X, Y, Z, U) such that $X \leftrightarrow X_1 \vee X_2$, $Y \wedge X \leftrightarrow X_2$;

– *if* ZU likewise to *if* $(Z_1 \vee Z_2)Z_2$.

Now, as *if* ZU is non-derivable, Z_1 cannot be empty. By itself, a single world verifying some state description from Z_1 would already falsify the conclusion – being a closest $Z_1 \vee Z_2$-world to itself which is not Z_2. But, the premise imposes the condition that closest $X_1 \vee X_2$-worlds must be

X_2-worlds; i.e., for every X_1-world, there must be some closer X_2-world. This condition is only operative when

$$Z_1 \subseteq X_1.$$

And even then, the obvious dodge is to pick a Z_1-world, with respect to some vantage world in $X_2 - Z_2$. This will fail to falsify the conclusion only if

$$X_2 \subseteq Z_2.$$

But then, putting together these two assertions in M, we have

(i) $if(X_1 \vee X_2)X_2$

(ii) $if(Z_1 \vee X_2)X_2$ (from (i) and (5); removal of counter-examples)

(iii) $if(Z_1 \vee (X_2 \vee (Z_2 \wedge \neg X_2)))(X_2 \vee (Z_2 \wedge \neg X_2)))$ (from (ii) and (4): addition of confirming instances)

By suitable Boolean identities, then, the conclusion $if(Z_1 \vee Z_2)Z_2$ follows after all: a contradiction. □

The counter-examples obtained in the above argument are even connected in the sense of Lewis (1973); whence M is also the one-premise fragment of the full original Lewis logic. Moreover, our conjecture is that, by a similar kind of argument, S equals the *(meta-)disjunction-free* fragment of that same Lewis logic.

As for the connection between S and the preferential logic, again, we have a

CONJECTURE. $P \subseteq S$.

By way of evidence, observe that the Δ-principle of Section 7 is indeed valid in Lewis models.

APPENDIX: 'BASIC CONDITIONAL LOGIC'

In contrast with the above presentation, here are some salient points from the analysis of conditionals given in Chellas (1975). The basic semantic framework in that paper has possible worlds models

$$\langle W, f \rangle;$$

where f is a function assigning sets of subsets of W to couples (w, X) $(w \in W, X \subseteq W)$. The explication of the conditional then becomes

$$if\, XY \text{ is true at } w \text{ iff } [\![Y]\!] \in f(w, [\![X]\!]).$$

The corresponding minimal logic L_1 has only one principle, viz. Replacement of Equivalents. A next plausible constraint, in Chellas' analysis, is that f be representable through some function f^* assigning subsets of W to couples (w, X), with

$$if\, XY \text{ is true at } w \text{ iff } f^*(w, [\![X]\!]) \subseteq [\![Y]\!].$$

The corresponding logic L_2 now becomes L_1 plus (upward right) Monotonicity (principle (3) above) and Conjunction (principle (9)). Finally, on this basis, Chellas considers various extensions, notably a logic L_3 having the additional principles

(a) $if \bot X$ ($\bot$ is any contradiction)

(b) $\dfrac{if(X \vee Y)Z}{if\, XZ}$ $\qquad \dfrac{if(X \vee Y)Z}{if\, YZ}$

(c) $\dfrac{if\, XZ \qquad if\, YZ}{if(X \vee Y)Z}$

Evidently, this perspective is at an oblique angle to the above. Now, as was stressed in Section 1, there is no need to make a choice here: both approaches may be illuminating. Therefore, we only mention some points of comparison.

The first, most general perspective is technically equivalent to the relational setup in Section 2. (Possible iterations, involving the parameter w, are not taken into account here. Indeed, significantly, not a single iterated conditional axiom occurs in Chellas' paper.) L_1 is what would have been obtained here as well, if no further constraints had been formulated in Section 3.

The intuitive constraints leading to the minimal logic M do not arise on Chellas' approach. For instance, even such a weak principle as reflexivity ($if\, XX$) remains beyond L_3; while a pervasive principle (in the present paper) such as Antecedence is only mentioned marginally. On the other hand, the f^*-representation illustrates an interesting line of thought not pursued in this paper, viz. the investigation of various simpler representations for the most general conditional relation.

The specific principle of Conjunction has not been postulated on our approach because of its great strength (cf. Theorem 11.4 below). Moreover, there is also a plausible philosophical critique of Conjunction as a principle of conditional reasoning: witness the discussion of 'Aggregation' in Jennings and Schotch (1980).

Finally, L_3 is an interesting logic in that it has both disjunction (c) and strengthening of antecedents (b), though lacking transitivity as a postulate. Analogously, in our earlier set-up, one could consider M together with L_3, forming a 'classical' conditional logic without transitivity. It is easy to see that this logic is a proper extension of S. In our framework, however, the extension even collapses into the earlier classical logic. For, from *if* XY, *if* YZ, one may infer *if* XY, *if* $(X \wedge Y)Z$ (strengthening of antecedents), and thence *if* XZ, by a familiar derived rule of S. Thus, transitivity is forthcoming after all.

11. GENERAL PATTERNS OF CONDITIONAL INFERENCE

The preceding section ended on an all too familiar track in intensional semantics: a proliferation of logics. The main virtue of the perspective taken in this paper is the following, however. Not only does it generate specific privileged logics, but it also provides the means for investigating possible conditional inference patterns, without being tied to exclusive 'natural' clusters. Thus, one may look at arbitrary inferential theories, asking for the range of conditionals validating at least, or just these. This is not just the common question of modelling some given 'logic': we are after the entire range of possible modellings, so to speak.

First, consider pure *if*-patterns without Boolean operators (cf. Section 2). These may be thought of as expressing familiar conditions on the binary conditional relation. A quick survey of n-premise schemata ($n = 0, 1, 2, \ldots$) yields only *reflexivity* and *transitivity* as serious candidates for conditional principles. (For propositional operators in general, the situation is much more diverse; cf. van Benthem (1983b).) Let us explore the latter combination.

11.1. THEOREM (Antecedence). Every reflexive, transitive conditional is *transmitting*, in the sense of validating both strengthening of antecedents and weakening of consequents.

Proof. Here is the second case (the first is analogous). Assume that *if* A, B, while $B \subseteq C$. Then *if* B, B (reflexivity), *if* $B, B \cap C$, *if* B, C (Antecedence), and hence *if* A, C (transitivity). □

Modulo one more constraint, these two requirements determine even just one single conditional.

11.2. THEOREM (Antecedence, Activity). The only reflexive, transitive conditional is *all*.

Proof. Let *if* be any reflexive transitive conditional. We show that *if* A, B iff $A \subseteq B$.

'If': this direction goes as in the preceding argument, by reflexivity and Antecedence. 'Only if': as before, since $A - B \subseteq A$, we have *if* $A - B, A$. So, by transitivity, *if* $A - B, B$, and hence *if* $A - B, \emptyset$ by Antecedence. Activity then implies that $A - B = \emptyset$; i.e., $A \subseteq B$. □

Actually, this argument would also go through with a weaker condition:

11.3. COROLLARY (Antecedence, Activity). *All* is the only conditional allowing strengthening of antecedents.

Proof. The step to be replaced in the above is this: 'from *if* A, B *directly* to *if* $A - B, B$'. □

This result explains why failure of 'left-monotonicity' is the hall-mark of all current non-classical conditional logics.

AN APPLICATION: THE SCOTT PRINCIPLES

The above cluster of requirements appears in the paper by Scott (1971). In the present terminology, Scott presents the following fundamental properties of conditionals: Reflexivity, Transitivity, Left- and Right-Monotonicity. Assuming Antecedence, Theorem 11.1 tells us that these postulates are not independent: monotonicity already follows from the first two requirements. Moreover, assuming also Activity, Theorem 11.2 adds the insight that the only conditional to satisfy these three principles is modal entailment. Accordingly, we understand why most current conditional logics violate the Scott principles: they have to.

In addition to pure patterns, there are also 'mixed' patterns of conditional inference, involving the conditional together with some other logical constants, whose meanings are fixed beforehand. (Even the latter could be treated as variables eventually, of course.) For instance, the above monotonicity principles are of this mixed variety, involving $\wedge$, $\vee$ with their standard meanings.

By way of illustration, Conjunction may be investigated in the above manner.

11.4. THEOREM (Antecedence, Activity, Quantity). On the *finite* models, *all* is the only conditional satisfying Conjunction.

Proof. The idea is this. If *if* satisfies Conjunction, and *if* A, B holds without $A \subseteq B$, then $B \cap A \subsetneq A$ and also *if* $A, B \cap A$. But then, by intersecting $B \cap A$ with equally large distinct sets $C \cap A$ (for which *if* A, C – by Quantity), one obtains *if* A, D for ever smaller sets $D \subseteq A$, and in the end *if* $A, \emptyset$. Therefore, $A = \emptyset$ (Activity), and hence $A \subseteq B$ after all: a contradiction. □

Another result in this connection again relates Conjunction to the earlier pure relational conditions.

11.5. THEOREM (Antecedence, Quantity, Extension). Every reflexive transitive conditional satisfies Conjunction.

Proof. Let *if* be reflexive and transitive. Consider a situation with *if* A, B, *if* A, C, as in Figure 14(i). By Antecedence, it suffices to consider $A, A \cap B, A \cap C$.

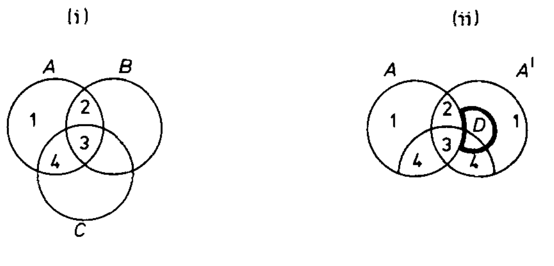

Fig. 14.

CASE 1. $2 \leqslant 1 + 4$.

Then, consider a new set A' consisting of $A \cap B$ with $1 + 4$ new worlds added. So, $A' \cap A = B \cap A$, and hence *if* A, A' (a). Moreover, by Quantity, the symmetry in numbers implies that *if* $A', B \cap A$. Now, pick any subset D of $A' - (B \cap A)$ of cardinality 2, as in Figure 14(ii). Again by Quantity, *if* $A', (B \cap A \cap C) \cup D$ (b). Transitivity then yields, from (a), (b), that *if* $A, (B \cap A \cap C) \cup D$; and hence *if* $A, B \cap A \cap C$, *if* $A, B \cap C$, by Antecedence.

CASE 2. $2 > 1 + 4$.

In this case, $4 \leqslant 1 + 2$ – and the argument of Case 1 may be repeated with respect to C. □

A direct syntactic deduction of Theorem 11.5 exists as well.

Other mixed patterns arise when negation is introduced. The ubiquity of modal entailment shows again in the behaviour of the best-known inference of this kind.

11.6. THEOREM (Antecedence, Activity, Extension). The only reflexive conditional satisfying *contraposition* is *all*.

Proof. As always, $A \subseteq B$ implies *if* A, B (reflexivity, Antecedence). Conversely, suppose that *if* A, B. As negation involves complements with respect to the universe, the context might be important in general. But here, thanks to Extension, strong Antecedence holds, allowing a complete retreat into A. And thus, *if* A, B, *if* $A - B, A - A$ (Contraposition), *if* $A - B, \emptyset, A - B = \emptyset$ (Activity): $A \subseteq B$. □

From these sample results, the flavour of the present study will have become clear.

One important variation upon the above theme is the *completeness* issue of, given a set of inference patterns, which conditionals validate precisely these, and no more. For instance, in the preceding, no specific conditional relation has been found yet validating precisely the minimal conditional logic.

QUERY. To find the range of conditionals validating *M*, or *M* together with the non-inferential constraints of Section 3, and no more.

Clearly, the systematic study of conditional inference has only just begun.

12. PROSPECTS

This has been a foundational paper, asking perhaps untimely questions about the nature of conditional logic. Even so, there is a potential here for further research, and some relevant possibilities will be mentioned.

Within the bounds of our first intuitions (Section 3), there remain many questions concerning conditional logics (Section 10) or conditional inference in general (Section 11); questions which may be pursued both in the finite and the infinite realm (Section 6).

As various intuitions are relaxed or modified (notably Quantity), the above questions can again be studied in intensional or inductive settings (Sections 7 and 8). But also, these frameworks themselves could be analyzed in more depth, perhaps in terms of new intuitions.

Then there is the limitation, explained in Section 2, to non-iterated conditionals. A good starting-point for lifting this restriction would be the study of world-indexed conditional relations $if_{E,w}$, again on the pattern of the previous investigation. Especially, in this way, one would hope to bridge the gap between the earlier-mentioned 'horizontal' and 'vertical' directions in conditional logic.

But, even within the present bounds, the illumination provided by the generalized quantifier perspective of this paper seems real enough to warrant further investigation.

NOTE

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