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# Thick Grating Focussing-Device-Design Using Poynting-Vector-Optics

P. St. J. Russell

Department of Engineering Science, University of Oxford, Parks Road, Oxford, England

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**Abstract.** Novel thick grating focussing and de-focussing devices are described which employ uniform phase gratings with special boundary shapes. The analysis used is based upon an eigenmodal approach to Kogelnik's coupled-wave equations, akin to the dynamical theory of x-ray diffraction. The relationship between the direction of phase progression of the coupled-waves at Bragg incidence, and the direction of the Poynting vector is carefully delineated. As a consequence, a new technique – Poynting Vector Optics – is suggested as potentially an important means of designing thick gratings to fulfil certain beam processing roles, especially in integrated optics applications. The two-dimensional coupled-wave equations are briefly employed to illustrate the effectiveness of a particular focussing device.

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Volume phase diffraction gratings have been of late much studied following Kogelnik's well-known article [1] analysing their properties. They appear in a variety of contexts, volume holography, acoustooptical interaction and integrated optics (the grating written on the guiding film) being the main examples. Kogelnik's coupled- wave theory is valid only for the 1-D case in which the incident and Bragg diffracted waves are infinite plane, the grating itself uniform, and the boundaries flat. There have been numerous contributions [2] generalising Kogelnik's theory to greater or less degrees. Solymar and his associates have extended it to 2 and 3-D, allowing for non-uniform grating strength, non-plane non-uniform waves, and non-flat boundaries. Interesting results have been reported for non-uniform gratings with flat boundaries [3], including the design of high efficiency volume grating beam synthesizers [4], the discovery of wave-guiding effects closely related to the Borrmann effect in x-ray diffraction [5], and the relationship between non-uniformity of grating strength and the fidelity and efficiency of beam reconstruction [6].

It is the aim of the present contribution to describe preliminary results of an exploration into the effect of non-flat boundaries on the performance of uniformly modulated gratings. This study will shed light on the relationship between the group velocity and phase velocity of the fields within a volume phase grating for incidence in a Bragg regime. In particular, it is shown that the direction of *power flow* is different from the direction of *phase progression* of the coupled waves in the grating. The relationship which exists between these directions is not an obvious one, and is elaborated upon using an eigen modal (or characteristic-modal) approach closely akin to that used in the dynamical theory of x-ray diffraction [7]. A new technique for analysing the properties of volume diffraction gratings – the "Poynting Vector Optics" of volume gratings – is suggested as a potentially important design tool when the grating strength is high. Finally, following the principles of Poynting Vector Optics, novel power focussing and defocussing devices are designed in which the boundary shape of a uniform phase grating is varied in a special manner. These devices are likely to have important applications as beam expanders and reducers in integrated optics.

## 1. Eigen-modes and the Direction of Power Flow

Kogelnik's equations, expressed in a slightly different notation, read (for exact Bragg incidence) as follows

$$E(\mathbf{r}) = V_0(X)\exp(-\mathrm{j}\bar{\mathbf{k}}_0\cdot\mathbf{r}) + V_{-1}(X)\exp[-\mathrm{j}(\bar{\mathbf{k}}_0-\mathbf{k}_g)\cdot\mathbf{r}]\,, \quad (1)$$

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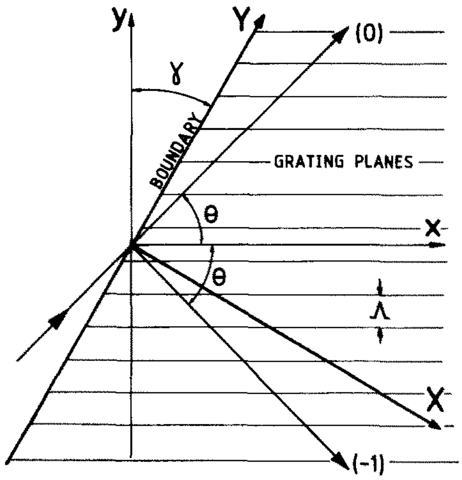


Fig. 1. The grating geometry; an infinite plane wave is incident on a straight boundary at the first Bragg angle $\theta$ with respect to the grating planes. Continuity of mean dielectric constant is assumed across the boundary. Coordinate axes $(x, y)$ are parallel and perpendicular to the planes, and the axes $(X, Y)$ are at an angle $\gamma$ to $(x, y)$. The line $X=0$ is the input boundary

where

$$\begin{bmatrix} \Gamma s_0 & j\kappa \\ j\kappa & \Gamma s_{-1} \end{bmatrix} \begin{bmatrix} V_0 \\ V_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2}$$

$\Gamma \equiv \partial/\partial X\,, \quad \kappa = \varepsilon_m \beta / 4\varepsilon_{oc}\,,$

$(X, Y)$ are coordinates perpendicular and parallel to the input boundary, the relative dielectric constant

$\varepsilon_r = \varepsilon_{oc} + \varepsilon_m \cos(\mathbf{k}_g \cdot \mathbf{r})\,,$

$\mathbf{k}_g = (2\pi/\Lambda)\hat{\mathbf{y}}$

is the grating vector, $(x, y)$ are coordinates parallel and perpendicular to the grating planes, $\bar{\mathbf{k}}_0$ is the refracted illuminating wave vector in the absence of a grating (i.e. when $\varepsilon_m = 0$) and $V_0$ and $V_{-1}$ are the coupled-wave amplitudes. The slope factors $s_0$ and $s_{-1}$ are related to the trajectory angle $\theta$ and the boundary angle $\gamma$ (Fig. 1) by

$$s_0 = \cos(\theta + \gamma) \quad \text{and} \quad s_{-1} = \cos(\theta - \gamma)\,, \tag{3}$$

and the propagation constant $\beta = (2\pi/\lambda)\sqrt{\varepsilon_{oc}}$ where $\lambda$ is the free space wavelength of the monochromatic illuminating wave. The Bragg condition is exactly satisfied when

$|\bar{\mathbf{k}}_0 - \mathbf{k}_g| = |\bar{\mathbf{k}}_0| = \beta\,;$

this condition has been assumed in (1) and (2). Characteristic modes of (2) may be found by requiring the determinant of coefficients to equal zero, yielding eigenvalues

$$\Gamma^{\pm} = \pm j\kappa / \sqrt{(s_0 s_{-1})} \tag{4}$$

and normalised mode shapes

$$\mathbf{e}^{\pm} = \begin{bmatrix} \mp\sqrt{s_{-1}} \\ \sqrt{s_0} \end{bmatrix} \cdot (1/\sqrt{s_0 + s_{-1}})\,, \tag{5}$$

where $\mathbf{e}$ is a column matrix. Thus the general solution of (2) may be written:

$$\begin{bmatrix} V_0(X) \\ V_{-1}(X) \end{bmatrix} = A^+ \mathbf{e}^+ \exp\left(j \frac{\kappa X}{\sqrt{s_0 s_{-1}}}\right) + A^- \mathbf{e}^- \exp\left(-j \frac{\kappa X}{\sqrt{s_0 s_{-1}}}\right), \tag{6}$$

where $A^-$ and $A^+$ are the modal amplitudes. Only *one* of the two modes will be excited if the ratio $[V_0(0)/V_{-1}(0)]$ at the input boundary is equal to $\mp\sqrt{s_{-1}/s_0}$. This excited mode will then progress through the grating *without further change*, that is, the coupled-wave amplitudes $V_0$ and $V_{-1}$ will be decoupled, remaining constant in magnitude. The only significant change is in their phase; the $X$-component of the wave-vector of the excited eigen-mode will differ by an amount $\pm(\kappa/\sqrt{s_0 s_{-1}})$ from its value in the absence of a grating, $(\bar{\mathbf{k}}_0 \cdot \hat{\mathbf{X}})$.

Suppose now that only one eigenmode is excited. The power flow for this mode may be based upon a simple intuitional definition of the Poynting vector

$$\mathbf{S} = \sum_{n=0,-1} |V_n|^2 \hat{\mathbf{u}}_n\,, \tag{7}$$

where $\hat{\mathbf{u}}_n$ is a unit vector in the direction of phase progression of coupled wave $(n)$. From (7) it is easy to show that the angle between $\mathbf{S}$ and $\hat{\mathbf{x}}$ (a unit vector parallel to the planes), $\alpha$, is given by

$$\alpha = \arctan(\tan\gamma \tan^2\theta)\,, \tag{8}$$

independently of which eigenmode ($\mathbf{e}^+$ or $\mathbf{e}^-$) is excited. The theoretical basis for the definition (7) of $\mathbf{S}$ is well established in x-ray diffraction theory (von Laue [8]). It is in fact the definition of *macroscopic* energy flow, i.e. the actual microscopic flow averaged over a single grating period. Kato's result [9], that the direction of power flow associated with an eigenmode is perpendicular to the dispersion surface at that eigenmode's point of excitation ("tie-point"), is equivalent to (8). For a clear description of the dynamical theory of x-ray diffraction, see the Battermann and Cole review article [7]; for a delineation of the link between the present theory and it, see [2].

An important question still to be answered is how $\mathbf{S}$ is affected for the common un-balanced boundary condition $V_0(0) = E_0$, $V_{-1}(0) = 0$. It is clear that in this case both eigenmodes will be excited in equal amounts. Is the Poynting vector therefore a simple sum $\mathbf{S}^+ + \mathbf{S}^-$ of the power flows of each mode? A glance at (6) will show that this cannot be so since the (+) and (−) modes have different rates of phase progression in the $X$-direction. This wave-vector splitting $(2\kappa/\sqrt{s_0 s_{-1}})$ means that the modes beat together giving the familiar "pendellösung" sinusoidal solutions of Kogelnik's theory. Therefore, the direction of power flow will oscillate between the (0) and (−1) directions at a spatial frequency proportional to $2\kappa/\sqrt{(s_0 s_{-1})}$, with a *mean* direction (i.e. that obtained by averaging over a pendellösung period) at an angle $\alpha$ to the grating planes. In order to clarify this, take the solutions of (6) for the unbalanced boundary condition mentioned above:

$$V_0(X) = E_0 \cos(\kappa X/\sqrt{s_0 s_{-1}}),$$
$$V_{-1}(X) = E_0 \sqrt{s_0/s_{-1}} \sin(\kappa X/\sqrt{s_0 s_{-1}}). \tag{9}$$

Hence, the Poynting vector, according to (7), is

$$\mathbf{S}(X) = (|V_0|^2 + |V_{-1}|^2)\cos\theta\, \hat{\mathbf{x}} + (|V_0|^2 - |V_{-1}|^2)\sin\theta\, \hat{\mathbf{y}}. \tag{10}$$

It may easily be shown via algebraic manipulation that in this case the angle $\alpha$ depends upon position, and is given by

$$\alpha(q) = \arctan\left\{\tan\theta \cdot \frac{\tan\gamma_n + \cos(2q)}{1 + \tan\gamma_n \cos(2q)}\right\}, \tag{11}$$

where

$$q = \kappa t(x_n - y_n \tan\gamma_n)/\sqrt{(1 - \tan^2\gamma_n)}, \tag{12}$$

$\gamma_n$ is a generalised boundary angle defined by

$$\gamma_n = \arctan(\tan\theta \tan\gamma), \tag{13}$$

$(x_n, y_n)$ are normalised coordinates in the $x$ and $y$ directions:

$$x_n = x/t\cos\theta, \qquad y_n = y/t\sin\theta \tag{14}$$

and $t$ is a characteristic length along a trajectory in the grating. The normalisation procedure in (13) and (14) allows elimination of the angle $\theta$ explicitly from the expressions[1]. Integrating (11) in the form

$$dy_n/dx_n = dy/dx \cdot \cot\theta = \cot\theta \cdot \tan[\alpha(q)]$$

yields for the integrated path of power flow:

$$y_n(\zeta) = y_n(0) + \left\{\zeta \tan\gamma_n + [\sqrt{(1 - \tan^2\gamma_n)}/2\kappa t] \cdot \sin[2\kappa t\zeta/\sqrt{(1 - \tan^2\gamma_n)}]\right\}/(1 - \tan^2\gamma_n), \tag{15}$$

[1] Note however that a two-wave regime [in which only the (0)th and (−1)th coupled-wave amplitudes are non-negligible, all higher-order waves being very weakly excited] must be in force. This restricts the angle $\theta$ to values which are not too small, via the condition that the parameter $\varrho = (2\varepsilon_{oc}/\varepsilon_m)(\lambda/\Lambda)^2$ must be greater than unity [10].

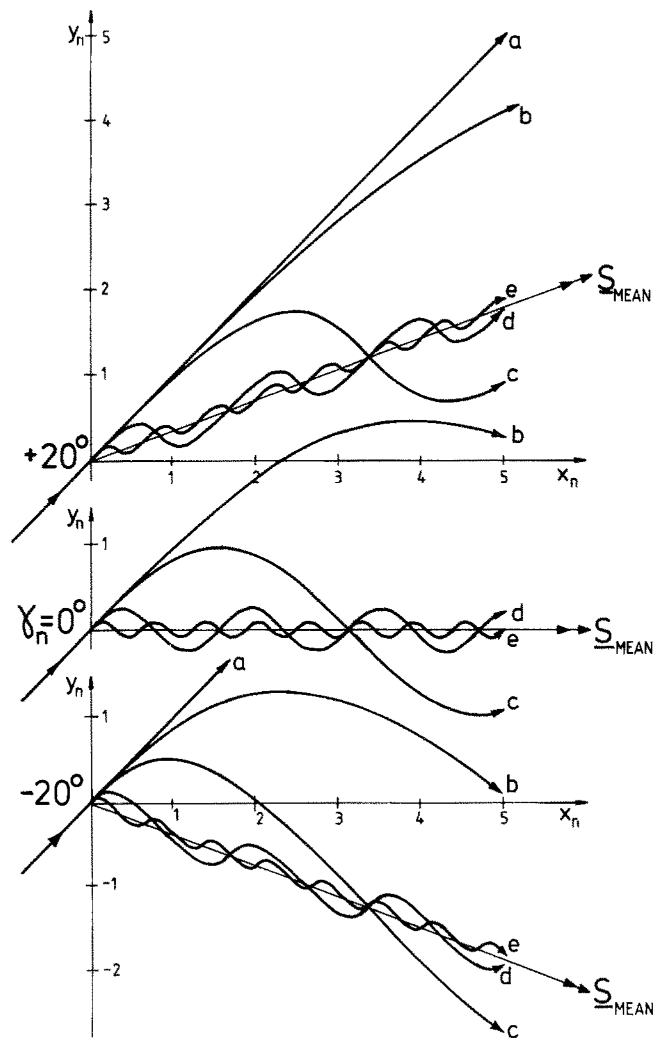


Fig. 2a–e. Poynting vector paths for three boundary angles, $\gamma_n = 0°$, $+20°$, and $-20°$, plotted against the normalised axes $(x_n, y_n)$; thus the grating planes are parallel to $x_n$. Values of grating strength and corresponding characters are: $\kappa t = 0$(a), 0.2(b), 0.5(c), 2.0(d), and 5.0(e). The input ray passes straight through for zero grating strength (a), and fluctuates about the mean direction $\mathbf{S}_{\text{mean}}$ at an angle $\gamma_n$ to the $x_n$ axis for $\kappa t > 0$

where

$$\zeta = x_n - y_n \tan\gamma_n$$

is proportional to $X$, see (12) and (9). This equation gives the "path" of a ray of power entering at $y_n = y_n(0)$, $\zeta = 0$. Equation (15) clearly shows that the mean path (averaging over one period of the sine function) starting at $y_n(0) = 0$ is

$$y_n = x_n \tan\gamma_n, \tag{16}$$

which corresponds to the power flow direction of a single eigenmode, see (8). For $\kappa = 0$ (no grating) it is

$$y_n = x_n, \tag{17}$$

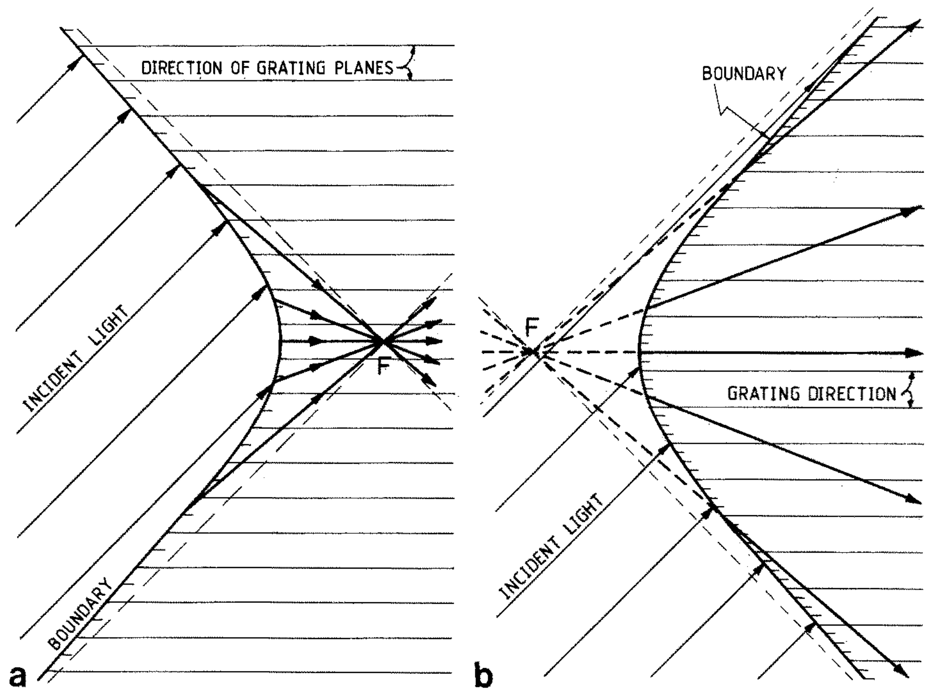


Fig. 3. (a) Focussing of an input wave by Poynting vector optics. The direction of $S_{mean}$ at every point on the boundary (assuming Bragg incidence and continuity of average dielectric constant) is always towards a point F within the grating. Hence for high values of grating strength, focussing of the incident power will occur. (b) The converse geometry to (a); a virtual focus F exists such that the direction of $S_{mean}$ at every point on the boundary is away from F. For high values of grating strength, this device would function as a beam expander

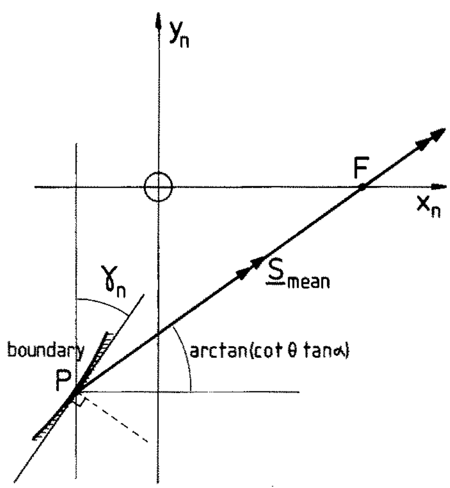


Fig. 4. The geometry used to derive the correct boundary shape for focussing to occur. The local tangent to the boundary is at an angle $\gamma_n$ to the $y_n$ axis such the $S_{mean}$ passes through a fixed point F. In the normalised coordinate system $(x_n, y_n)$ the angle $\arctan(\cot\theta \tan\alpha)$ is in fact equal to $\gamma_n$

that is, the direction of phase progression of the incident wave (no diffraction), and for $\kappa \to \infty$ (the high coupling limit) it is

$$y_n = x_n \tan\gamma_n, \tag{18}$$

that is, the mean path. The magnitude of the fluctuation from the mean path is $1/[2\kappa t \sqrt{(1-\tan^2\gamma_n)}]$, inversely proportional to the grating strength. Equation (15) is plotted in Fig. 2 for a variety of boundary angles and grating strengths. For very low coupling, the incident wave travels through relatively undiffracted; at intermediate values, the power flow path fluctuates over a range on either side of the mean path (18).

It is interesting to enquire how the power flow operates if the illuminating wave is of finite extent. Poynting vector optics suggests that (for high $\kappa t$), power will flow predominantly at an angle $\alpha$ to the grating planes, dependent via (8) upon the angle between the (flat) boundary and the grating planes. This general result has been substantiated (using the two-dimensional coupled-wave theory [5]) for Gaussian beam incidence on a flat boundary, the degree of power flow confinement to the direction $\alpha$ being proportional to the coupling level. From the discussion in the last paragraphs, it seems highly likely that the correct condition for good guiding of power is that the spatial rate of change of the incident amplitude distribution should be much slower than the rate of pendellösung coupling between the waves. Or, stated in another way, that the change in the incident amplitude over a distance equal to the fluctuation $1/[\kappa t \sqrt{(1-\tan^2\gamma_n)}]$ of the Poynting vector path from its mean should be very slight. An alternative approach is based upon Fourier plane-wave decomposition of the incident beam, arguing that the wider this spectrum relative to the off-Bragg angular sensitivity of the grating, the poorer is the guiding of the incident power along the mean path $y_n = x_n \tan\gamma_n$ [2].

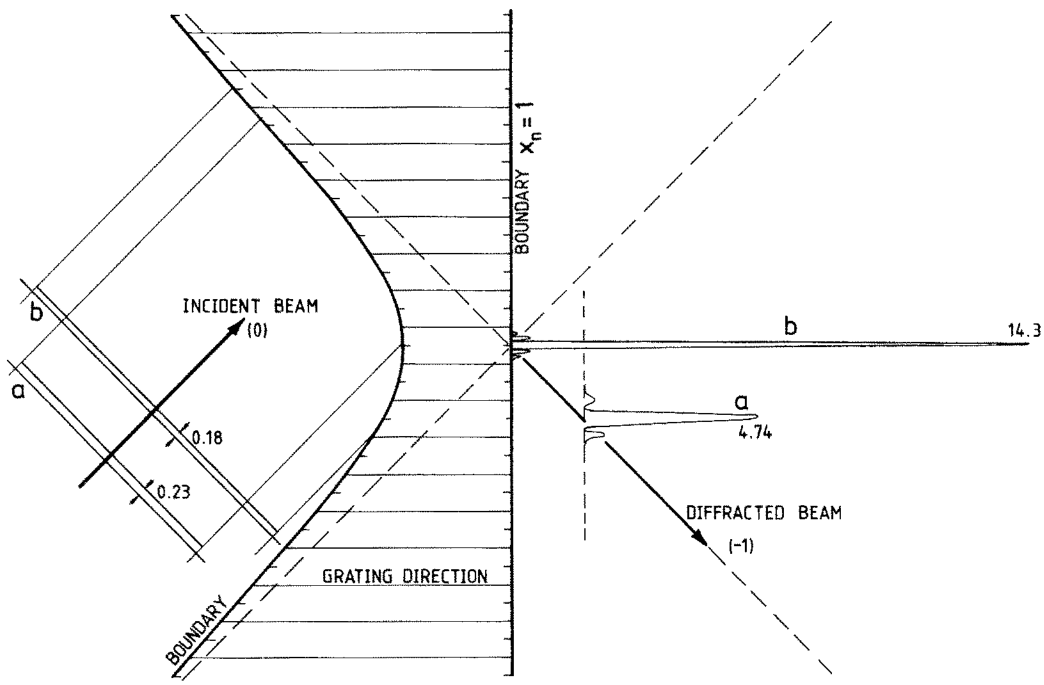


Fig. 5a and b. The results of a two-dimensional analysis of the focussing device (to be presented more fully in a subsequent article). The numbers adjacent to each intensity profile refer to peak intensities. The power in each case [i.e. (a) and (b)] is unity, and only the *diffracted* profiles at the output face are depicted. For clarity the profile *a* is given a short way along the diffracted beam trajectory. Values of grating strength and corresponding characters are: $\kappa t = 7.1$ (a), 16.5 (b). Intensity magnifications of 20× (a) and 80× (b) are obtained. The grating strengths were selected so that maximum power is diffracted into the $(-1)$th beam. Upon leaving the output boundary $x_n = 1$, the light travels in the $(-1)$th direction, with profiles as depicted (note that these profiles are taken at an angle of 45° to the direction of travel)

## 2. Poynting Vector Optics as a Design Tool for Grating Devices

Intriguing power focussing and defocussing devices may be designed using Poynting vector optics. From (8), a special boundary shape can be found such that a ray incident on any part of it will be diffracted such that power will always flow (on average in the high-coupling limit) either (a) towards a real focus or (b) away from a virtual focus (Fig. 3). In terms of normalised coordinates, (8) may be written $\tan\alpha \cot\theta = \tan\gamma_n$, and (Fig. 4) the equation of the correct boundary shape will be

$$dy_n/dx_n = \tan\theta \cot\alpha = (1 - x_n)/(-y_n) \tag{19}$$

giving upon integration

$$y_n^2 = -x_n(2 - x_n). \tag{20}$$

The dimension $t$ in this context is the "along-trajectory" length between the origin and the focal plane $x_n = 1$. Equation (20) takes a simpler form if coordinate axes perpendicular to the trajectory directions,

$$\xi_n = (x_n - y_n) \quad \text{and} \quad \eta = -(x_n + y_n)$$

are used; the boundary curve is then

$$\eta_n = \xi_n/(1 - \xi_n) \quad \text{or} \quad \xi_n = \eta_n/(1 + \eta_n). \tag{21}$$

The lines $\xi_n = 1$ and $\eta_n = -1$ are trajectories through the focus. In Fig. 5, field intensity profiles in the focal plane, obtained using the 2-D theory, are given for incidence of a uniform finite beam on this boundary. The values of the coupling, $\kappa t = 7.1$ and 16.5 mean that there are approximately 2 and 5 periods of pendellösung between the origin $x_n = 0$, $y_n = 0$ and the focus (1, 0). These values were selected (using the 2-D theory) such that maximum power is diffracted into the $(-1)$th coupled wave. The device is an efficient non-converging (i.e. non-lens-like) beam concentrator, the degree of concentration (or contraction of the incident finite width) being proportional to the coupling level. A full analysis of this device will be available in a later article.

The alternative virtual focus geometry (with the grating present on the other side of the boundary) seems likely to offer non-divergent beam expansion, but has not yet been analysed.

As a general technique, Poynting vector optics offers a useful method for designing (in the high coupling case) thick grating beam processors. Probably the most important application area is integrated optics, where thick gratings can be used for spatial processing of guided modes, the modulation depth of the grating being controlled electro-optically at will (system of interdigital electrodes), and nonplane grating boundaries being easy to fabricate. It is hoped in future articles to explore the possibilities of Poynting vector optics more rigorously, applying it also to gratings with non-planar diffracting fringes and non-uniform modulation depths.

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