# GENERALIZED LYAPUNOV CHARACTERISTIC INDICATORS AND CORRESPONDING KOLMOGOROV LIKE ENTROPY OF THE STANDARD MAPPING

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**Abstract.** Lyapunov characteristic indicators are currently defined as the mean, i.e. the first moment, of the distribution of the local variations of the tangent vector to the flow. Higher moments of the distribution give further informations about the fluctuations around the average.

**Key words:** Lyapunov characteristic numbers – Kolmogorov entropy – standard mapping

## 1. Introduction

It is well known that chaoticity of a dynamical system (in the sense of exponential divergence of nearby trajectory) is usually estimated by the largest Lyapunov indicator (see for a review Froeschlé 1984). However, this quantity, even for $t$ very large, does not give a full description of the chaotic flow since it is an asymptotic quantity. Actually the Lyapunov characteristic indicators are nothing but the first moment of the distribution of the local variations of the tangent vectors to the flow (see Benettin 1980, Froeschlé 1984). Since we have no prior knowledge about the mathematical forms of the distribution, we will try to characterize this distribution using not only the two first moments (the mean and the $r.m.s.$) but also the Fisher coefficients $\gamma_1$ and $\gamma_2$, which measure respectively the asymmetry and the flatness with respect to the normal distribution. Actually generalized Lyapunov exponents of order 2 have been already introduced (Crisantin et al. 1988, Fujisaka 1983) in the context of theoretical physics mainly for dissipative flows.

In Section 2, we describe briefly the definition of local Lyapunov indicators and the statistical tools used to study their distribution. In Section 3, we give in the first subsection the variations with time of the generalized Lyapunov indicators (GLI) for typical orbits of the standard map. Using the GLI we make an exploration of a section of the standard map in the second subsection, and, in the third one, we perform a global study of the standard map by means of Kolmogorov like entropy varying the parameter.

## 2. Generalized Lyapunov Indicators (GLI)

For a given dynamical system it is well known that two orbits initially close diverge either linearly or exponentially depending on whether the two starting points lie in

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an integrable region or in a stochastic one.

The Lyapunov characteristic exponents (LCE) provide a more precise quantitative definition of stochasticity. Let us recall the essential features of the theory (see Benettin et al. 1980, Froeschlé 1984). Let $M$ be an $N$-dimensional compact differentiable manifold, $\mu$ a normalized measure on it and $\Phi^t$ a measure-preserving flow. The set $(M, \mu, \Phi^t)$ is called a dynamical system. Let $P \in M$; $T_P(M)$ denotes the tangent space of $M$ at the point $P$ and $D\Phi^t_P$, the tangent mapping of $\Phi^t$ from $T_P(M)$ onto $T_{\Phi^t(P)}(M)$. Given a nonzero vector $w \in T_P(M)$, one defines the quantity

$$\gamma^t_w(P) = \| w \|^{-1} \, ln \, \| D\Phi^t_P(w) \|$$

and, if the limit exists,

$$\chi(P, w) = \lim_{t \to \infty} \frac{1}{t} \gamma^t_w(P)$$

(here $\| \, . \, \|$ denotes the norm associated with the metric $\mu$). The limit is shown to exist for almost all initial points $P$ and all vectors $w$ and it is called the LCE of the flow $\Phi^t$ relative to $P$ and $w$. Of course the compactness of the phase space is rarely realized in celestial mechanics and therefore we are dealing with what could be called heuristic Lyapunov characteristic indicators. Moreover, as the vectors $w$ scan the whole tangent space, $\chi(P, w)$ takes at most $N$ distinct values $\chi_i(P)$, $i = 1, ..., N$, and there exists at least one basis $(e_1, ..., e_N)$ of $T_P(M)$ such that

$$\chi_i(P) = \lim_{t \to \infty} \frac{1}{t} \gamma^t_{e_i}(P)$$

For the sake of simplicity we shall denote $\gamma^t_{e_i}$ by $\gamma^t_i$ and choose the indices such that $\chi_1(P) \geq \chi_2(P) \geq ... \geq \chi_N(P)$. Furthermore, for a Hamiltonian system or for a symplectic mapping, $\chi_i(P) = -\chi_{N+1-i}(P)$. This set of $\chi_i(P)$ is a sensitive indicator of stochasticity in the sense that, if there exist $p$ isolating integrals i.e. uniform integrals functionally independent and in involution. Then there are $2p$ vanishing $\chi_i(P)$. As a consequence, in an integrable situation, all the $\chi_i(P)$ vanish. However for the standard map we are interested in the computation of the largest and unique positive LCE. Let us consider a mapping $F$ and the corresponding tangential mapping defined as follow

$$X_{n+1} = F(X_n) \quad , \quad Y_{n+1} = (\frac{\partial F}{\partial X_n}) Y_n \, .$$

We iterate simultaneously these two mappings, taking the norm of the initial vector $Y_0$ equal to 1, the evolved vector $Y_n$ are renormalized at arbitrary times $j\tau$ (here $\tau = 1$ is the period of the mapping and $j = 1, ...n$). Then we get :

$$\chi(X_0, n) = \frac{1}{n\,\tau} \sum_{j=1}^{n} ln(\alpha_j) \quad , \quad \chi_1(X_0) = \lim_{n \to \infty} \chi(X_0, n) \, ,$$

where $\chi_1(X_0)$ is the largest LCE and is of course approximated with a finite $n$. The Lyapunov characteristic indicator $\chi(X_0, n)$ is no more than the first moment, i.e the mean value of the distribution of local variations $ln(\alpha_j)$. See Froeschlé 1984 for the mathematical definition of $\alpha_j$, which result from the Gram-Schmidt orthonormalization of the vectors $Y_n$.

Actually this distribution summarizes a lot of informations concerning the dynamics of an orbit. For instance the presence of cantori around invariant zones will be reflected in the distribution, since for a while the motion is roughly quasi periodic. Besides the mean $m$ and the $r.m.s.$ $\sigma$ given by the first and second moment of the distribution :

$$m = E(X) \quad , \quad \sigma = \sqrt{E(X-m)^2} \; ,$$

we compute the fisher coefficients

$$\gamma_1 = \mu_3/\sigma^3 \quad , \quad \gamma_2 = \mu_4/\sigma^4 - 3 \; ,$$

with $\mu_p = E(X-m)^p$.

In the case of a normal distribution $\gamma_1$ and $\gamma_2$ vanish; when the shape of the distribution is symmetric $\gamma_1$ is equal to zero, in the unimodal case while $\gamma_2$ reflects the flatness of the distribution with respect to a normal one. These quantities could be considered as generalized Lyapunov exponents (GLE). However like for the definition of Lyapunov characteristic exponents we necessarily consider the results of a finite number of iterations (or integrations for a continuous system). Therefore we will rather consider the concepts of generalized Lyapunov indicators, which will be the values of $m$, $\sigma$, $\gamma_1$, $\gamma_2$ estimated from a finite number $n$ of iterations, and will be called in the following GLI.

## 3. Numerical Results

### 3.1. Convergence of the Generalized Lyapunov Indicators (GLI)

We study orbits of the standard map

$$x_1 = x_0 + a\sin(x_0 + y_0) \quad , \quad y_1 = x_0 + y_0 \qquad (mod 2\pi) \; ,$$

for different initial conditions and different values of the parameter $a$.

Fig. 1 a, b, c, d show respectively the variations with the number of iterations of $m$, $\sigma$, $\gamma_1$, and $\gamma_2$ previously defined. In the well studied case (see Lichtenberg et al. 1983) $a = -1.3$ for a regular orbit, i.e an invariant curve. We notice as expected the convergence of $m$ to the value zero, the fast convergence of $\sigma$ to a constant value. While the quantity $\gamma_1$ and $\gamma_2$ converge more slowly to a non-zero value. This is not surprising since the distribution for $10^5$ iterations (Fig. 5 a) is bimodal, then consequently neither symmetric unimodal nor normal.

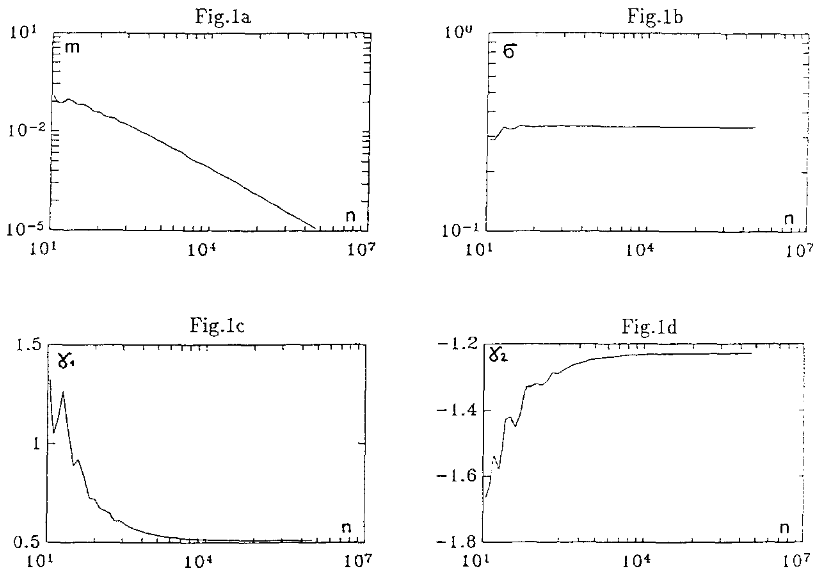


Fig. 1. (a), (b), (c), (d) show respectively the variation of the generalized indicators $m$, $\sigma$, $\gamma_1$, $\gamma_2$, for initial conditions corresponding to an invariant curve ($a = -1.3$, $x_0 = 1.$, $y_0 = 0.$)

The variations of the GLI $m$, $\sigma$, $\gamma_1$ and $\gamma_2$ for a chaotic orbit ($a = -1.3$, $x_0 = 2$, $y_0 = 0$) are respectively shown on Fig. 2 a, b, c and d. As it is well known, the mean does not go to zero, and a good approximation to a constant value is not reached before $10^4$ iterations. Conversely the limit of $\sigma$ is reached very quickly ($10^2$ iterations). Moreover the oscillations displayed by $m$ and $\gamma_2$ (a and d) reveal the complicated structure of the "Chaotic zone", when the motion takes place, that is around invariant curve the presence of cantori slows down the diffusion. Like $\sigma$, the convergence of $\gamma_1$ (c) is quite fast ($\sim 10^2$ iterations), however the small value does not reflect a symmetric distribution as seen on Fig. 5 b. In this case the non zero value of $\sigma$ may be explained in the chaotic region by the presence of a dense set of hyperbolic points which may have different eigenvalues and different orientations of the eigen vectors.

### 3.2. Transversal Exploration along the x-axis

Since the standard mapping exhibits all the features of conservative Hamiltonian systems with two degrees of freedom, i.e. invariant curves, islands, cantori, chaotic orbits, we have plotted on Fig. 3 a, b, c, d. the values of the estimators $m$, $\sigma$, $\gamma_1$, and $\gamma_2$ after 10 000 iterations with the following initial conditions: $y_0$ fixed equal to zero and $x_0$ spacing the $x$ axis from 0 to $\pi$ with a step equal to 0.02. On Fig.3 a, we recover the well known results taking LCEs as indicators of stochasticity, i.e. zero

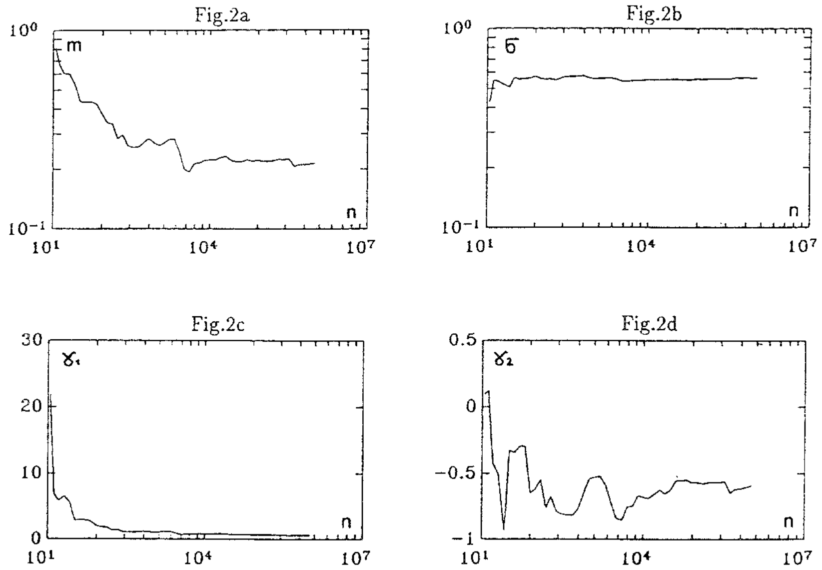


Fig. 2. Same as Figure 1 but for a chaotic orbit ($a = -1.3$, $x_0 = 2.$, $y_0 = 0.$)

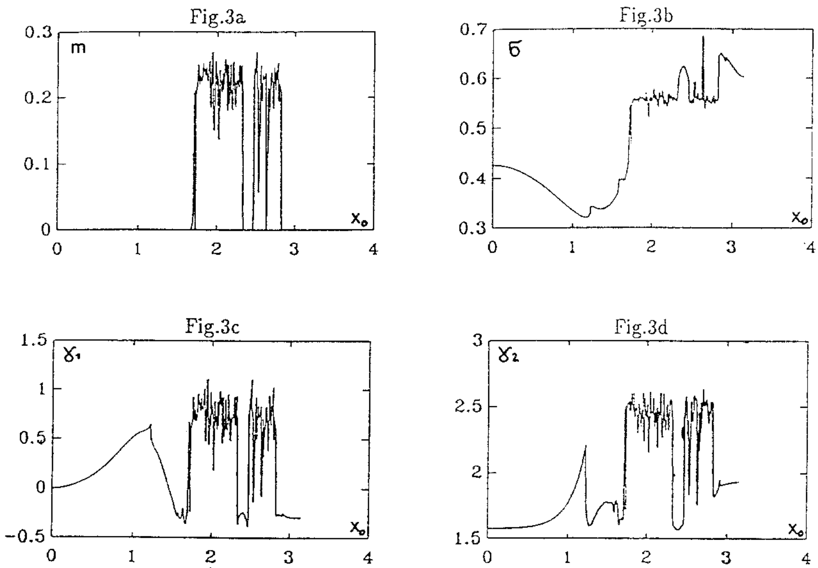


Fig. 3. (a), (b), (c), (d) show respectively the variations along the $x$ axis of the generalized Lyapunov indicators (GLI) $m$, $\sigma$, $\gamma_1$, $\gamma_2$ computed at 10 000 iterations at $y_0 = 0$ for a step 0.05 for the coarse graining.

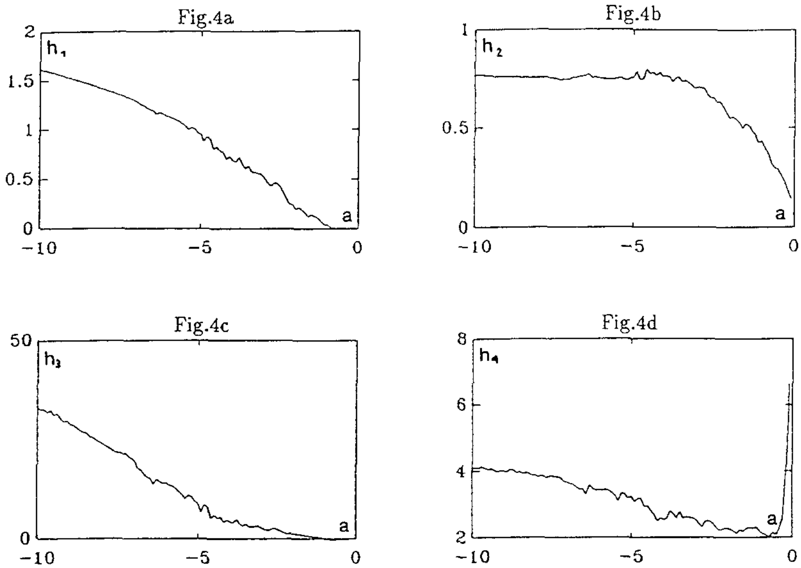


Fig. 4. Generalized Kolmogorov entropies $h_1, h_2, h_3, h_4$ versus the parameter axis $a$. Coarse graining step equal to $-0.2$. For each value of the parameter $a$, $M = 50 \quad N = 2000$.

values indicate ordered region and important variations of the values of $m_{10000}$ even for orbits starting close to each other. The same behaviour, as far as continuity is concerned appears for the variations of $\gamma_1$ and $\gamma_2$. Let us notice the smallest and more discriminating variation of $\sigma$. In the ordered region, the continuous decrease of $\sigma$ to $\sim 0.32$ when $x$ goes to 1 may be due to both the flatness of the elliptic like invariant curve and the variation of the rotation number. More studies on simple models are needed to discriminate between these two effects.

## 4. Kolmogorov like Entropy

Pesin's formula gives the relation between the LCEs and the Kolmogorov entropy of a system. Let $\varrho(P)$ denote the sum of all positives LCEs. In our case, one has

$$\varrho(P) = \chi_1(P)$$

The formula states that the total entropy is given by

$$h = \int_M \varrho(P)\, d\mu \,.$$

Consequently, the local quantity $\varrho(P)$ defines a density of Kolmogorov entropy. It is related to the exponential stretch of a small volume of the phase space, in the directions corresponding to the positive LCEs and the Kolmogorov entropy gives

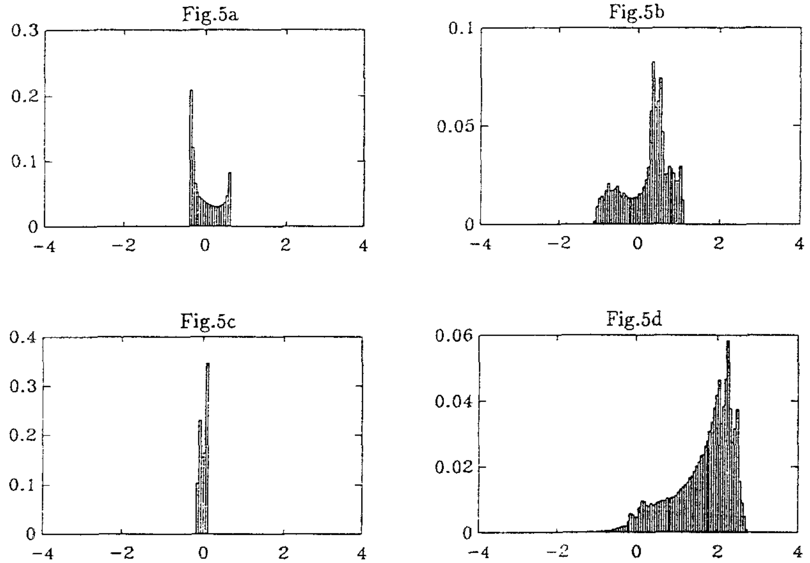


Fig. 5. Typical distributions of the local Lyapunov exponents $ln(\alpha_i)$ for different orbits of the standard mapping

| | | | |
|---|---|---|---|
| a | $x_0 = 1.$ | $y_0 = 0.$ | $a = -1.3$(invariant curve) |
| b | $x_0 = 2.$ | $y_0 = 0.$ | $a = -1.3$ (a chaotic orbit) |
| c | $x_0 = 1.$ | $y_0 = 0.$ | $a = -0.1$ (invariant curve in the circulation case) |
| d | $x_0 = 1.$ | $y_0 = 0.$ | $a = -10.$ (strong chaotic orbit) |

an average over the phase space of this density. Following the same philosophy we compute using a Monte Carlo method the quantities

$$h_k = \frac{1}{NM} \sum_{j}^{M} \sum_{i=1}^{N} ln(\alpha_{ij})$$

$j$ refers to an orbit $j$ whose initial conditions have been taken at random and $i$ refers to the number of iterations. The subscript $k$ $(k = 1, 4)$ defines the quantities $h_1$, $h_2$, $h_3$ and $h_4$. Where $h_1$ is an estimation of the Kolmogorov entropy, $h_2$ is a measure of the dispersion of this entropy. While $h_3$ and $h_4$ estimate respectively the mean over the phase space of the asymmetry and the flatness parameters defined previously. The variations along the parameter axis $a$, of $h_1$, $h_2$, $h_3$ and $h_4$ are plotted respectively on Fig. 4 a, 4 b, 4 c and 4 d.

The three quantities $h_1$, $h_3$ and $h_4$ increase with the absolute value of the parameter $a$, while $h_2$ reaches a constant value for $\mid a \mid > 5$, which means that the distribution of the local Lyapunov indicators $[ln(\alpha_{ij})]$ is more or less translated towards the positive axis.

## 5. Conclusion

The first moments of the distribution of the local Lyapunov exponents, and particularly the $r.m.s.$ $\sigma$ seem to provide good hints of the global stochasticity of orbits of hamiltonian systems (or symplectic mappings). We plan to continue this preliminary study in two directions.

First, using simpler ad hoc mappings we intend to characterize, by the four first moments of the distribution, the geometry and kinematics of regular orbits, and as far as chaotic orbits are concerned we will try to measure the influence of the magnitude and orientation of hyperbolic invariant manifolds.

Second, since many distributions are far from the normal one and show, for instance, bimodal feature, the tools of artificial intelligence may provide better parameters to characterize the slope of such distributions.

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