FREQUENCY ANALYSIS OF A DYNAMICAL SYSTEM

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Abstract. Frequency analysis is a new method for analyzing the stability of orbits in a conservative dynamical system. It was first devised in order to study the stability of the solar system (Laskar, Icarus, 88, 1990). It is a powerful method for analyzing weakly chaotic motion in hamiltonian systems or symplectic maps. For regular motions, it yields an analytical representation of the solutions. In cases of 2 degrees of freedom system with monotonous torsion, precise numerical criterions for the destruction of KAM tori can be found. For a 4D symplectic map, plotting the frequency map in the frequency plane provides a clear representation of the global dynamics and describes the actual Arnold web of the system.

Key words: Frequency analysis – chaotic motion – symplectic maps

1. Frequency Analysis

The method of numerical analysis of the fundamental frequencies was introduced in the study of the stability of the solar system, as modeled by a reduced (but nevertheless complicated) 15 degrees of freedom system (Laskar, 1990). In that case, frequency analysis permitted numerical estimates of the size of chaotic zones in all directions of the 15 degrees of freedom, and revealed that for the inner planets (Mercury to Mars), the chaotic zones were relatively large, while for the outer planets (Jupiter to Neptune), these zones were much smaller.

More generally, the frequency analysis method can be applied to study the stability of the solutions of a conservative dynamical system, and is based on a refined numerical search for a quasiperiodic approximation of its solutions over a finite time span (Laskar, 1990, 1992, Laskar *et al.*, 1992). If $f(t)$ is a function with values in the complex domain, obtained numerically over a finite time span $[-T, T]$ the frequency analysis algorithm will consist in the search for a quasiperiodic approximation for $f(t)$ with a finite number of periodic terms of the form

$$\tilde{f}(t) = \sum_{k=1}^{N} a_n e^{i\sigma_k t} \; .$$

The frequencies σ_k and complex amplitudes a_k are found with an iterative scheme. To determine the first frequency σ_1, one searches for the maximum of the amplitude of

$$\phi(\sigma) = \langle f(t), e^{i\sigma t} \rangle$$

where the scalar product $\langle f(t), g(t) \rangle$ is defined by

$$\langle f(t), g(t) \rangle = \frac{1}{2T} \int_{-T}^{T} f(t) \bar{g}(t) \chi(t) dt \; ,$$

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and where $\chi(t)$ is a weight function, that is, a positive function with

$$\frac{1}{2T}\int_{-T}^{T}\chi(t)dt = 1 .$$

In all computations, the Hanning window filter was used, that is

$$\chi(t) = 1 + \cos(\pi t/T) ,$$

although some other weight functions could be used. Once the first periodic term $e^{i\sigma_1 t}$ is found, its complex amplitude a_1 is obtained by orthogonal projection, and the process is started again on the remaining part of the function $f_1(t) = f(t) - a_1 e^{i\sigma_1 t}$. As all the different functions $e^{i\sigma_k t}$ are not orthogonal, it is also necessary to orthogonalize the set of functions $(e^{i\sigma_k t})_k$, when projecting f iteratively on these $e^{i\sigma_k t}$. In the case of an hamiltonian system with n degrees of freedom, the frequency analysis of the solutions will give its quasiperiodic expansion and in particular will determine the vector $(\nu_i)_{i=1,n}$ of the fundamental frequencies of the system. In the case of nonintegrable systems, not all the solutions are quasiperiodic, but under certain conditions, for example under the hypotheses of KAM theorems, there still exist many of these. For such solutions, the frequency analysis over a finite time span $[0, T]$ will give the same kind of results as for integrable systems.

Even if an orbit is not regular (quasiperiodic), in case of nearly integrable systems the solution will look very regular on a finite time span. More precisely, this will be the case if the time span is smaller than the characteristic time of divergence of nearby orbits. In this case, the frequency analysis gives a quasiperiodic approximation to the solution which holds only locally in time. In other words, it will give us a frequency vector $(\nu_i(t))_i$ for each value of t, obtained by applying the frequency analysis algorithm over the time span $[t, t+T]$. In the case of a quasiperiodic solution, $\nu_i(t)$ does not depend on t, while for non-regular solutions, $\nu_i(t)$ will evolve with time, revealing the diffusion of the orbit in phase space. The frequencies are used here instead of the action variables for a more accurate monitoring of the diffusion of the orbit.

2. Two Dimensional Twist Map.

Let us first consider the case of a symplectic twist map on R^2. As an example, we shall consider the Standard Map,

$$\begin{cases} x' = x - a\sin y & \mod(2\pi) \\ y' = x' + y & \mod(2\pi) \end{cases}$$

As a dynamical system, it is not integrable, and gives rise to the usual features of conservative dynamics, with invariant curves, chaotic regions, and elliptic islands. A simple criterion for the disappearance of irrational curves, based on Birkhoff's theory, was derived from the frequency analysis of such a monotone (increasing) twist map (Laskar *et al.*, 1992).

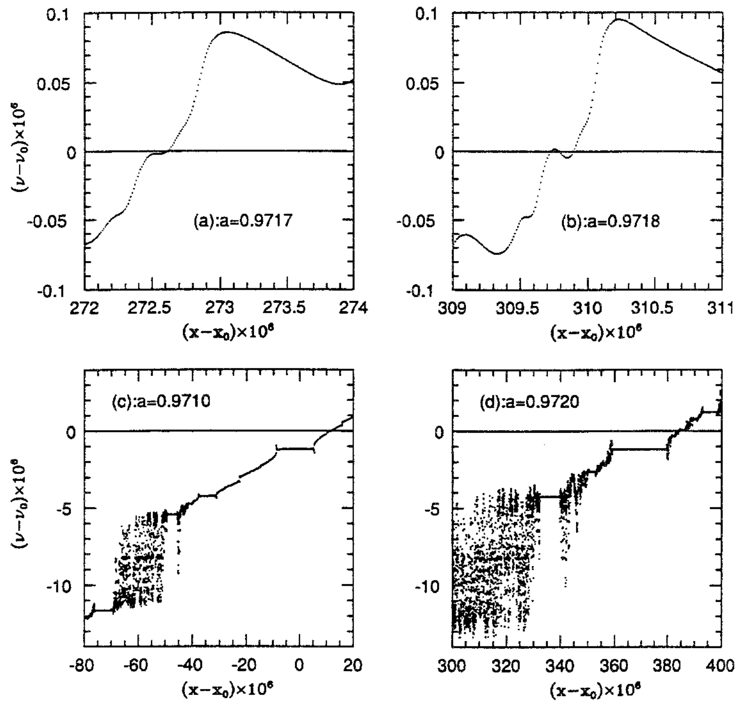

Fig. 1. Variation of the fundamental frequency ν for the 2D Standard Map for different values of the parameter a, in the vicinity of the golden rotation number ν_0 which corresponds to the zero dotted line. The origin in the x scale is arbitrarily taken to be $x_0 = 4.176550$. The origin of frequencies is the golden value $\nu_g = (3 - \sqrt{5})/2$. The unit for ν and x is 10^{-6}. If $x_1 < x_2$ and $\nu(x_1) > \nu(x_2)$, we can conclude that there exist no KAM invariant curves of irrational rotation number between $\nu(x_2)$ and $\nu(x_1)$. In Fig. 1b, we can see that that the golden invariant curve does not persist for $a = 0.9718$.

For each initial condition x on the vertical line $y = 0$, we call γ_x the orbit obtained by iterating the mapping and $\nu(x)$ the frequency given by the frequency analysis of this orbit during a given time span.

If there exist two values $x < x'$ on the vertical line $y = 0$ for which $\nu = \nu(x) > \nu' = \nu(x')$, then there are no invariant KAM curves of irrational rotation number ν'' with $\nu' < \nu'' < \nu$.

This criterion provides a simple way of knowing whether a KAM curve has disappeared by looking at the graph of the frequency map $\nu(x)$, obtained on a given time span $[0, T]$. The figure (1b) was obtained with $T = 12516$, and shows the disappearance of the golden curve for the value $a = 0.9718$ of the parameter, which is very close to and compatible with the value $a_c = 0.971635$ derived by Greene (1979).

3. Higher Dimension

We shall consider the case of a symplectic map on R^{2n} written in coordinates (x, y) which are close to angle-action variables.

The angle-like variables x_0 are fixed. If we take some initial conditions y, we can carry out the frequency analysis for the orbits corresponding to initial conditions (x_0, y) (at $t = 0$) over the time span $[t, t+T]$. We thus define a map

$$\begin{array}{llll} F_T : & R^n \times R & \longrightarrow & R^n \\ & (y,t) & \longrightarrow & f(y,t) \end{array}$$

For a given value of t, let us denote F_T^t the restriction of F_T to $R^n \times \{y\}$, and let $\mathcal{A}$ be the set of y-values which correspond to invariant tori of dimension n.

a) If $y \in \mathcal{A}$ then $F_T(y, .)$ is constant on R (up to the precision of the determination of the frequencies).

b) In the case $n = 1$, for a monotone twist map and for a given value of t,

$$\begin{array}{llll} F_T^t : & \mathcal{A} & \longrightarrow & R \\ & y & \longrightarrow & f(y,t) \end{array}$$

is monotone.

The property a) was already used to study the stability of the solar system (Laskar, 1990); b) was used to study the destruction of golden tori for the two dimensional standard map (Laskar *et al.*, 1992).

The frequency map is exactly defined on the Cantor set of the invariant tori. It can be thought of as a diffeomorphism on this set, which could be extended in some sense to a diffeomorphism on R^2 (cf. Pöschel, 1982). Chaotic zones will therefore appear as a loss of regularity for the frequency map. This can be clearly seen around the golden curve for the two dimensional standard map (Fig.1) . As the parameter increases, there are some distorsions in the frequency curves. These distorsions permit statements about the non-existence of KAM tori, but these distorsions also eventually produce complete loss of regularity of the frequency map, which can be taken as an indication of chaotic motion. Moreover, this loss of regularity of the frequency map can be generalized to higher dimensions.

4. Application to the 4D Standard Map

We will use the frequency analysis to study the global dynamics of a 4 dimension symplectic map which was first studied by Froeschlé (1972).

$$\begin{cases} x_1' = x_1 + a_1 \sin(x_1 + y_1) + b \sin(\frac{1}{2}(x_1 + y_1 + x_2 + y_2)) & \mod(2\pi) \\ y_1' = x_1 + y_1 & \mod(2\pi) \\ x_2' = x_2 + a_2 \sin(x_2 + y_2) + b \sin(\frac{1}{2}(x_1 + y_1 + x_2 + y_2)) & \mod(2\pi) \\ y_2' = x_2 + y_2 & \mod(2\pi) \end{cases}$$

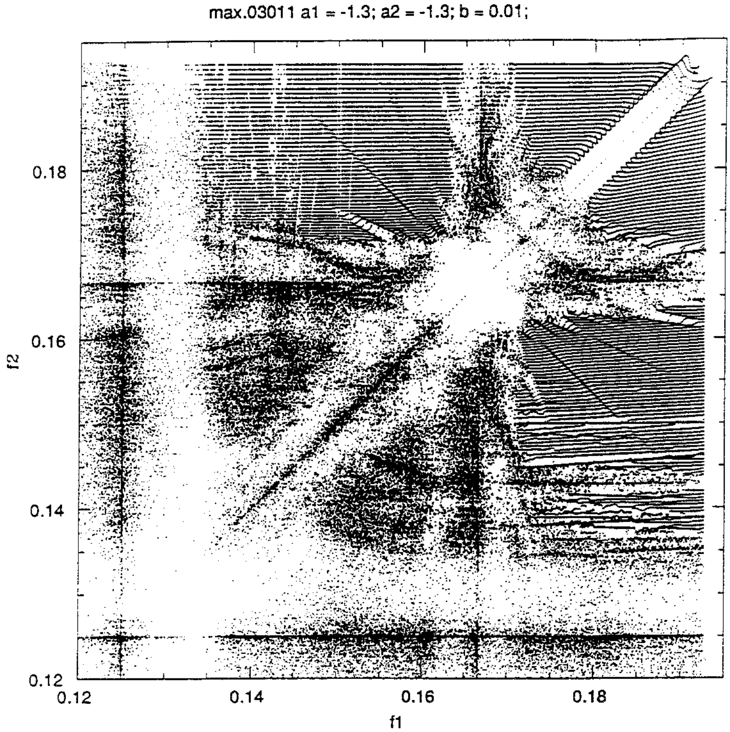

Fig. 2. Visualisating in the frequency plane (f_1, f_2) of the frequency application $a_1 = a_2 = -1.3, b = 0.01$.

We shall carry out the frequency analysis for the parameter value $a_1 = a_2 = -1.3, b = 0.01$and we shall consider orbits with initial conditions on the plane $x_1 = x_2 = 0$. We can visualize the complete frequency map F_T^0 which is a map from R^2 to R^2. In the regular regions, the behavior is not very wild, and it will be possible to visualize the map by drawing the images of the lines of initial conditions $R \times y_1$ for various values of y_1 (Fig 2). For each initial condition (y_1, y_2), the two main frequencies f_1 and f_2 of the orbit are determined with the frequency analysis over 516 iterations, and the point of coordinates (f_1, f_2) is indicated on the graph. In a regular region, the image of a line will appear as a smooth curve, which will not be the case in chaotic regions.

In fact, what is pictured here is the Arnold web of the mapping, with the description of the actual strength associated to each of the resonant lines. Resonant lines exist on a dense set of frequencies, but most of them have a negligible effect and are not visible here. Different zones appear on these plots. The first ones are the regular zones, with very smooth, non-distorted frequency curves. The motion will be very regular in these regions. Next are some resonance regions, such as the top of the vertical $f_1 = 1/6$ zone, where the points are regularly spaced in f_2, but more erratically in f_1. This corresponds to the product of a chaotic motion in f_1 with something more regular in f_2. In these zones, Arnold diffusion is probably possible. There are also zones of the pictures where the points seem to be erratically distributed in all directions, which ought to correspond to completely chaotic motion. This is the case for example, in the outside zone, for small f_1 and f_2, which corresponds to the large-scale chaotic motion where most tori are destroyed, even in the uncoupled problem.

The analysis of the regularity of the frequency application presented here allows one to obtain a global picture, in two dimensions, of the dynamics of a 4 dimensional symplectic map, or a 3 degrees of freedom hamiltonian system. This method can also be applied in higher dimensions, and I am convinced that this new method of frequency analysis will become an important tool for the study of many kinds of conservative dynamical systems.

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