NUMERICAL RESULTS TO THE SITNIKOV-PROBLEM

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Abstract. We present numerical results of the so-called Sitnikov-problem, a special case of the three-dimensional elliptic restricted three-body problem. Here the two primaries have equal masses and the third body moves perpendicular to the plane of the primaries' orbit through their barycenter. The circular problem is integrable through elliptic integrals; the elliptic case offers a surprisingly great variety of motions which are until now not very well known. Very interesting work was done by J.Moser in connection with the original Sitnikov-paper itself, but the results are only valid for special types of orbits. As the perturbation approach needs to have small parameters in the system we took in our experiments as initial conditions for the work moderate eccentricities for the primaries' orbit ($0.33 \leq e_{primaries} \leq 0.66$) and also a range of initial conditions for the distance of the 3^{rd} body (= the planet) from very close to the primaries orbital plane of motion up to distance 2 times the semi-major axes of their orbit. To visualize the complexity of motions we present some special orbits and show also the development of Poincaré surfaces of section with the eccentricity as a parameter. Finally a table shows the structure of phase space for these moderately chosen eccentricities.

Key words: Sitnikov-Problem - surfaces of section - chaotic motion

1. Introduction

A very interesting dynamical problem is known under the name Sitnikov- problem which is cited quite often as a model case for the appearance of chaotic motion. The dynamical model was first described by Pavanini (1907): a massless body is moving in the z-direction perpendicular to the plane of two equally massive primary bodies, which move on Keplerian orbits around their center of gravitation. The circular problem (where the primaries move on circles) was discussed in details by McMillan (1913) where he showed the integrability of the equations of motion with the aid of elliptic integrals which has been rediscussed in detail by K.Stumpff(1965). This is also evident from the fact that in this form it can be regarded as a special case of the Two-Body Fixed Center Problem, which is known to be integrable since Euler (1760).

Much more interesting is the case, where the primaries move in eccentric orbits. Then we can observe periodic orbits, quasi periodic orbits and unbounded motions and additionally the recently rediscovered chaotic motions (already Poincaré mentioned such orbits ,1892)

As first rough definition one can say that two originally very close orbits separate from each other hyperbolically. It is interesting to note that the whole complexity of phase space is already present for very small eccentricities $e \sim 0.0001$ of the primaries' orbit, although it is so close to the integrable circular problem. (J. Liu and S.Sun, 1991). First qualitative results were derived by Sitnikov (1960) himself for special orbits and later by J. Moser (1973). C.Marchal (1990) discussed

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the problem also in a qualitative way . H.Juranek (1991) developed a 1^{st} order perturbation theory valid for small eccentricities and small oscillations, while J.Hagel (1992) used a perturbation method up to the 3^{rd} order in the eccentricities. J.Hagel and T.Trenkler (1992) adapted a technique to find integrals of motion for all eccentricity values of the primaries' orbit but only for small oscillation and derived interesting qualitative results.

We were interested in the structure of phase space for cases not yet studied well; therefore we did the numerical experiments in the range of ($0.33 \leq e_{primaries} \leq 0.66$). In what concerns the other initial conditions they will be precised later.

2. The Formulation of the Problem

Let us know give the equations of motion:

$$\ddot{z} + \frac{z}{\sqrt{r^2 + z^2}^{\,3}} = 0 \tag{1}$$

$$r(v) = \frac{1 - e^2}{2(1 + e\cos v)} \tag{2}$$

where z is the distance from the plane of the primaries' orbit; the distance r of the barycenter to one primary varies with the time t according to Keplers 1^{st} law (v is the true anomaly).

We used for integration a modified equation developed by K.Wodnar (1991) where T is defined through the following equations:

$$T := \frac{z}{2r} = z\frac{1 + e\cos v}{1 - e^2} \tag{3}$$

$$\frac{dT}{dv} := T' = \dot{z}\,\frac{\sqrt{1 - e^2}}{1 + e\cos v} - z\frac{e\sin v}{1 - e^2} \tag{4}$$

$$z = T\frac{1 - e^2}{1 + e\cos v} \tag{5}$$

$$\dot{z} = \frac{1}{\sqrt{1 - e^2}}[Te\sin v + T'(1 + e\cos v)] \tag{6}$$

The T-value has the following geometrical meaning: it is half of the tangent of the angle of view of the planet seen from one of the primaries. Finally we are lead to a differential equation of the following form:

$$T'' + \frac{1/\sqrt{1/4 + T^2}^{\,3} + e\cos v}{1 + e\cos v} \cdot T = 0 \tag{7}$$

This equation of motion was integrated with a Lie-integration method with variable step length (e.g. A.Hanslmeier and R. Dvorak, 1984). The time scale was 1000 orbits of the primaries for most of the cases for; then had enough points in the Poincaré surfaces of section, which was defined as T versus T' for every pericentric position of the primaries. In exceptional cases, when we discovered a fractal structure of islands we increased the integration time up to 5000 revolutions of the primaries.

After some test calculations with different initial conditions we fixed the following ones:

- the true anomaly $v_{ini} = 0^o$, that means the starting point was always when the primaries are at their pericenter.
- we always set $T' = 0$ and varied only T, which corresponds to a variation of the planet's distance to the barycenter.
- as mentioned above $e_{primaries}$ was chosen between 0.33 and 0.66

It should be kept in mind that this is only a necessary restriction because of limiting computer time available at the moment. Nevertheless it is hoped that we have found the main structures of phase space for moderate eccentricities and the motions not too far away from the primaries' orbital plane.

An appropriate method to find out the structure of phase space is to plot the different surfaces of section and compare them for various values of the eccentricities and the initial T-value. The method was introduced for numerical experiments by Hénon and Heiles (1964) for a simple model of a galactic potential and it is still the most powerful tool to present such numerical experiments. A more rigorous way to determine regions of chaotic motions in phase space is calculate the Liapunov characteristic exponents (e.g. Froeschlé 1984). But this is still a very "expensive" (from the point of view of computer-time) procedure and therefore it was kept for a future project on the same topic.

3. The Numerical Results

Before we discuss the global results we want to give some interesting details: the dependance of the location of the periodic orbits (=PO) on the parameter of the system (eccentricity of the primaries) , the decay of periodic orbits in form of bifurcations and the onset of chaos close to a separatrix.

3.1. THE CHANGE OF POs WITH THE PARAMETER

Quite well known is the pitchfork-diagram studied extensively at first in the logistic equation. The phenomenon of splitting of 1 PO into 3 is shown in fig.1 for invariant curves surrounding stable POs.

We can see in fig.1 the invariant curves in the SOS starting from the point T=1.13 (T'=0) for 6 different e-values. It is interesting to see the shift of the location of the PO (generally at the center of the island) outwards to greater T-values with

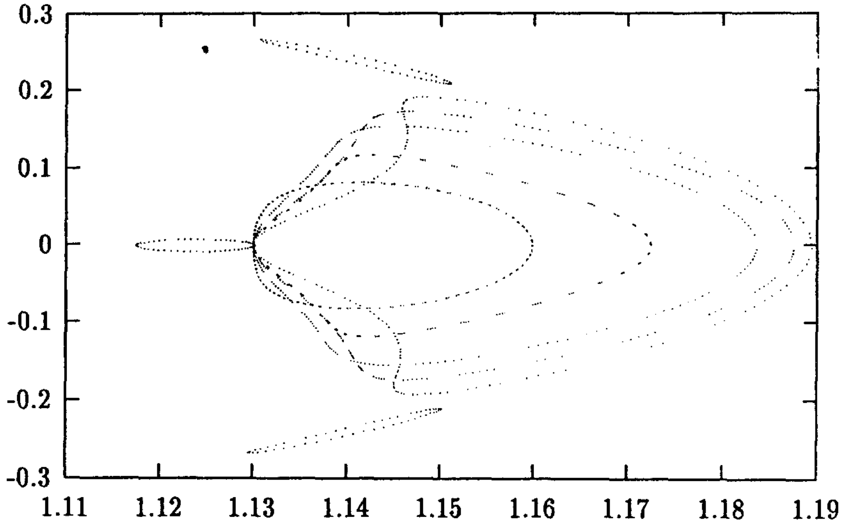

Fig. 1. Decay of a 1 PO to 3 PO with an increasing value of the parameter of the Sitnikov-system

increasing e-values. Then we suddenly observe a splitting from a 1 PO to a 3 PO (islands around them). The most interesting orbit very close to the separatrix is shown in fig.2. The accumulation of points close to a hyperbolic point (an unstable PO) is also visible on this graph. The starting point was the same as in fig.1, the eccentricity was chosen in between the eccentricity where we derived the last one and the one where we derived 3 islands in the surface of section.

As another example we show the decay of an island around a PO into a chain of islands. We fixed the initial conditions for T ($T = 1.05$, T' is always zero in our experiments) and varied the parameter e again. Fig.3a shows a well defined invariant curve (e=0.23) which decays for e=0.28 int o 7 islands (fig.3b). Then for e=0.33 we see even 17 islands replacing the one from fig.3a. Fig.3d shows the 2 small islands on the left bottom corner of fig.3c on a smaller scale.

3.2. MOTION ON A SEPARATRIX

Sometimes the initial conditions were chosen such that the motion is close to a separatrix, a curve connecting the hyperbolic fixed-points (unstable POs) in the surface of section. On fig.4 we see such an example where the consecutive points of intersection are surrounding tiny islands lying around stable POs. It is evident that the very complex shape on the SOS is difficult to derive with any analytical method.

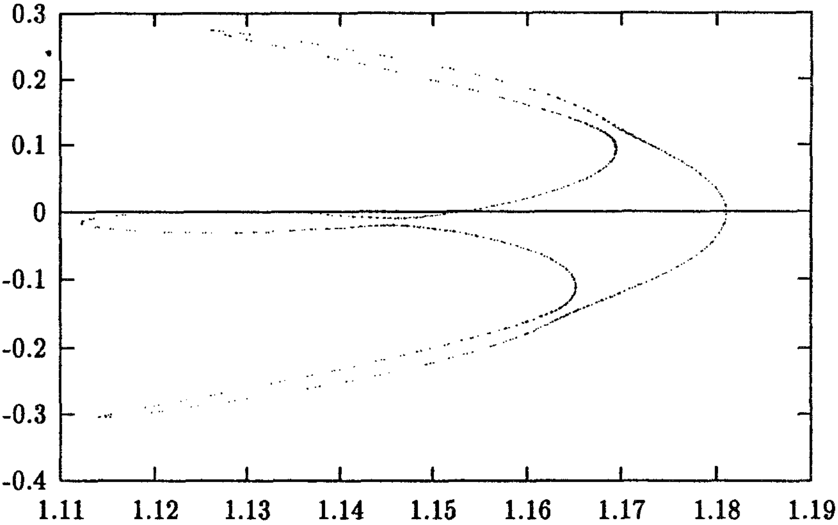

Fig. 2. motion close to the separatrix surrounding the 3 PO

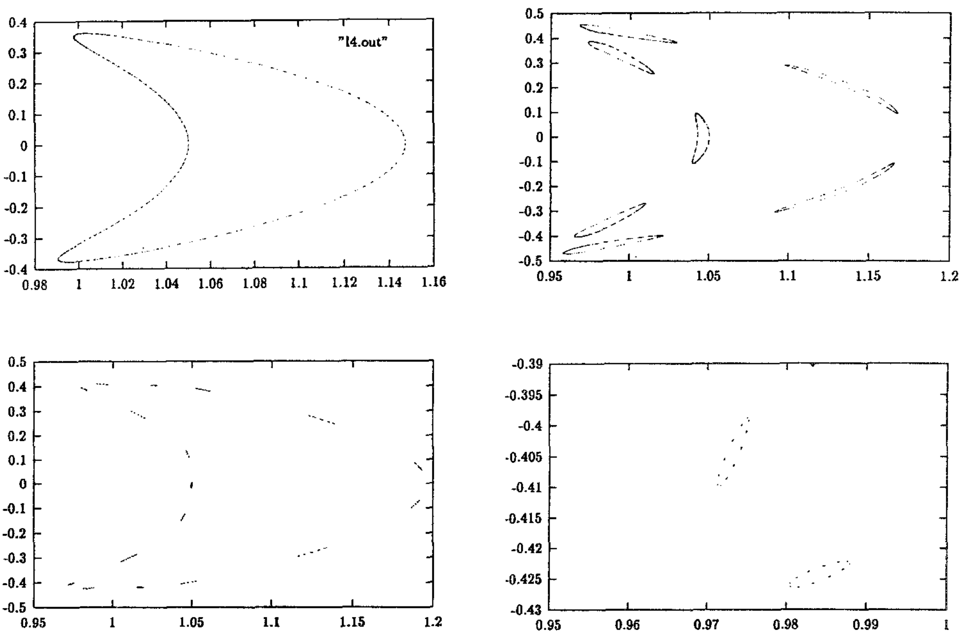

Fig. 3. The decay of an island: fig.3a (top,left) invariant curve for e=0.23; fig.3b(top, right): 7 invariant curves for e=0.28; fig.3c(bottom,left):17 invariant curves for e=0.33; fig.3d(bottom,right) 2 islands from the former graph on a smaller scale; all the plots show surfaces of section T - versus - T'

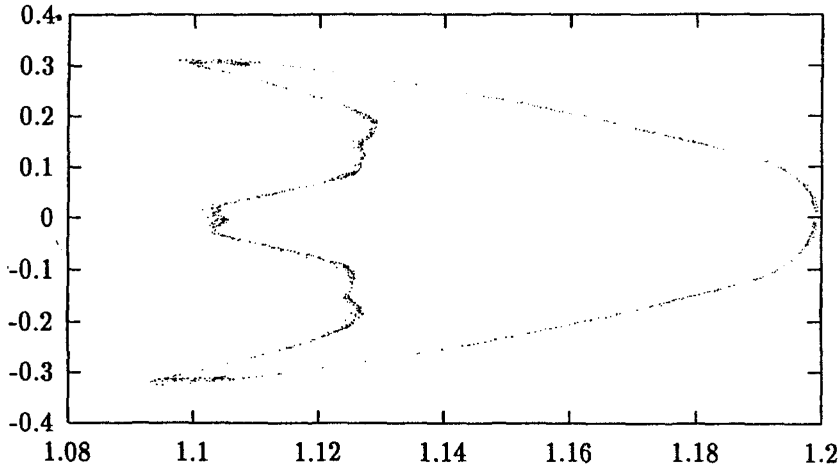

Fig. 4. chaotic motion "on" the separatrix close to multiple periodic orbits

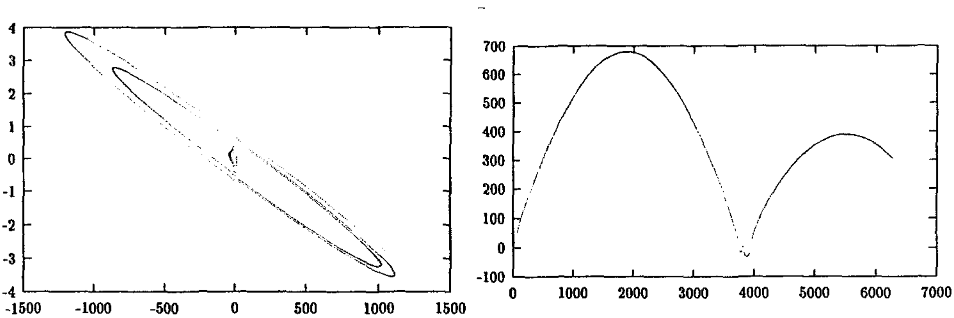

Fig. 5. typical chaotic motion for e=0.66, (left side surface of section T versus T', right side z versus the true anamoly v

3.3. A HIGHLY CHAOTIC MOTION

Varying just a little bit the initial conditions of the one orbit "on" the separatrix we have a full chaotic orbit which is shown in fig.5a in the surface of section and fig. 5b in a plot of T (the distance from the barycenter of the massless body) versus the time scaled in $2\ \pi$ corresponding to one whole orbit of the primary bodies.

It is evident, that the moments of the passage of the third body through the barycenter are very important. The time interval of such events is more or less periodic on an invariant curve and it is practically undistinguishable from a motion in a thin chaotic layer. Great differences of such time intervals can occur for

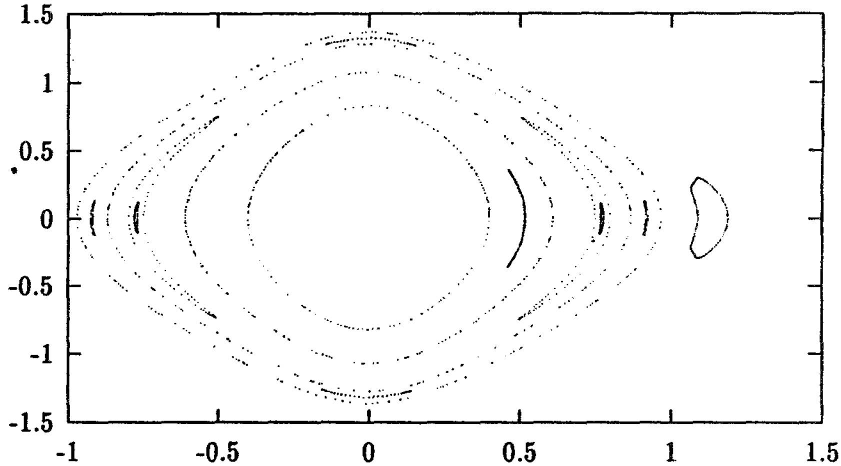

Fig. 6. Overall picture of the Poincare surface of section for e=0.37; T is plotted versus the velocity T'

motions in the global stochastic zone. One example is shown in fig.5b, where we can recognize small variations in T after quite a large one, and then again a very large one. This kind of motion is excluded in Sitnikov (1960), but is in principle included in Moser's book (1973). Although the qualitative picture of the variety of orbits which exist in the Sitnikov problem is quite interesting and complete no method of finding such special initial conditions is given explicitly.

3.4. GLOBAL RESULTS

In table I we have plotted the main characteristics of the motions for the initial conditions of the primaries for the eccentricities e of the primaries for which the experiments have been undertaken. The interval in the T-direction was $\Delta T = 0.01$ from T=0.4 to T=1.1, covering the most interesting parts of phase space. Note that we started always in the pericenter position of the primaries. A careful inspection of the Poincaré surface of section lead to the following results of the structure of phase space given in table 1. We marked there the qualitatively different orbits in the following way

- "o" orbit on an invariant curve
- "a" orbit on an invariant curve with some accumulation of points, indicating that the motion is close to a separatrix
- "n" marks the number of islands of the respective orbit
- "s" states that the motion is (on or) very close to a separatrix
- "k" orbit has a chaotic character
- "*" 10 and more islands

TABLE I
Results of the systematic research of the Sitnikov-Problem

T / e	0.4	0.5	0.6	0.7	0.8	0.9
0.33	oooooooooo	1ooooooaoo	oaoo22222o	oaoo11111o	oooooooooo	o44ooo669k
0.37	oooooooooo	oo1ooooooo	oooooooooo	ooooa22222	2ooooooooo	oo44?5?okk
0.41	oooooooooo	ooo1oooooo	oooooooooo	oooooo2222	222ooooooo	oo44ooo6kk
0.44	oooooooooo	oooo1ooooo	oooooo4ooo	oaoooo2222	22222ooooo	3oo444s6kk
0.47	oooooooooo	ooooo1oooo	ooooooo4oo	ooooooa222	222222oooo	o3oa44skkk
0.51	oooooooooo	oooooo11oo	oooooooooo	oooooooo22	22222222so	oo3*k44kkk
0.55	oooooooooo	ooooaoo111	oooooooooo	oooaoooos*	2222222222	2d8k3kk4kk
0.58	oooooooooo	ooooooooo1	1ooooooooo	ooooooooos	2222222222	2sssk3kkkk
0.61	ooooaooooo	aoooaooooo	111oooooo6	ooo4oooooo	k2*2222222	222kkkkkkk
0.64	oooooooooo	oooooooooo	o1111ooooo	oooo44oo3o	kkkkk22222	22222kkkkk
0.66	oooooooooo	oooooooooo	o11111oooo	ooooo444o3	kkkkkk2222	2222o*kkkk

TABLE II
The last invariant curve

e	0.33	0.37	0.41	0.44	0.47	0.51	0.55	0.61	0.64	0.66
T	0.99	0.98	0.98	0.97	0.92	0.91	0.92	0.80	0.80	0.80

At first sight we can see that invariant curves exist for greater values of T when the eccentricity e is smaller. As a consequence from low eccentricities on towards higher ones the appearance of the 1^{st} island is shifted more and more outwards; also the area (meaning here number of initial conditions) where islands can be observed is becoming more extended with increasing initial T values. The onset of global chaos is in contrary shifted more and more towards smaller T-values. But we shou ld keep in mond that the transformation from T to z and vice versa given in eqs. (3) - (5).

In table 2 we listed the last island for the specific e-value; from here on global chaos can arise. But it is also visible from fig.1 that in this zone of chaoticity there are still regions of regular motions visible through (sometimes very strange formed) islands. Their existence is due to high order resonances.

3.5. Conclusions

What can be said from the systematic numerical study of the Sitnikov problem? First of all we emphasize that this problem of Celestial Mechanics is the most simple problem after the integrable two body problem and the integrable Two-body-Fixed center problem. In this sense it can be regraded as a generic problem! In fig.6 we see a complete picture of the SOS for e=0.33. From that results we can deduce the following: Close to the linear problem, for very small oscillations around the barycenter, we can observe closed invariant curves which were expected to exist because of the KAM-theorem. These closed curves exist up to a certain value of the initial conditions and then they break and unbounded and chaotic motion is possible. But already in the domain of closed invariant curves we can observe islands which exist around stable periodic orbits of the problem. It is known since years (e.g. Henon and Heiles, 1964 Contopoulos, 1968) that in between such island we will have hyperbolic points - separatrices and sometimes only very thin layers of chaotic motion (e.g. Lichtenberg and Lieberman, 1983). Nevertheless this motion is bounded and can never lead to an escape: it is therefore quite important having determined the "1^{st} chaotic orbit" which will be close to the last "KAM-Torus". Well visible in fig.6 is the island in the chaotic sea which is also due to motion close to a stable periodic resonant orbit. The structure of such islands is very complicated as on sees from fig. 4.

Finally it has to be said that this first systematic numerical study of the Sitnikov problem has to be extended to smaller Δe and sometimes - in interesting areas to a smaller ΔT. Another point is that for some orbits in the chaotic zone we should calculate also the Liapunov-characteristic exponent; this tool is especially important to determine the zone of the onset of global chaotic motions.

But we emphasize, that the purpose of this paper was to show for the first time explicitely the great variety of possible orbits in the Sitnikov-Problem.

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References

Contopoulos,G. 1967: 'Resonance Phenomena and the Non-Applicability of the "Third" Integral', *Contr. Astr. Dep. Univ. Thessaloniki*, 31

Euler, L. 1760: *Mém. de l'Acad. de Berlin*

Hagel, J., 1992: 'An New Analytical Approach to the Sitnikov Problem', *Celest.Mech.*, **53**, 267–292

Hagel, J., Trenkler T.: 1992, 'A Computer aided analysis of the Sitnikov Problem', *Celest.Mech.*, this volume

Hanslmeier A., Dvorak, R. 1984:' Numerical Integration with Lie- series', *Astron.Astrophys.*, **132**, 203

Hénon, M., Heiles, C.: 1964: 'The applicability of the third integral of motion, some numerical experiments', *Astron.J.*, **69**, 73

Juranek, H.: 1990, *Zum Sitnikov Problem: Störungsrechnung erster Ordnung*, Diploma Thesis, University of Vienna

Lichtenberg, A.J. and Liebermann, M.A.: 1983, *Regular and Stochastic Motion*, Applied Mathematical Sciences, Volume 38, Springer, New York

Mac Millan, W. D.: 1913, 'An integrable case in the restricted problem of three bodies', *Astron.J.*, **27**, 11

Marchal, C., 1990: *The Three Body Problem*, Studies in Astronautics, 4, Elsevier, Amsterdam, p.419ff
Moser, J., 1973: *Stable and Random Motions in Dynamical Systems*, Annals of Mathematics Studies Number 77, Princeton University Press and University of Tokio Press, Princeton, New Jersey
Pavanini, G.: 1907, 'Sopra una nuova categoria di soluzioni periodiche nel problema dei tre corpi', *Annali di Mathematica*, Serie III, Tomo XIII
Poincaré, H. ,1892: *Les méthodes nouvelles de la Mécanique Céleste*, Gauthier-Villars
Sitnikov, K.: 1960, 'Existence of oscillating motions for the three-body problem', *Dokl. Akad. Nauk. USSR*, **133**, 303–306
Stumpff, K., 1965: *Himmelsmechanik*, Band II, VEB Deutscher Verlag der Wissenschaften, Berlin
Lin, J.and Sun, Yi-Sui: 1991, *Celest.Mech.*, **49**, 285
Wodnar, K.: 1991, 'New Formulations of the Sitnikov Problem', in Archie E.Roy (ed): Predictability, Stability, and Chaos in N-Body Dynamical Systems, Plenum Press, p 457