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INVESTIGATING ERRORS OF BEGINNING MATHEMATICIANS

Abstract. Diagnostic interviews were conducted with children in their third year at a Melbourne primary school to determine why they had made errors on sixteen arithmetical tasks. The Newman method of analyzing errors was modified in order that the children's errors could be sensibly classified in terms of the sequence of steps they used when attempting to solve the problems. The diagnostic interviews clearly showed where children were making their errors, and this enabled their teacher to devise more appropriate teaching procedures.

It is common practice for teachers of beginning readers to hear individual children read, to ask questions of what they read, to analyze any errors in their reading and answers, and then to organize their teaching to overcome any weaknesses revealed.

In teaching 'beginning mathematicians' however, a common practice in Australia is for a teacher to present a short test to the class, count the number of wrong answers made by each child, and then to follow one of several procedures. The test results may be written on a form and filed away to be later shown to parents or the principal. Mathematics lessons for the class then proceed as if the test had not been given. If the children are a little more lucky the teacher may conclude from the test results that some children in the class showed weaknesses in say, division or subtraction. The subsequent mathematics lessons for the class will perhaps repeat earlier instructions on how to do division or subtraction, with practice sums.

If the children are even more lucky they might be allocated to maths groups on the basis of their test scores. Those with low scores will be in the 'slow at maths' group, and probably will receive repetitions of earlier lessons on all the basic topics – Johnny hears again how to do division though he might understand well the concepts involved; Susie is told how to do subtraction though up till now she had been pleased how she could always do take-away sums.

The luckiest children may have a teacher who uses the test to note what type of questions each child did incorrectly. Johnny and Susie both gave wrong answers for the two multiplication sums so the teacher may run through the procedure for doing multiplication sums with the slow group, or with Johnny and Susie by themselves, if he gets a chance. But, in fact, Johnny gave

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wrong answers for the multiplication sums because he miscounted on his fingers, and Susie because she misread + for the ×.

It is clear that all these common practices in teaching mathematics at the early primary level are not satisfactory. Indeed, the recognition of similar difficulties in teaching young children to read led reading educators to develop methods which included further forms of evaluation and assistance. While there may be class lessons, or reading groups, for teaching beginning readers it is usual for teachers to hear each child read individually. Most teachers go to great lengths to ensure that each child is heard and questioned and helped individually. Yet most of those teachers would not sit down and watch a child work through a math problem, then ask questions about it, and then try to overcome any weaknesses shown.

The aim of the present study was to 'hear' beginning mathematicians, in their third year at school, in a way analogous to hearing reading, and to discover if this was a feasible and useful practice for classroom teachers of elementary mathematics.

1. DIAGNOSTIC INTERVIEWS IN MATHEMATICS

Hollander (1978) has reviewed many studies of children's thinking in solving verbal arithmetic problems. Most of these, which date from as early as the 1920's, have involved questioning the pupils or noting their verbalizations while solving problems, yet these studies are few in number in comparison with those that have relied upon written responses. The studies described by Hollander, and others, notably Erlwanger's as described by Davis (1973), Brownwell's cited as pre-eminent by Glennon and Wilson (1975), and more recently, the study by Lankford (1974), have clearly shown that the interview technique reveals far more of mathematical thinking than does analysis of written responses to a test. More general works on evaluation in mathematics, such as those of Dutton (1969) and Suydam (1974), have also advocated the use of diagnostic interviews to discover the reasons for errors by children.

Yet diagnostic interviews seem to have little place in most primary school classrooms. Why is this? There are several possible answers, some more obvious than others. It is clear that the use of a written test is far quicker and easier than questioning each child about how he did a particular problem. A classroom teacher would not have the time to do an 'in-depth' interview of the sort used by Erlwanger for example, and most studies report interviews taking an hour or more. A less obvious reason is that the interviews in many cases do not provide information that can be readily understood or used by the teacher. Hollander believed that most studies she reviewed have produced a list

of broad, poorly-defined error types that do not indicate the causes of errors. A teacher is unlikely to use a list of errors made by a child if it does not readily suggest ways of helping the child.

The teachers of beginning readers have been aided in the use of diagnostic interviews by the development of error or miscue analysis by Goodman (1965, 1969) and others. These provide a model or framework for the types of errors in the reading process so the teacher can more readily see the reasons for the errors made. Knowing the reasons for errors, the teacher can adjust the teaching to overcome the weaknesses. For example, if the child makes syntactic errors the teacher can develop a program to help the child use syntax correctly or more appropriately.

The present study used an error analysis technique for mathematics that seemed to hold all the advantages and utility of the Goodman miscue analysis in reading.

2. THE NEWMAN ERROR HIERARCHY

The error-classification used was that devised by Newman (1977). The advantage of this classification for the teacher is that it provides a clear framework for questioning the pupil and for analyzing any errors. The teacher can discover where and why the pupil made a mistake. Newman has successfully used the classification and interviewing with grade six pupils, and Clements (1980) gives a full report on its use with children from grades five to seven.

The Newman classification can be used with a wide range of mathematical problems and is based on a plausible model of how a child goes about solving a problem. The model postulates a sequence of steps, and failures at different stages are shown as different errors. The sequence of steps is seen as:

Reading → Comprehension → Transformation

→ Process skills → Encoding.

The associated 'Criterion for Error Causes' can be summarized as follows (with sub-categories under each main category of error also given):

1. *Reading ability* – can the pupil read the question?
   (i) Word recognition. (ii) Symbol recognition.

2. *Comprehension* – can the pupil understand the question?
   (i) General understanding. (ii) Understanding of specific terms or symbols.

3. *Transformation* – can the pupil select the mathematical processes which are required to obtain a solution?

4. *Process skills* – can the pupil perform the mathematical operations necessary for the task?
   Subcategories (for arithmetical skills only):
   (i) Random response
   (ii) Wrong operation
   (iii) Faulty algorithm
   (iv) Faulty computation
   (v) No response

5. *Encoding* – can the pupil write the answer in an acceptable form?

As well as these five categories of error there are three other types of errors that do not reflect the sequence of steps. They are:

6. *Motivation* – the pupil could have correctly solved the problem had he or she tried.

7. *Carelessness* – the pupil could do all the steps but made a careless error which is unlikely to be repeated.

Both motivation and carelessness could produce errors at any of the five stages outlined above. Newman recognized the existence of a further cause of error:

8. *Question form* – the pupil makes an error because of the way the problem has been presented. (A question may be ambiguous, for example.)

Newman found with 124 sixth grade low achievers aged between 11 and 13 years, that forty-seven percent of errors occurred in the first three stages of solving the problems, that is, before the process skills were called upon. Clements found similar proportions for grade 5 children and for low achievers in grade 7. Both Newman's and Clements' findings raise the question of whether similar results would be found with children at an earlier stage of learning mathematics. Do the children at the second grade level, for example, have similar difficulties as older children? Could the difference in level of mathematics produce a different pattern of errors? Before such questions can be answered it has to be shown that the error analysis can successfully be used with the younger children.

3. METHODOLOGY

Thirty children in grade two (their third year at primary school) in a Melbourne outer-suburban school were given a test of sixteen arithmetical problems, and over the next three weeks fifteen of the children were interviewed. The Newman error analysis was used in an attempt to discover why the children had not

obtained correct solutions to problems. The children's ages ranged from $6\frac{1}{2}$ to $7\frac{1}{2}$. The testing and interviewing was done by the writer, the teacher of twenty-five of the children. The other five children were selected by another grade two teacher as having general difficulties with mathematics.

The Test

There were sixteen test items involving the processes of addition, subtraction, multiplication and division. With the limited time available (all testing and interviewing had to be done in school time as part of normal lessons), it was decided to concentrate on one particular process and division was chosen. So five of the sixteen items involved the notion of division.

As many of the children were known to be very slow readers, there were no written verbal problems such as those used by Newman. The verbal problems were presented orally by the teacher and repeated as often as requested by the children in the group. When all children had finished a verbal problem (or had said they could not finish it), the next one was presented. Then at the finish of question six the written questions were distributed and children worked through them at their own pace. The oral part of the test took about fifteen minutes, and the written part was completed in times varying from five to twenty-five minutes. (One girl was asked to stop after thirty minutes when she had attempted five of the written questions.) The children had the choice of using a bead frame or counters if they wished, and all used one or the other, or counted on fingers.

The test items were:

1. Mother baked 12 scones. She shared them so each child got the same number. There are 4 children. How many scones did each child get?
2. A boy has six lollies and buys seven more. How many lollies does he have altogether?
3. At the fruit stop there are 4 apples in each bag. How many apples are there in 3 bags?
4. Jim starts with 12 cards and then loses 5. How many cards has he got left?
5. Fourteen, how many two's, equals?
6. Mary has 12 dolls. She stands them in groups so there are the same number in each group. There are 4 groups. How many dolls are in each group?

7. $12 - 5 = \square$ 8. $7 + 6 = \square$ 9. $3 \times 4 = \square$ 10. $20 + 50 = \square$

11. $12 \div 3 = \square$ 12. $5 + \square = 12$ 13. $14 \div 2 = \square$ 14. $4 \times \square = 12$

15. $13 - \square = 7$ 16. $\frac{1}{3}$ of $12 = \square$

(The last item was included to see how the children would respond to something that most would not have seen before.)

Diagnostic interview

The interviews were done over three weeks in school time by the writer at times when the rest of the grade was being taken by another teacher. The fifteen children chosen for the interviews were selected on the basis of their previous success in maths by the two teachers involved. Unlike Newman's study, where only low achievers were tested, it was decided to analyze the errors of five children who had shown few difficulties with mathematics and ten children who were seen by the teachers as having persistent difficulties with mathematics. The criteria for deciding who went into each group were purposely left at this general level of teacher's opinion to see if the two groups, regarded as different by the teachers, did show general differences in the types of errors they made. Clements found there were some differences between low achievers and average achievers in grade 7. The time taken by the interviews ranged from ten minutes for those with few errors, to nearly one hour for a girl who was very slow to respond. (She was interviewed over two sessions.) Usually interviews took from twenty to thirty minutes.

The questions which children were asked by the interviewer in the present study were sometimes different, in form, from those recommended by Newman. She firstly asked the pupils to read the question. In the present study the oral question was repeated by the teacher before the pupil reworked it, and the written questions were read aloud by the pupil. The second request made by Newman was: 'Tell me what the question is asking you to do'. With the younger children it was clear after a few attempts that this was inappropriate. Greg replied: "To get the answer" and other children ummed and ahhed, and then went ahead doing the problem with counters or bead frame. Thus it was decided to ask the children to show how they did the question rather than to verbalize about what it meant and what they were doing. In all cases the children then proceeded to manipulate the counters or beads, except for one girl who drew marks on paper instead of using counters.

If it was not clear why a child counted out a certain number of counters or arranged them in a certain way, they were asked: 'Why did you do that?' or 'Why did you use that number of counters?'. The teacher in this case did not of course try to influence the child's responses, and tried to avoid cues as to whether the answer or step was right or wrong. So although the younger children could not express their thinking verbally, their steps in tackling the problems could usually be seen in the manipulation of the concrete aids, and

unclear cases were clarified with occasional questions. The children were not told that they were reworking only problems that they answered wrongly in the test.

The types of errors made were recorded as were any comments of the child and the manipulations of the counters were noted. For example, with the question '3 x 4' Joe read it correctly, but then counted out three counters and four more and wrote '7' as his answer. (In the original test he wrote the answer '6'.) When asked why he did this he replied that it said to add up. So his errors were recorded as 2(ii) – misunderstanding symbols, and as 4(ii) – wrong operation. Lynne, with '14 ÷ 2', could not read ÷ aloud, counted out fourteen counters and said 'twenty-eight', but could not write it. When asked how she got twenty-eight she said she counted by two's. So she had errors under 1(i) – reading symbols, 2(ii) – misunderstanding symbols, 4(ii) – wrong operation, and 5 – encoding.

4. RESULTS AND DISCUSSION

Table I shows the number of wrong answers or 'no replies' in the test and in the interview, indicating that the children regarded as less able at maths did have far more mistakes. It is also clear that the children performed better in the interview situation, probably because of being in a one-to-one situation with the teachers in a room with few distractions.

TABLE I

Total number of incorrect answers or 'no replies' given in Test and Interview for 10 children Less Able at maths and 5 children More Able at maths. Figures in parentheses are mean numbers of errors per child.

	Total Number Errors Made by Less Able Group ($N = 10$)	More Able Group ($N = 5$)
Test	102 (10.2)	24 (4.8)
Interview	80 (8.0)	10 (2.0)

The numbers of errors of the different types are given in Table II. The total number of errors refers to the errors made when the children worked through the problems. In the examples given earlier, Joe had two errors with '3 x 4' and Lynn had four errors in all with '14 ÷ 2'. The initial errors are those errors that were taken as the major cause of the wrong answer. Thus each problem answered incorrectly had an initial error for each child getting it wrong. Joe's initial error with '3 x 4' was 2(ii) – Misunderstanding Symbols, and Lynne's initial error with '14 ÷ 2' was 1(ii) – Misreading Symbols. Newman (1977) obtained the total number of errors in her study but reported the initial errors only.

In most cases the initial error was the first one made on the problem, but in a few cases the child misread the question aloud but their working showed that the correct meaning of the symbol was being used. In these cases the misreading was not taken as the initial error. The error categories Motivation and Question Form are not shown on the table as there were no cases where these were thought to cause errors during the reworking of the problems. There are also no errors for reading words as the verbal problems were presented orally as described earlier.

TABLE II

Classification of Total and Initial errors for the Less Able group (10 children) and the More Able group (5 children).

Categories of Errors	Less Able Group		More Able Group	
	Total	Initial	Total	Initial
1. Reading (i) Words				
(ii) Symbols	22	18	3	3
2. Comprehension (i) General	15	15	4	3
(ii) Symbols	22	19	3	3
3. Transformations	9	5	0	0
4. Processes (i) Random	4	0	0	0
(ii) Wrong Operation	26	7	4	0
(iii) Algorithm	5	2	1	1
(iv) Computation	17	8	0	0
(v) No response	16	1	1	0
5. Encoding	7	1	0	0
6. Carelessness	4	4	0	0
Totals	146	80	16	10

Table II clearly shows that for both groups the large majority of initial errors were made at the stages of Reading and Comprehension. This result was caused largely by the written problems from number 11 onwards. The verbal problems, apart from number 5 (which is really a symbolic equation in verbal form) were handled more successfully, with the few errors being spread evenly over the six categories. For the Less Able group, twenty-three percent of initial errors occurred during the stage of applying Processes, indicating that they had greater difficulty at this level than the More Able group. The younger children showed a far higher percentage of errors at the stages of Reading and Comprehension than did the older children studied by Newman and Clements. Many of the Less Able group, if they could read the question in symbolic form, did not know what those questions required them to do.

One aspect of the results that needs to be considered is the low proportion of errors due to Carelessness or Motivation when compared with Newman's

and Clements' data. In their studies, conducted with older children, it was found that these types of errors accounted for twenty-two to thirty-eight percent of the errors made. This difference may be due in part to the greater willingness or enthusiasm of the younger children, but the major reason would be that there was a difference in procedure for the error analysis. Newman and Clements classified errors in the original test given, and if a child obtained the correct answer during the interview the error in the test was assumed to be due to Carelessness or Motivation. In the present study however, the errors given are those made during the interview; if the child was correct during the interview no error was recorded. This was done because it was often observed that the children varied their way of working out a problem. A child would do one division sum one way, another in a different way, or he would change his mind half way through working out a problem. There were fifty-one cases where children obtain different wrong answers in the test and interview. So it did not seem justified to argue that a difference between the test and the interview was due solely to Carelessness or lack of Motivation during the test. It seemed wiser to make no assumptions about reasons for errors in the original test when the child's thinking could not be assessed. (If the Newman and Clements procedure is followed, the data in Tables I and II show that thirty-two percent of errors in the original test would be classed as due to Carelessness or Motivation.)

The most interesting feature of the results, from the classroom teacher's point of view, is that it was possible, from the classification of errors for each child, to see precisely how he or she approached the problem, to see where strengths and weaknesses lay. In all cases, too, it was possible to tell the child at some stage that they were successful, if only at the reading of the problem. This means that the child could receive some justified reinforcement whether the final answer was correct or not. John, for example, gave the wrong answer for '20 + 50'. If the teacher was relying on the final answer only, he would have to tell John that he was wrong. But with the use of the error analysis the teacher could honestly reinforce John by pointing out that he was correct with all the sum but for a careless slip in the final stages. Sometimes it was a pleasant surprise to both the teacher and the child to discover how much was successfully achieved. Even though the writer had been teaching these children for several months at the time of the interview, it was very enlightening to see and hear how each child tackled the problem.

5. THE USE OF THE DIAGNOSTIC INTERVIEW IN THE CLASSROOM

The study has shown that an interview using the Newman error analysis can be used with children at an early stage of mathematical development. The

diagnostic interview was successfully adapted for use with children who could not express their thinking 'out loud'. The variations to the procedure used by Newman with older children were: (1) the use of concrete aids such as counters and bead frames, (2) asking the children to show, using the aids, how they did the question, and (3) requesting verbalizations when the reasons for the manipulations could not easily be deduced.

The Newman model of the sequence of steps in solving mathematical problems provided a very useful framework for the diagnosis of strengths and weaknesses. It helped the teacher assess the child's skills at all the stages of 'doing a sum'. This is important in the classroom because with the pressures of time the teacher can easily overlook some steps where the child might be succeeding or failing.

The interview is also useful in that it forces the teacher to 'hear' the child doing mathematics. The teacher is not tempted to hurry the process by telling the child a way to get the answer so he can speak to Fred who is calling out that he wants help also. The child is not left with a rushed explanation that could be more confusing than enlightening. Even if it is not clear during the interview how a particular error should be classified, if the teacher has noted down what was done or said, that information can provide a firm base for giving the child further help.

Many teachers may say that because of the pressures of time mentioned above it is not possible to give diagnostic interviews in the classroom situation. This is undoubtedly true if the interviews are to be on the scale of Newman's or Clements', or even some of those in this study. But there is no reason why much more restricted interviews could not be equally useful. If the teacher, for example, is about to introduce a new topic for that year, a short test could be given and those children who do poorly could then be interviewed, with the error analysis done over two or three problems. This would take only five minutes or so for each child, and would provide the teacher with a clear picture of where these children are making errors. The interview may show that some children understand the concepts and made errors at later stages involving computation. Knowledge such as this could save time overall because the teacher would then know that he does not have to spend time introducing the concepts to those children.

Or, if a few children in a class have been experiencing persistent difficulties in mathematics, the teacher could conduct more wide-ranging interviews with those few children, over several short sessions if necessary. Not only would this clearly show where the difficulties lay, but the sequence of steps postulated by Newman could then be used to devise lessons to help the children, one step at a time. The child would have a much greater probability of success than if

he had to go through all five steps at the one attempt, as would normally be required. Even the slowest child in the present study was correct on at least one step of solving the problems and could be reinforced for it. For this reason, if no other, the diagnostic interview with an error analysis such as Newman's should be tried by all teachers of 'beginning mathematicians'.

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