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Amplitude Modulation of the cw Laser Light in the TEM_{mn} Mode by Means of a Mechanical Chopper

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Abstract. Amplitude modulation of the laser beam in a TEM_{mn} mode by means of a mechanical chopper is investigated on the basis of the chopping model represented by a moving system of infinitely long, parallel slots and mark spaces. The cases of rectangular and axial symmetry of the laser beam are both treated. The explicit expressions for the waveform of the modulated normalized transmitted laser power are deduced and their consequences investigated. It is found that in the case of rectangular symmetry, unlike the case of axial symmetry, the TEM_{mn} modes give, for a constant value of the mode number n and for any value of m, the same time dependence of the amplitude modulated laser power. The notion of the equivalent modulation widths is introduced and conditions for the efficient amplitude modulation are found.

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Amplitude modulation of the cw laser beam by means of a mechanical chopper is frequently used in photophysics and particularly in photoacoustic spectroscopy [1–4]. Usually, the TEM_{00} (Gaussian) mode is employed. For such a case the complete modulation theory, corresponding to the case depicted in [Ref. 5, Fig. 6] as well as the simplified model depicted in [Ref. 5, Fig. 2] of the amplitude modulation by a mechanical chopper were given in [5]. Basically, in both cases the appropriate laser-light intensity was integrated over the system of moving slots, thus yielding the transmitted laser power as a function of time. It was found that for the laser light in the TEM_{00} mode, for all practically important values of the chopping disc slot and mark space widths relative to the radius of the Gaussian beam, the difference between the complete modulation theory and the model is effectively insignificant, thus making unnecessary the use of the precise and rather cumbersome treatment. The chopping model was used in [6] to obtain the conditions for the harmonic-like and at the same time efficient amplitude modulation of the cw Gaussian laser beam by means of a mechanical chopper (the simultaneous fulfilment of these two features being of interest in photoacoustic applications).

In this paper the chopping model is utilized to obtain the waveform of the amplitude-modulated laser light in a TEM_{mn} mode in the case of either rectangular (Sect. 1) or axial (Sect. 2) symmetry. The predictions of the model in the two cases, some further considerations and discussion are presented in Sect. 3. The results obtained are of potential interest in all applications of the amplitude modulation of a cw laser light by a mechanical chopper.

1. Modulation Theory in the Case of Rectangular Symmetry

Consider a cw laser beam in the TEM_{mn} oscillation mode for a system with a rectangular geometry propagating along the direction normal to the plane containing moving parallel slots and mark spaces. It is convenient for the present purpose to use dimensionless parameters and variables. This is achieved by measuring all lengths in units of the radius a of the

Gaussian laser beam (including a itself, thus $a \equiv 1$) and the time t in units of a/v (v being the velocity of the system of parallel slots and mark spaces). In [6] these dimensionless quantities were distinguished by a prime which is here omitted for the sake of simplicity. The laser light intensity $I_{mn}(x, y)$ corresponding to the TEM_{mn} mode in the case of rectangular symmetry is [7, 8]

$$I_{mn}(x,y)=\frac{P_{mn}}{\pi 2^{m+n-1}\Gamma(m+1)\Gamma(n+1)} \cdot \mathrm{H}_m^2(2^{1/2}x)\mathrm{H}_n^2(2^{1/2}y)e^{-2(x^2+y^2)}. \quad (1)$$

Here P_{mn} denotes the power of the incident laser light in the TEM_{mn} mode while H_m and Γ denote the Hermite polynomial of order m and the usual gamma function, respectively. The mode numbers m and n give, as is well known, the number of zeros in a mode patern along the x and y direction, respectively. The constant factor in (1) is such that the intensity $I_{mn}(x, y)$, when integrated over all values of x and y, gives (with the aid of the orthogonality property for the Hermite polynomials, [9]) exactly the incident laser power P_{mn}.
At any instant t, the laser power $P_{mn}(t)$ transmitted by a mechanical chopper is obtained by integrating the laser light intensity over the system of moving slots

$$P_{mn}(t)=\sum_k \int_{-\infty}^{+\infty} dx \int_{y_{k-}(t)}^{y_{k+}(t)} dy\, I_{mn}(x,y). \quad (2)$$

Here the quantities

$$y_{k\pm}(t) \equiv k(b+\beta)-t \pm b/2, \quad (3)$$

denote the y-coordinates of the upper (+) and lower (−) edges of the k^{th} slot ($k=0, \pm 1, \pm 2, \ldots$) at time t. The b and β are the dimensionless slot and mark space widths respectively. Combining (1 and 2) and performing integration over x, one obtains for the normalized transmitted laser power in the TEM_{mn} mode of the rectangular symmetry as a function of time the following expression

$$\frac{P_{mn}(t)}{P_{mn}}=\frac{1}{\pi^{1/2}2^n\Gamma(n+1)} \cdot \sum_k [K_n(2^{1/2}y_{k+}(t))-K_n(2^{1/2}y_{k-}(t))], \quad (4)$$

where the quantity K_n is introduced via the following indefinite integral

$$K_n(\xi) \equiv \int \mathrm{H}_n^2(\xi)e^{-\xi^2}d\xi. \quad (5)$$

With the aid of the well known properties of the Hermite polynomials the following relation is established ($n=1, 2, \ldots$)

$$K_n(\xi)=2nK_{n-1}(\xi)-e^{-\xi^2}\mathrm{H}_n(\xi)\mathrm{H}_{n-1}(\xi), \quad (6)$$

which together with

$$K_0(\xi)=\frac{\pi^{1/2}}{2}\operatorname{erf}\xi, \quad (7)$$

gives any $K_n(\xi)$. The first few K_n's are

$$K_1(\xi)=\pi^{1/2}\operatorname{erf}\xi-2\xi e^{-\xi^2}, \quad (8)$$

$$K_2(\xi)=4\pi^{1/2}\operatorname{erf}\xi-4\xi(2\xi^2+1)e^{-\xi^2}, \quad (9)$$

$$K_3(\xi)=24\pi^{1/2}\operatorname{erf}\xi-8\xi(4\xi^4-4\xi^2+9)e^{-\xi^2}. \quad (10)$$

Using these, one can obtain the corresponding $P_{mn}(t)/P_{mn}$ expressions from (4). In the case $m=n=0$, (4), together with (7 and 3), reproduces the expression given in [5, 6] for the amplitude modulated laser light in the TEM_{00} mode.
The remarkable property of (4) is its independence of the mode number m; for given, constant value of the mode number n (which defines the number of zeros in a mode pattern in the direction across the slots) all TEM_{mn} modes with $m=0, 1, 2, \ldots$ have the same time dependence of the amplitude modulated laser power. This is the consequence of
(i) assumed symmetry of the chopping model (i.e., of the assumption that the slots are parallel and infinitely long along the x-axis; this is, as was shown in [5], in all cases of practical interest a very good approximation) and
(ii) the fact that all these modes (with any m and constant n), have the same functional dependence of the light intensity on the y-coordinate, (1).
Thus, in particular, any of the TEM_{m0} modes with $m=1, 2, \ldots$ gives the same time dependence of the amplitude modulated laser power as the TEM_{00} mode, the case which has been already extensively discussed [5, 6].
Figure 1 presents the waveforms of the amplitude modulated laser light in the case of rectangular symme-

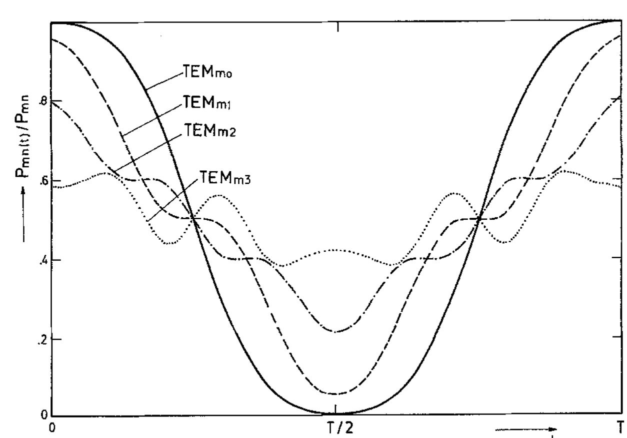

Fig. 1. Waveforms of the amplitude modulated laser light in the case of rectangular symmetry for $n=0, 1, 2$, and 3 (any m) and $b=\beta=2.77$

try for $n=0, 1, 2$, and 3 (any m) and $b=\beta=2.77$ which were obtained using (3, 4, 7–10) as well as the rational approximation for the error function [10]. Defering detailed discussion to the Sect. 3 we only note here that the modulation depth is decreasing with increasing n value reflecting the fact that the cross sectional area occupied by a mode increases with the mode number.

2. Modulation Theory in the Case of Axial Symmetry

The treatment is in this case somewhat more complex due to the fact that the axial symmetry of the laser beam combines with the rectangular symmetry of the system of slots and mark spaces (as assumed by the chopping model). Consequently the expression obtained for the amplitude modulated laser light is more complex and the relative simplicity of the $P_{mn}(t)/P_{mn}$ expression in the rectangular case is here lost, of course. The derivation in the case of axial symmetry generally follows the rectangular case. The starting point is the expression for the intensity of the laser light in the TEM_{mn} mode which is in the case of axial symmetry given by (employing cylindrical coordinates r, ϕ, and z, [7,8])

$$I_{mn}(r,\phi)=\frac{4\Gamma(m+1)P_{mn}}{\pi(1+\delta_{n0})\Gamma(m+n+1)} \cdot (2r^2)^n[L_m^n(2r^2)]^2 e^{-2r^2}\cos^2 n\phi . \tag{11}$$

Here, r is the dimensionless radius, δ_{n0} is the usual Kronecker delta symbol and L_m^n are the associated Laguerre polynomials. The m and n are the mode numbers (such that m and $2n$ give the number of zeros in a mode pattern in the radial and the azimuthal direction, respectively). The constant factor in (11) is such that the intensity $I_{mn}(r,\phi)$, when integrated over all values of r and ϕ, gives (with the aid of the orthogonality property for the associated Laguerre polynomials, [9]) exactly the incident laser power P_{mn} in the corresponding mode.

Firstly, consider the case when the axis of the laser beam impinges on a chopper mark space. The treatment analogous to that of Sect. 1 gives, with the aid of the expression for the associated Laguerre polynomial given in [10], the expression for the normalized transmitted laser power in the TEM_{mn} mode of the axial symmetry as a function of time in the form

$$P_{mn}(t)/P_{mn}=f_{mn}(t), \tag{12}$$

where

$$f_{mn}(t)\equiv M(m,n)\sum_k \sum_{l=0}^{m}\sum_{l'=0}^{m} N(m,n,l,l') \cdot \int_0^{\pi} P(n,l,l',r_{k\pm})\cos^2 n\phi\, d\phi , \tag{13}$$

$$M(m,n)\equiv \Gamma(m+1)\Gamma(m+n+1)/\pi(1+\delta_{n0}), \tag{14}$$

$$N(m,n,l,l')\equiv \frac{(-1)^{l+l'}}{\Gamma(l+1)\Gamma(l'+1)\Gamma(n+l+1)\Gamma(n+l'+1)\Gamma(m-l+1)\Gamma(m-l'+1)}, \tag{15}$$

$$P(n,l,l',r_{k\pm})\equiv \operatorname{sgn}\{r_{k+}\}R(n+l+l',2r_{k+}^2) - \operatorname{sgn}\{r_{k-}\}R(n+l+l',2r_{k-}^2), \tag{16}$$

$$r_{k\pm}=r_{k\pm}(t,\phi)\equiv y_{k\pm}(t)/\sin\phi , \tag{17}$$

and

$$R(j,\xi)\equiv \int \xi^j e^{-\xi} d\xi = -e^{-\xi}\left[\xi^j+\sum_{i=1}^{j}\frac{\Gamma(j+1)}{\Gamma(j-i+1)}\xi^{j-i}\right]. \tag{18}$$

The remaining integration in (13) must be done by numerical methods.

When the axis of the laser beam passes through a chopper slot it is more convenient to find the normalized laser power which is not transmitted by the chopper. It turns out that it is given by $|f_{mn}(t)|$ and therefore in such a case

$$P_{mn}(t)/P_{mn}=1-|f_{mn}(t)| . \tag{19}$$

The TEM_{00} mode is the same for both rectangular and axial symmetry and therefore in the case $m=n=0$ Eqs. (12, 19) together with (13–18) must reproduce the expression given in [5, 6]. That this is indeed the case one can see using the following relation

$$1-\frac{1}{\pi}\int_0^{\pi} e^{-\xi^2/\sin^2\phi} d\phi = \operatorname{erf}|\xi| , \tag{20}$$

the validity of which can be easily established.

We see that in the case of the axial symmetry the $P_{mn}(t)/P_{mn}$ expression depends on both mode numbers unlike the case of the rectangular symmetry. Figures 2–4 present the waveforms of the amplitude modulated laser light in the case of axial symmetry for $m=0,1,2$, $n=0,1,2,3$, and $b=\beta=2.77$ which were obtained using (12–19).

3. Further Considerations

In both cases (of rectangular, Sect. 1, or axial, Sect. 2, symmetry) the amplitude modulated laser light by a mechanical chopper is periodic function with the period

$$T=b+\beta . \tag{21}$$

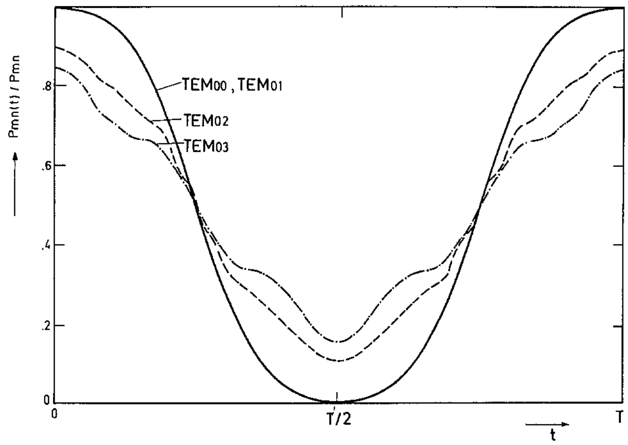

Fig. 2. Waveforms of the amplitude modulated laser light in the case of axial symmetry for $m=0$, $n=0, 1, 2$, and 3, and $b=\beta=2.77$

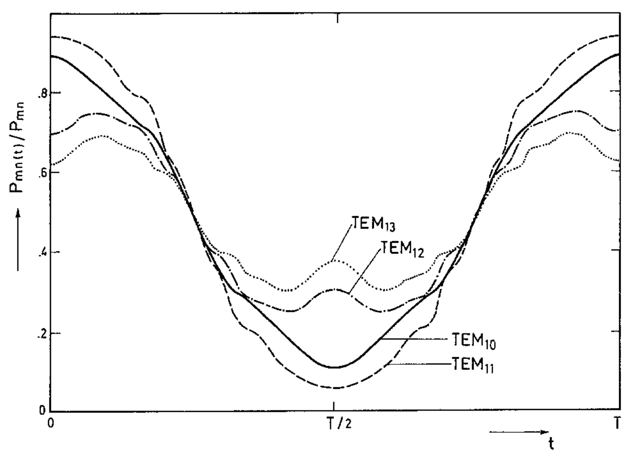

Fig. 3. Waveforms of the amplitude modulated laser light in the case of axial symmetry for $m=1$, $n=0, 1, 2$, and 3 and $b=\beta=2.77$

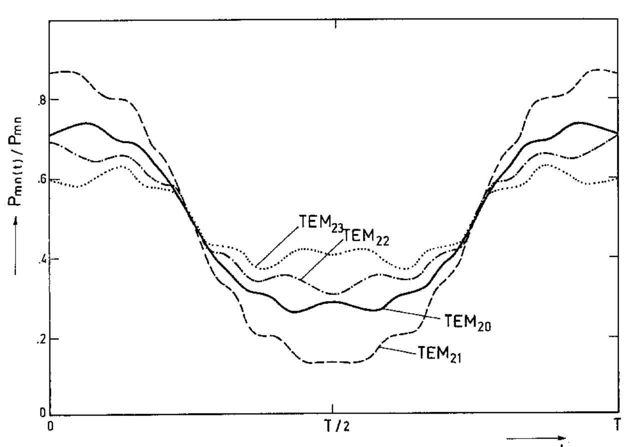

Fig. 4. Waveforms of the amplitude modulated laser light in the case of axial symmetry for $m=2$, $n=0, 1, 2$, and 3, and $b=\beta=2.77$

In the case when $b, \beta \gg 1$ the laser beam impinges completely on one slot at most at any instant. In such a case in the sum over k in (4) or (13), as the case may be, only one term contributes significantly; the one corresponding to the slot which is at that particular moment centered on the laser beam. This is the k'^{th} slot where

$$k' = \left(\frac{t}{b+\beta}\right). \tag{22}$$

[Here (ξ), ξ denoting $t(b+\beta)^{-1}$, is used to denote the nearest integer to ξ.]

On the other hand, when $b, \beta \ll 1$ the laser light passes at any instant through many slots simultaneously. In (4) or (13) many terms contribute and although the pattern is moving the transmitted power is constant [e.g., in the case of rectangular symmetry, for the slots k which are contributing significantly, the arguments of the K_n functions appearing in (4) are small so that series expansion and rejection of the small higher-order terms indeed gives a constant]. In fact when $b, \beta \ll 1$ one has

$$\frac{P_{mn}(t)}{P_{mn}} \approx \frac{b}{b+\beta} = \frac{\text{transmitting area}}{\text{total area}} = \text{const}. \tag{23}$$

In the intermediary case $b, \beta \approx 1$ several slots contribute significantly (the contribution of others being negligible) and the modulation depth is reduced when compared with the case $b, \beta \gg 1$.

In all cases considered, the time averaged value $\overline{P_{mn}(t)/P_{mn}}$ of the transmitted laser power is also given by (23) irrespectively of the sizes of the slot and mark space widths relative to the laser beam diameter.

The largest and the smallest value of the normalized transmitted laser power, $[P_{mn}(t)/P_{mn}]_{\max}$ and $[P_{mn}(t)/P_{mn}]_{\min}$, respectively, are the quantities which measure the modulation depth. These two quantities are depicted on Fig. 5 for the case of rectangular symmetry and on Fig. 6 for the case of axial symmetry as a functions of the mode numbers and for $b=\beta=2.77$. In the case when $b=\beta$ one has

$$[P_{mn}(t)/P_{mn}]_{\min} = 1 - [P_{mn}(t)/P_{mn}]_{\max}.$$

One can see that the modulation depth is decreasing with increasing values of the mode numbers, reflecting the fact that the cross sectional area occupied by a mode increases with the mode numbers. Obviously, in order to achieve the same modulation depth as in the case of the TEM_{00} mode one must, in the case of a TEM_{mn} mode, employ larger values of the slot and mark space widths. One can introduce the notion of the equivalent modulation widths $b_e=\beta_e$ as the chopper slot and mark space widths which achieve the same prescribed modulation depth for different TEM_{mn} modes. Figures 7 and 8 give typical values of the

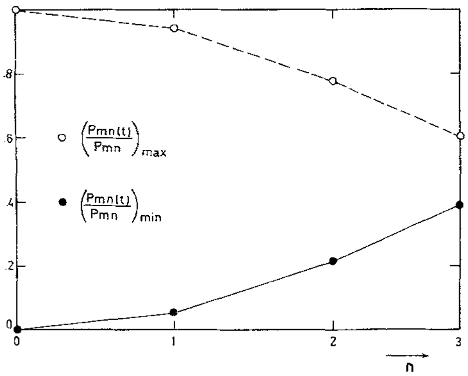

Fig. 5. Largest and smallest normalized transmitted laser power for the case of rectangular symmetry as functions of the mode number n (any m) and for $b=\beta=2.77$

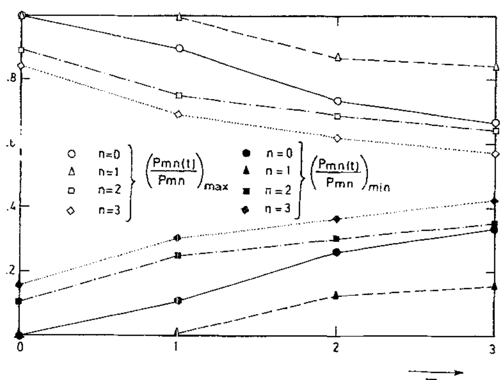

Fig. 6. Largest and smallest normalized transmitted laser power for the case of axial symmetry as functions of the mode numbers m and n and for $b=\beta=2.77$

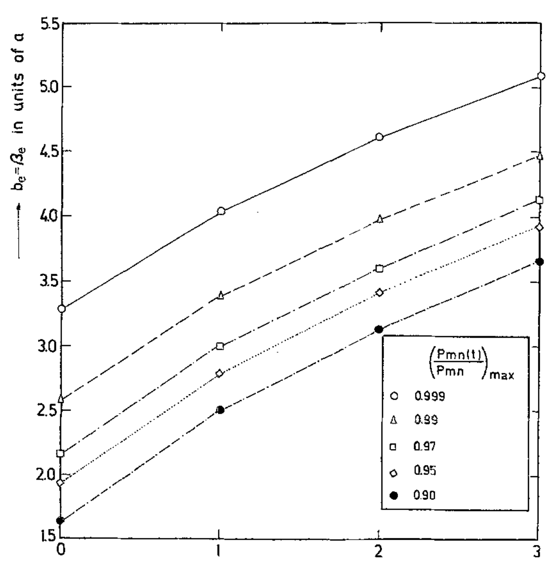

Fig. 7. Equivalent chopper slot and mark space widths $b_e=\beta_e$, measured in units of the radius a of the Gaussian laser beam, as functions of the mode number n (any m) in the case of rectangular symmetry and for various prescribed modulation depths indicated on the figure

equivalent widths as a functions of the mode numbers for the two symmetry cases. One should bear in mind that the equivalent widths ensure only the same modulation depth; the waveform of the modulated laser light being generally different for different modes.

4. Conclusions

The waveform of the amplitude modulated cw laser light in a TEM_{mn} mode by means of a mechanical chopper, based on the chopping model [5], is given by (4) in the case of rectangular symmetry and by (12 and 19) in the case of axial symmetry of the laser beam. In the case of rectangular symmetry, unlike the case of axial symmetry, the TEM_{mn} modes with a constant n value and for any m value give the same time dependence of the amplitude modulated laser power. In particular, any of the rectangular TEM_{m0} modes with $m=1, 2, \ldots$ gives the same waveform as the TEM_{00} mode, the case which has been already thoroughly investigated [5, 6]. The modulation depth as a function of mode numbers m and n, for given slot and mark space widths, is presented for the two symmetry cases in Figs. 5 and 6. It is found that the modulation depth is generally decreasing with increasing mode numbers; the larger values of the chopper slot and mark space widths must be employed in such cases in order to achieve the same modulation depth. The notion of the equivalent chopper slot and mark space widths is introduced and their values obtained for various

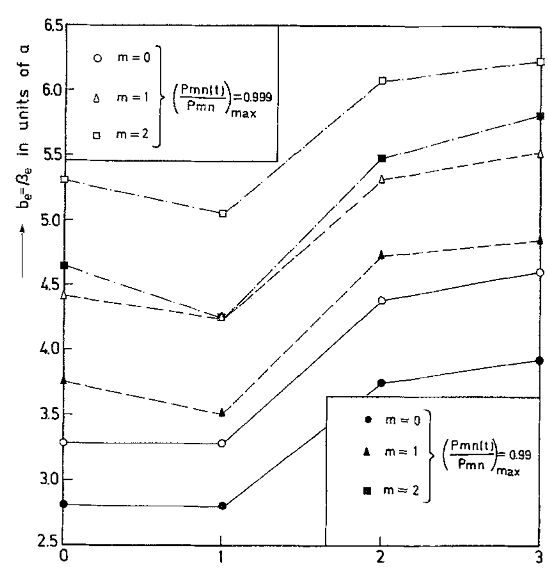

Fig. 8. Equivalent chopper slot and mark space widths $b_e=\beta_e$, measured in units of the radius a of the Gaussian laser beam, as functions of the mode numbers m and n in the case of axial symmetry and for the two modulation depths indicated on the figure

prescribed modulation depths (Figs. 7 and 8). The results obtained are of potential interest in all applications of the amplitude modulation of a cw laser light by means of a mechanical chopper.

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