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On Some Relations for the Inverse Blackbody Radiation Problem

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Abstract. Some relations between the area-temperature distribution of a blackbody and its radiated power spectrum are derived. These relations shall be useful in verifying the adequacy and the accuracy of the various procedures recently developed for the numerical solution of the inverse blackbody radiation problem.

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Lately, the electromagnetics community has begun to pay attention to the blackbody radiation phenomenon due to its occurrence in the area of remote sensing [1]. The forward problem has been investigated quite extensively from around the turn of this century, but the inverse problem has only recently been the subject of exploration [2–5]. However, the inverse problem is quite ill-conditioned [6], and the various numerical algorithms proposed for its solution do rely on the nonunique inverse Laplace transform. Consequently, it is necessary to develop algorithm-independent checks in order to verify the accuracy and the adequacy of the numerical results obtained on a digital computer. This is precisely the objective of this communication where three such relationships will be derived in the last section.

The Inverse Blackbody Radiation Problem

It is well-known that when a body's surface is at an absolute temperature T, it emits "blackbody radiation" whose spectrum is given by Planck's law as

$$W(v)=(2hv^3/c^2)[\exp(hv/KT)-1]^{-1}, \tag{1}$$

where v is frequency, h is Planck's constant, c is the speed of light *in vacuo*, and K is Boltzmann's constant. If, however, all parts of the body's surface are not at the same temperature, and $a(T)$ is the area-temperature distribution function, then (1) must be modified to read

$$W(v)=(2hv^3/c^2)\int_0^\infty dT\, a(T)\, [\exp(hv/KT)-1]^{-1}. \tag{2}$$

The inverse blackbody radiation problem is constituted by the determination of $a(T)$ provided $W(v)$ is known, experimentally or otherwise [7]. As stated earlier, several new procedures for solving this inherently ill-conditioned problem have recently come to light and shall now be briefly discussed.

Solution of the Inverse Problem

On defining a "coldness" function u such that

$$u=(h/K)(1/T) \tag{3}$$

and an auxiliary spectrum $g(v)$ as

$$g(v)=(c^2/2hv^2)W(v) \tag{4}$$

the integral equation (2) can be reconstituted in the form

$$g(v) = v \int_0^\infty du\ a(u)\ [\exp(uv) - 1]^{-1}\,. \tag{5}$$

Hamid and Ragheb [3] considered the case at microwave frequencies where the Rayleigh-Jeans approximations $uv \ll 1$ holds. By expanding the denominator of the integrand in (5) and discarding the high-order terms, they obtained

$$g(v) = \mathscr{L}\{a(u)/u\}\,, \tag{6}$$

where $\mathscr{L}\{\cdot\}$ denotes the Laplace transform. From (6) the solution of (2) in the Rayleigh-Jeans approximation is then given by

$$a_{HR}(u) = \mathscr{L}^{-1}\{g(v)\}\,, \tag{7}$$

where $\mathscr{L}^{-1}\{\cdot\}$ is the inverse Laplace transform.
On the other hand, should the Wien approximation $\exp(uv) \gg 1$ hold, then (5) transforms to [4]:

$$g(v) = v\mathscr{L}\{a(u)\}\,, \tag{8a}$$

whence the solution given by Lakhtakia and Lakhtakia [4] in the Wien regime is

$$a_{\mathrm{LL}}(u) = \mathscr{L}^{-1}\{g(v)/v\}\,. \tag{8b}$$

Two solution procedures, valid everywhere, have also been proposed recently. The earlier of the two [2] uses an iterative procedure whose m^{th} stage gives a solution

$$a_{(m+1)}(u) = a_{\mathrm{LL}}(u) - \sum_{n=2}^{\infty} (1/n)a_{(m)}(u/n)\,, \qquad m > 0 \tag{9}$$

with the initial guess being the Wien regime solution (8b).
The second of the two "universal" procedures is rather complicated but avoids iterations, and only the final result is stated here [5]:

$$\begin{aligned} a(u) = {} & a_{\mathrm{LL}}(u) - (1/2)a_{\mathrm{LL}}(u/2) - (1/3)a_{\mathrm{LL}}(u/3) \\ & - (1/5)a_{\mathrm{LL}}(u/5) + (1/6)a_{\mathrm{LL}}(u/6) - (1/7)/a_{\mathrm{LL}}(u/7) \\ & + (1/10)/a_{\mathrm{LL}}(u/10) - (1/11)a_{\mathrm{LL}}(u/11) \\ & - (1/13)a_{\mathrm{LL}}(u/13) + (1/14)a_{\mathrm{LL}}(u/14) \\ & + (1/15)a_{\mathrm{LL}}(u/15) - (1/17)a_{\mathrm{LL}}(u/17) - \ldots\,. \end{aligned} \tag{10}$$

Checks on the Inversion Procedures

The point to be noted in the previous section is that in all of the four procedures mentioned above, the inverse Laplace transform must be utilized at some stage or the other. Because (5) is an integral equation of the first kind [6, 8] and because the inverse Laplace transform is not unique [9], all of the solution procedures suffer from a certain lack of confidence in them (but, see Appendix). Hence, it is necessary to develop from (2) itself some properties of the blackbody radiation phenomenon which can serve to check the accuracy of the computed solutions (6, 8b, 9 or 10). This is what is now going to be described and developed.
Beginning with (2) and using the definition (3) one obtains

$$(c^2/2h)W(v)/v^3 = \int_0^\infty du\ a(u)/[\exp(uv) - 1]\,.^1 \tag{11}$$

Operating now on both sides of (11) by the integral operator

$$\int_0^\infty (\cdot)v^{2n-1}dv\,, \qquad n = 1, 2, 3, \ldots \tag{12a}$$

yields

$$\begin{aligned} & (c^2/2h) \int_0^\infty dv\ W(v)v^{2n-4} \\ & = \int_0^\infty dv\, v^{2n-1} \int_0^\infty du\ a(u)\ [\exp(uv) - 1]^{-1}\,. \end{aligned} \tag{12b}$$

The order of integration on the rhs of (12b) can be reversed since u and v are independent variables. Consequently,

$$\begin{aligned} & (c^2/2h) \int_0^\infty dv\ W(v)v^{2n-4} \\ & = \int_0^\infty du\ a(u) \int_0^\infty dv\ v^{2n-1}[\exp(uv) - 1]^{-1}\,. \end{aligned} \tag{12c}$$

But the v-integral on the rhs of (12c) can be expressed in closed-form as [Ref. 10, Eq. (3.411-2)]

$$\begin{aligned} & \int_0^\infty dv\ v^{2n-1}[\exp(uv) - 1]^{-1} \\ & = (-)^{n-1}(2\pi/u)^{2n}(\mathrm{B}_{2n}/4n)\,, \end{aligned} \tag{13}$$

where B_{2n} is the Bernoulli number of order $2n$ satisfying the relations

$$\begin{aligned} \mathrm{B}_{2n} = {} & -\frac{1}{2n+1} + \frac{1}{2} \\ & - \sum_{k=2}^{2n-2} \frac{2n(2n-1)\ldots(2n-2k+2)}{k!}\mathrm{B}_k\,, \end{aligned} \tag{14a}$$

$$\mathrm{B}_0 = 1\,, \quad \mathrm{B}_1 = -(1/2)\,, \quad \mathrm{B}_3 = \mathrm{B}_5 = \mathrm{B}_7 = \mathrm{B}_9 = \ldots = 0\,. \tag{14b}$$

As a result of using (13), (12c) converts to

$$\int_0^\infty dv\ W(v)v^{2n-4} = F_n \int_0^\infty du\ a(u)u^{-2n}\,, \tag{15a}$$

where, the number

$$F_n = (-)^{n-1}(2\pi)^{2n}(h/2n)(\mathrm{B}_{2n}/c^2)\,, \tag{15b}$$

[1] $a(u) = a(T)T/u$ is called the area-coldness function

thereby leading to a family of relations between $a(u)$ and $W(v)$.
A more general property of a similar nature is given by

$$\int_0^\infty dv\, W(v)v^{p-4} = (2h/c^2)\,\Gamma(p)\zeta(p) \cdot \int_0^\infty du\, a(u)u^{-p}, \quad p>0 \tag{16}$$

for all non-zero, positive real p. This property can be derived in the same manner as (15) was derived from (11), the difference being that the integral operator this time will be

$$\int_0^\infty (\cdot)v^{p-1}dv \tag{17a}$$

in place of (12a), and use is made of the integral [Ref. 10, Eq. (3.411-22)]

$$\int_0^\infty dv\, v^{p-1}[\exp(uv)-1]^{-1} = u^{-p}\,\Gamma(p)\zeta(p), \quad p>0 \tag{17b}$$

in place of (13). In (16 and 17), $\Gamma(p)$ is the gamma function and $\zeta(p)$ is the Weierstrass function defined in [Ref. 10, Sect. 8.17].
It is interesting to note that (15a) relates the $(2n)^{\text{th}}$ moment of $W(v)v^{-4}$ to the $(-2n)^{\text{th}}$ moment of $a(u)$, $n=1,2,3,\ldots$. This relationship is generalized in (16) to between the p^{th} moment of $W(v)v^{-4}$ and the $(-p)^{\text{th}}$ moment of $a(u)$ for all positive real p.
Finally, in this section, provided $a(u)=0\ \forall u<u_0>0$, then using the operator

$$\int_0^\infty (\cdot)v\exp(\mu v)dv, \qquad \mathrm{Re}\{\mu\}>u_0>0 \tag{18}$$

in place of (12a), and proceeding likewise, leads to the relation [Ref. 10, Eq. (3.411-24)]

$$\int_0^\infty dv\; W(v)v^{-2}\exp(\mu v) = 2h(\pi/c)^2 \cdot \int_{u_0}^\infty du\; a(u)\;[u\sin(\mu\pi/v)]^{-2}, \tag{19}$$

which is a relation between the complex moments of the area-coldness distribution and the radiated power spectrum.
These three families of relations (15a, 16, and 19) thus developed are independent of the specific solution procedure used to solve the inverse problem. Hence, their chief utility lies in ascertaining the accuracy of the solution $a(u)$ obtained from the specific algorithm implemented on a digital computer.

Appendix

If two functions $f_1(t)$ and $f_2(t)$ have the same Laplace transform

$$F(s) = \int_0^\infty dt\exp(-st)f_1(t) = \int_0^\infty dt\exp(-st)f_2(t),$$

then

$$f_2(t) = f_1(t) + N(t),$$

where $N(t)$ is a null function such that

$$\int_0^{t'} dtN(t) = 0$$

for every positive t'. This statement is known as Lerche's theorem [9], and it somewhat broadens the conditions for the uniqueness of the inverse Laplace transform of $F(s)$. However, the inverse blackbody radiation problem is bedevilled by two culprits. The first is the error in the measurement of $W(v)$ or $g(v)$ itself. The second one is the error buildup in a digital computer. Whereas the latter can be surmounted to some degree by using as high precision arithmetic as possible, the effect of the former could possibly be reduced by filtering out null functions from $a(u)$. A priori, that cannot be done; hence, the need for algorithm-independent checks.
A further word of caution: not every function of s is a Laplace transform. The class of functions $F(s)$ that are transforms is limited by several conditions of continuity and analyticity dealt with in detail in [Ref. 9, Sect. 63]. Thus $g(v)$ or $g(v)/v$ must be able to satisfy the conditions of inverse transformability for the solution algorithms to be effective.

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