

A CONTINUUM APPROACH TO DETERMINATION OF ELASTIC PROPERTIES OF SHORT FIBRE COMPOSITES

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Short fibre composites with polymer or metal matrices are a class of composites with wide applicability since they provide improvements in properties of neat materials and have the advantage of providing stiffness levels comparable to continuous fibre composites combined with the ease of processing and the ability to be moulded into complex shapes. These composites are handled as bulk suspensions that are moulded into parts by processes such as transfer, compression and injection moulding. As a suspension flows through a die or a mould, the deformation leads to a change of the orientation of the fibres. As the polymer matrix solidifies, these orientations are subsequently frozen in and become a governing factor in determining the properties of the composite. If the fibres are randomly orientated in a region, the properties become isotropic in that region of the composite. If the fibres are preferably orientated, the composite attains the largest deviation from matrix properties in the direction of the main orientation. The orientation distribution in a manufactured part is generally nonuniform and can vary from region to region.

There can be different approaches to the problem of determination of the composite properties, e.g. the elastic stiffness constants, for an arbitrary fibre orientation distribution. One of them, called the micromechanics approach, is to find the overall (effective) composite properties in terms of the phase geometry (fibre aspect ratio, fibre volume fraction, etc.) and the properties of the phases, i.e. their constitutive relations. The micromechanics solutions for unidirectional fibrous microstructure or for a single fibre surrounded by matrix are used together with an orientation averaging procedure to obtain the desired composite properties. A review of the micromechanics approach is given in [1] and [2].

An alternative approach, treated here, is the continuum approach. To model the orientation state of the short fibres, one of the approaches has been to adopt a continuum model and with its help, to ascribe a value of orientation state to every point of the composite. This assumption turns out to be good if length of the fibre is small compared to the scale over which the orientation of fibres and the overall dimensions of the composite change. To establish a way from processing to structure to properties of short fibre composites, one must track the orientation of the fibres during the flow of suspension which produces the composite, to predict the final frozen-in orientation structure. Knowing the orientation structure allows to predict its influence on the mechanical properties. Thus, the orientation of the fibres plays the role of a structural state variable which would permit to describe the nature of orientation at any point of the domain. Hence one must be able to characterize the orientation structure of a part in order to predict it during the flow and to evaluate the mechanical properties of the composite. In the continuum approach a composite is considered to be a homogeneous body with an internal field of distributed entities. These distributed entities could be a reinforcing phase such as the particles of fibres. Such internal field evolves during processing of the composite and is fixed when the manufacturing of a part of the composite has been completed. In a given state of the composite, the internal field is characterized by a set of spatially varying variables called the internal state variables. For short fibre composites we shall use certain statistical descriptors of the fibre orientation state as the internal state variables. We shall then use the well-developed continuum thermodynamics with internal state variables [3] to characterize the thermomechanical response of short fibre composites.

The approach employed here is general enough to characterize the thermomechanical response of short fibre composites from the bulk suspension state to the solidified one as well as under subsequent thermomechanical service loadings. However, the present treatment is limited to determining the elastic properties of a solidified part with given fibre orientation distribution and fibre volume fraction. Our aim is to demonstrate the usefulness of the approach and to illustrate the methodology involved. Extension to more general situations will be done in the future.

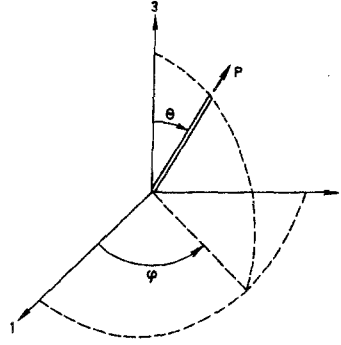


Fig. 1. Coordinate system used to define $\psi(\theta, \varphi)$ and $\psi(\rho)$ in spherical coordinates.

We shall first describe different ways of characterizing the fibre orientation state and of its use in predicting the effective elastic properties of the composite. We shall then apply the continuum approach, regarding the fibre orientation state as an internal one, and derive expressions for all coefficients of the anisotropic stiffness matrix. It will be shown that only nine independent constants are needed to complete determination of the coefficients of the anisotropic stiffness matrix. An experimental procedure for determination of the nine constants will be described.

Characterization of Fibre Orientation. To begin with, we assume that all fibres are rigid axisymmetric particles of uniform length and diameter and that their concentration is uniform throughout the composite. A direction can then be associated along the fibre length using a unit vector \mathbf{p} relative to some reference frame. Furthermore, one can also specify the orientation by the angles (θ, φ) , as shown in Fig. 1. The components of \mathbf{p} are related to θ and φ .

$$p_1 = \sin\theta \cos\varphi, \quad p_2 = \sin\theta \sin\varphi; \quad p_3 = \cos\theta,$$

and are also interrelated, since the vector length is fixed,

$$p_i p_i = 1,$$

where summation over repeated indices is implied. The set of all possible directions of \mathbf{p} corresponds to the unit sphere. For a certain class of moulded parts such as sheet moulded compounds where the fibres are much longer than the mould thickness, most of the fibres will lie in one plane. This makes it possible to describe the orientation of a fibre by one angle φ or a two-dimensional vector \mathbf{p} . The set of all possible directions for planar (i.e., two-dimensional) orientation is a unit circle.

In a short fibre composite, the fibre may be orientated in many different directions. Figure 2 is a redrawn image of a radiograph of a compression moulded plaque from a sheet moulding compound and reflects the complexity of the situation [4]. It is more advantageous to convey the idea of the use of many directions for the description of fibre orientation and call this the fibre orientation state than to apply the description of the direction of each individual fibre.

The most general description of the fibre orientation state is provided by the probability distribution function. If we consider a sufficiently small region of the composite, the distribution of fibre orientations in that region will represent the orientation state. The probability distribution function for orientation, also known as the orientation distribution function $g(\theta, \varphi)$ is defined so that the probability of the presence of fibres lying between angles θ and $\theta + d\theta$, φ and $\varphi + d\varphi$ is given by $g(\theta, \varphi) \sin\theta d\theta d\varphi$. The orientation distribution function must satisfy two physical conditions. First, one end of the fibre should be indistinguishable from the other, and therefore g must be periodic,

$$g(\theta, \varphi) = g(\pi - \theta, \varphi + \pi).$$

Second, every fibre must have some direction, so the integral over all possible directions of the orientation space must be equal to the unity:

$$\int_0^{2\pi} \int_0^\pi g(\theta, \varphi) \sin\theta d\theta d\varphi = 1. \quad (1)$$

This is known as the normalization requirement. If the orientation statistics changes with position, g will be a function of the spatial coordinates x_1, x_2, x_3 in addition to θ and φ .

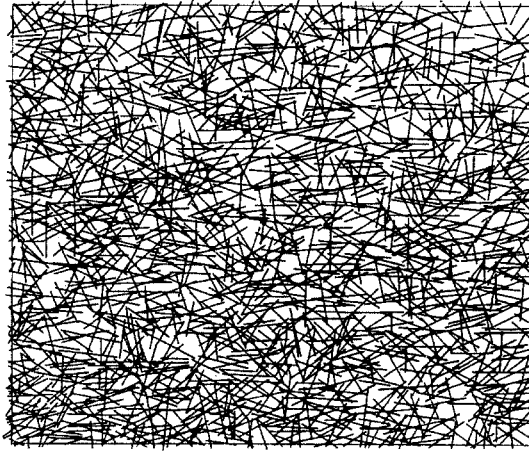


Fig. 2. Computer redrawn plot of fibre from the radiograph of a rectangular plaque made from SMC.

The distribution function can be approximated by measuring the orientation of a large number of fibres selected from a region where the distribution function does not change drastically. The distribution function provides a complete and unambiguous description of the fibre orientation state. The disadvantage of the distribution function as a means for orientation representation is that the calculations of predicting orientation in flowing suspensions become very complicated. Note that g is a function of two variables (either θ and φ or two of the p_i) and can also depend on the position in the part. Recently, one can say that processing this amount of information in a numerical simulation consumes much time, and considerable efforts are necessary. (For details see [5]). A more compact description would be extremely useful.

To provide easily interpretable measures of orientation, a number of orientation parameters have been defined. The most often used is the Hermann orientation parameter [6]. This parameter consists of a scalar which represents the magnitude of orientation in axisymmetric cases. It is not possible to derive such orientation parameters for general cases of fibre orientation nor is it possible to evaluate them from equations of change for flow induced orientation. A description that combines the generality of the distribution function and the compact nature of orientation parameters is a tensor description of orientation [5]. Second and fourth order orientation tensors can be defined as

$$a_{ij} \equiv \langle p_i p_j \rangle ; \quad a_{ijkl} \equiv \langle p_i p_j p_k p_l \rangle ,$$

where the angle brackets, $\langle \rangle$ denote the integral over all \mathbf{p} weighted by $g(\mathbf{p})$. (This notation expresses the average over all possible directions weighted by the orientation distribution function). There are a number of physical interpretations of these tensors. They can be thought of as generalization of orientation parameters, moments of the distribution function or as a series expansion of the distribution function. Advani and Tucker [5] provide a complete review. They have shown that these tensors are free from a priori assumptions about the shape of the distribution function and can be readily transformed from one coordinate frame of reference to another complying with the rules of tensor transformation. The orientation tensors are symmetric, that is,

$$a_{ij} = a_{ji} , \quad a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij} .$$

The normalization condition, Eq. (1), implies that the trace of a_{ij} is unity, i.e.

$$\text{tr } \mathbf{a} = a_{ii} = 1 . \quad (2)$$

The tensor description when replacing the distribution function allows to describe orientation using a fewer number of scalar quantities. For example, at planar orientation only two of the four components are independent. For a three-dimensional case only five of the nine components are independent. Figure 3 shows examples of planar orientation states represented by the distribution function, orientation parameters and orientation tensors. When a_{22} approaches unity all fibres are primarily along x_2 -axis and the distribution function tends to the Dirac delta function around that axis. When the fibres are randomly orientated, a_{11} and a_{22} are equal and if the principal orientation is not along coordinate directions a_{12} is nonzero. It is of interest to mention that orientation parameters (such as Herman parameters) erase vital information, e.g. the direction of the principal orientation if it differs from that of the coordinate.

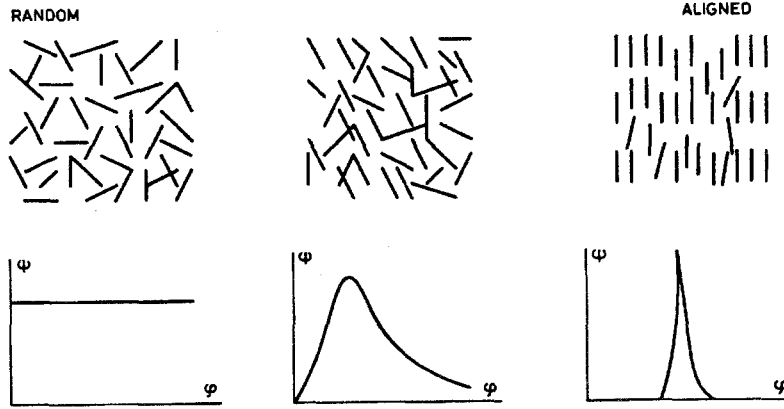


Fig. 3. Different representation of planar fibre orientation. For a - $f = 0$; $a_{ij} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$;
 b - $f = 0.75$; $a_{ij} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.7 \end{bmatrix}$; c - $f = 0.98$; $a_{ij} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.98 \end{bmatrix}$.

We shall now neglect the conditions concerning the geometry of fibres discussed at the beginning of this section. We will allow the fibres to be of any elongated shape (i.e., with the length much greater than the width and the thickness) and of varying length. The unit vector \mathbf{p} is now replaced by a vector $\mathbf{v} = \mathbf{v}\mathbf{p}$, where \mathbf{v} is the volume contained within the fibre surface. We define a second order fibre state tensor as

$$\alpha_{ij} = \langle v_i v_j \rangle / V^2$$

where V is the volume of the representative element around the point for which the fibre state is defined.

The fibre state tensor and the fibre orientation tensor are interrelated by

$$\alpha_{ij} = v_f^2 a_{ij} , \quad (3)$$

where v_f is the volume fraction of fibres in the composite. Also, we have, from Eqs. (2) and (3)

$$\text{tr } \alpha = v_f^2 .$$

Orientation Averaging. If any continuum model of mechanical properties is introduced the two-phase fibre-matrix material is replaced by a homogeneous continuum. A way to calculate the properties of that continuum is averaging the properties of the two phases. When one of the phases (let us say, fibre) introduces anisotropy, the averaging procedure must account for the orientation of that phase. Many micromechanics theories begin by estimating either the properties of a material with unidirectional microstructure or those of a single fibre and nearby matrix. The properties of the composite are then calculated as an average of these unidirectional properties over all possible directions weighted by the orientation distribution function. Such an averaging should be performed over a domain that is both large enough to contain many fibres and small enough compared to the length scale of the global stress or deformation field for the orientation averaging procedure to be valid. Properties calculated this way might include thermal conductivities, elastic stiffness or compliance, viscosity, etc. Micromechanics treatments when performing orientation averaging do not use the complete description of the orientation state. They require only a limited set of even-order orientation tensors up to the order of the property that is being averaged [5].

For example, thermal conductivity is a second order property and one would need orientation description up to only the second order orientation tensors to calculate the average thermal conductivity. Elastic stiffness or compliance are fourth order properties which require description of the orientation state up to the fourth order. This would make the idea of calculation of the fourth order orientation tensors from flow induced orientation equations of change very attractive. This requires solving three times as many equations as for the prediction of second order orientation tensors from processing conditions [5 and 7].

In the present approach orientation averaging of the properties is not required. The average properties are derived from a response function which is assumed to be a function of the fibre orientation state. The fibre orientation state is represented by the orientation tensor of the chosen order. Thus, the choice of the fourth order orientation tensor would provide a higher degree of fibre orientation description than that achieved with the second order tensor. However, a reasonable choice of the orientation tensor must seek a balance between the accuracy and easiness of use. We assume here that the second order orientation tensor

is sufficient to represent the fibre orientation state. Hence, the fibre state tensor a defined above, which contains certain information about the orientation state, is further assumed to be the field quantity for microstructure of short fibre composites.

Mechanical Response. The theory of thermodynamics with internal state variables advanced by Coleman and Gurtin [3] provides a natural framework for describing thermomechanical response of materials with evolving microstructure. The microstructure can generally consist of distributed discrete entities whose characteristic size falls within the range of the so-called microscale. A suitable average description of the microstructural entities in the neighborhood of a general point represents the internal state and the associated statistical descriptors are taken as the internal state variables. Several treatments of material response have been done using the internal state variable concept, see, e.g. [8] for metal plasticity, [9] for spallation of brittle solids, [10] for creep of polycrystalline solids, [11-13] for various aspects of damage of composite materials.

We shall here consider a short fibre composite as a continuum with a microstructure consisting of short fibres. Assuming the fibres to be sufficiently small to be regarded of a microscopic scale, and their length, diameter and concentration to be uniform throughout the composite, we shall take the fibre state, tensor a to represent, with sufficient accuracy, the internal state of the composite.

The Coleman–Gurtin formulation assumes the thermodynamic response to be given by five functions: the Cauchy stress σ_{ij} , the specific Helmholtz free energy Ψ , the specific entropy s , the heat flux q_i , and the internal variable evolution rate, here presented by $\dot{\alpha}_{ij}$. The thermodynamic state is given by the strain $\varepsilon_{ij} = 1/2(v_{i,j} + v_{j,i})$, for small displacement gradients $v_{i,j}$, the absolute temperature T , the temperature gradient $T_{,i}$ and the internal variables α_{ij} .

We shall here restrict the treatment to isothermal response of short fibre composites with a fixed fibre state which does not evolve during loading. A typical situation where this holds is that of a polymer matrix composite in a solidified form. Restricted to this, the set of response functions reduces to

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{kl}, \alpha_{mn}) \quad (4)$$

and

$$\Psi = \Psi(\varepsilon_{kl}, \alpha_{mn}) . \quad (5)$$

The thermodynamic restrictions yield, in addition [3],

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial \varepsilon_{ij}} , \quad (6)$$

where ρ is the mass density.

Equation (6) allows selection of one of the two response functions, (4) and (5), for describing mechanical response. Ψ being a scalar function is a more appropriate choice. We shall choose to expand this function as a polynomial one with respect to its arguments ε_{ij} and α_{ij} . With the assumption that α_{ij} forms an internal field in the matrix which is isotropic the function Ψ can be expressed in terms of the isotropic invariants, i.e. the full orthogonal group [14], given by

$$\text{tr}\varepsilon, \text{tr}\varepsilon^2, \text{tr}\varepsilon^3, \text{tr}\alpha, \text{tr}\alpha^2, \text{tr}\alpha^3 ; \text{tr}\varepsilon\alpha, \text{tr}(\varepsilon^2\alpha), \text{tr}(\varepsilon\alpha^2) , \text{ and } \text{tr}(\varepsilon^2\alpha^2),$$

where the product of two tensors is given by one index contraction, e.g. $(\varepsilon\alpha)_{ij} = \varepsilon_{ik}\alpha_{kj}$ and $(\varepsilon^2)_{ij} = \varepsilon_{ik}\varepsilon_{kj}$.

The most general polynomial form of Ψ is expressed by

$$\rho\Psi = k_1 + k_2(a) + f_1(\varepsilon, \alpha) + O_3(\varepsilon, \alpha), \quad (7)$$

where k_1 is a constant, k_2 is function only of α_{ij} , f_1 is a linear function of ε_{ij} , f_2 is a quadratic function of ε_{ij} and O_3 contains terms of the order three and higher in ε_{ij} .

From Eq. (5) and Eq. (6) we may write the rate equation as

$$\dot{\sigma}_{ij} = \rho \frac{\partial^2 \Psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \rho \frac{\partial^2 \Psi}{\partial \varepsilon_{ij} \partial \alpha_{kl}} \dot{\alpha}_{kl} . \quad (8)$$

The fibre orientation state in the solidified state of a short fibre composite is assumed to remain unchanged during deformation. Thus, the second term in Eq. (8) vanishes, giving

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\varepsilon}_{kl} , \quad (9)$$

where

$$C_{ijkl} = \rho \frac{\partial^2 \Psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} . \quad (10)$$

From Eq. (7) and Eq. (10) we have, neglecting terms in ε_{ij} of the order three and higher, i.e. restricting to small strains,

$$C_{ijkl} = \rho \frac{\partial^2 f_2}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} . \quad (11)$$

The function f_2 is a homogeneous quadratic function of ε_{ij} and contains terms of all orders in α_{ij} . We now restrict our treatment to composites with low concentration of fibres given by $v_f^2 \ll 1$. Most practical short fibre composites have fibre volume fractions satisfying this requirement. The polynomial expansion of f_2 can be restricted to quadratic terms in α_{ij} . The most general form of such polynomial in terms of the isotropic invariants [14] is given by,

$$\begin{aligned} f_2 = & b_1 (\text{tr} \varepsilon)^2 + b_2 \text{tr} \varepsilon^2 + b_3 \text{tr}(\varepsilon^2 \alpha) + b_4 \text{tr} \varepsilon \text{tr}(\varepsilon \alpha) + \\ & + b_5 [\text{tr}(\varepsilon \alpha)]^2 + b_6 \text{tr}(\varepsilon^2 \alpha^2) + b_7 \text{tr} \varepsilon \text{tr}(\varepsilon \alpha)^2 + b_8 \text{tr} \varepsilon^2 \text{tr} \alpha^2 + \\ & + b_9 (\text{tr} \varepsilon)^2 \text{tr} \alpha^2 , \end{aligned} \quad (12)$$

where the coefficients $b_1, b_2, b_3,$ and b_4 may be functions of the fibre volume fraction as they involve only the linear term describing the fibre state while the remaining b_i 's are constants as they are multiplied by the quadratic terms representing the fibre state.

From Eq. (11) and Eq. (12) the coefficients of the elastic stiffness matrix C_{ijkl} can be derived for an arbitrary fibre state given by α_{ij} .

Using the relations:

$$\begin{aligned} \varepsilon_{11} = \varepsilon_1 , \varepsilon_{22} = \varepsilon_2 , \varepsilon_{33} = \varepsilon_3 , 2\varepsilon_{23} = \varepsilon_4 , 2\varepsilon_{13} = \varepsilon_5 , 2\varepsilon_{12} = \varepsilon_6 , \\ \text{and } \alpha_{11} = \alpha_1 , \alpha_{22} = \alpha_2 , \alpha_{33} = \alpha_3 , \alpha_{23} = \alpha_4 , \alpha_{13} = \alpha_5 , \alpha_{12} = \alpha_6 , \end{aligned}$$

and the elastic stiffness matrix can be expressed in compact notation as

$$C_{pq} = \frac{\partial^2 f_2}{\partial \varepsilon_p \partial \varepsilon_q} ; \quad p, q = 1, 2, \dots, 6. \quad (13)$$

The coefficients C_{pq} are given in the Appendix. Note that for a given fibre state, the 21 different components of the anisotropic and symmetric matrix C_{pq} are given by nine unknown coefficients b_1 - b_9 .

Transforming C_{pq} from arbitrary Cartesian coordinate axes to those formed by the principal directions of α_{ij} , we have, in the new coordinate system, $\alpha_4 = \alpha_5 = \alpha_6 = 0$ and

$$C_{pq} = \left\{ \begin{array}{cccccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ \text{sym.} & & & & & C_{66} \end{array} \right\} .$$

Thus, the elastic stiffness matrix is orthotropic with nine independent constants determined by b_1 - b_9 and given by

$$\begin{aligned} C_{pq} = & 2b_1 + 2b_2 \delta_{pq} + (\alpha_p + \alpha_q) (b_3 \delta_{pq} + b_4) + \\ & + 2\alpha_p \alpha_q (b_5 + \delta_{pq} b_6) + (\alpha_p^2 + \alpha_q^2) b_7 + 2 (b_8 \delta_{pq} + b_9) r_a \end{aligned} \quad (14)$$

for $p, q = 1, 2$ and 3 , and where $\delta_{pq} = 1$ for $p = q$ and $\delta_{pq} = 0$ for $p \neq q$. Here $r_a = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$. For $p = 4, 5$ and 6 the coefficients are

$$C_{pp} = b_2 + r_p b_3 + s_p b_6 + b_8 r_a , \quad (15)$$

where no summation is implied on the repeated index and

$$r_p = \begin{cases} \frac{1}{2} (\alpha_2 + \alpha_3) & \text{for } p = 4 ; \\ \frac{1}{2} (\alpha_1 + \alpha_3) & \text{for } p = 5 ; \\ \frac{1}{2} (\alpha_1 + \alpha_2) & \text{for } p = 6 ; \end{cases} \quad (16)$$

and

$$s_p = \begin{cases} \frac{1}{2} (\alpha_2^2 + \alpha_3^2) & \text{for } p = 4 ; \\ \frac{1}{2} (\alpha_1^2 + \alpha_3^2) & \text{for } p = 5 ; \\ \frac{1}{2} (\alpha_1^2 + \alpha_2^2) & \text{for } p = 6 ; \end{cases} \quad (17)$$

It is now evident that once the coefficients of the orthotropic matrix C_{pq} are determined in the principal directions of α_{ij} , they can be transformed to any set of Cartesian axes by using the relations given in the Appendix.

The coefficients b_1 and b_2 associated with the quadratic functions of ϵ only in Eq. (9) may be written as

$$\begin{aligned} b_1 &= b_1^0 + c_1 [(\text{tr } \alpha + (\text{tr } \alpha)^2 + \dots)], \\ b_2 &= b_2^0 + c_2 [(\text{tr } \alpha + (\text{tr } \alpha)^2 + \dots)]. \end{aligned}$$

Note that b_1 and b_2 are still independent of fibre orientation state as $\text{tr } \alpha = v_f^2$. Hence, C_{pq} from (12) and (13) may be expressed as

$$C_{pq} = C_{pq}^0 = C_{pq}(\alpha),$$

where C_{pq}^0 is independent of the fibres. Thus, for $\alpha \equiv 0$, i.e. no fibres, we have

$$C_{pq} = C_{pq}^0.$$

The stiffness matrix C_{pq}^0 is isotropic and is determined by the two constants b_1^0 and b_2^0 . These constants are related to the engineering moduli E_m and ν_m , i.e. the Young's modulus and Poisson's ratio of the unreinforced matrix material, by the following equations

$$b_1^0 = \frac{E_m \nu_m}{2 (1 + \nu_m) (1 - 2\nu_m)}$$

and

$$b_2^0 = \frac{E_m}{2 (1 + \nu_m)}.$$

Material Constants. The theory described above shows that an unreinforced matrix material which is isotropic and requires two material constants, e.g., E_m and ν_m , for description of its elastic response, when reinforced by short fibres of arbitrary distribution, becomes orthotropic along the principal directions of all and therefore requires nine material constants for description of its elastic response. It is natural to expect that these constants, given here by b_1 - b_9 , can be determined experimentally in the same way as b_1^0 and b_2^0 , or equivalently, E_m and ν_m , determined, e.g., by mechanical tests. However, in a general case, the fibre orientation state will have a spatial distribution and consequently, the principal directions of all will also vary spatially. It is therefore desirable first to determine the constants b_1 - b_9 , by testing a specimen material of orthotropic symmetry having the same fibre volume fraction as that of the manufactured part of interest and then calculate its properties by using Eqs. of the Appendix. This procedure would, however, be feasible if a test material of orthotropic symmetry can be manufactured. We shall therefore describe a scheme for determining b_1 - b_5 using such test material with practically achievable fibre orientations.

Test 1. Unidirectional Composite. It may be possible using compression mould to manufacture a specific test material such as the long discontinuous fibre (LDF) material [15] which consists of fully impregnated aligned fibres in a ply form; so a test specimen with all fibres aligned along the same direction is obtained. The fibre state of such a unidirectional composite

is given by $\alpha_p = (v_f^2, 0, 0, 0, 0, 0)$ where the x_1 -axis is aligned with the fibre axis. Inserting this α_p in Eqs. (14)-(17) we obtain the following stiffness coefficients C_{pq}^1

$$\begin{aligned} C_{11}^1 &= 2 (b_1 + b_2) + 2v_f^2 (b_3 + b_4) + 2v_f^4 (b_5 + b_6 + b_7 + b_8 + b_9) ; \\ C_{12}^1 &= 2b_1 + v_f^2 b_4 + v_f^4 (b_7 + 2b_9) ; \\ C_{22}^1 &= 2 (b_1 + b_2) + 2v_f^4 (b_8 + b_9) ; \\ C_{23}^1 &= 2b_1 + 2v_f^4 b_9 \end{aligned} \quad (18)$$

and

$$C_{55}^1 = b_2 + \frac{v_f^2}{2} b_3 + \frac{v_f^4}{2} (b_6 + 2b_8) .$$

Additionally, the remaining nonzero coefficients are

$$C_{12}^1 = C_{13}^1, \quad C_{33}^1 = C_{22}^1, \quad C_{44}^1 = 1/2 (C_{22}^1 - C_{23}^1) \quad \text{and} \quad C_{66}^1 = C_{55}^1.$$

The composite is transversely isotropic, i.e. isotropic in the plane normal to the fibre axis. The five independent constants given in (18) can be easily determined from well established mechanical tests or by ultrasonic techniques.

Test 2. 2D Random Composite. A short fibre composite with fibres distributed randomly in one plane has the fibre state given by $\alpha_p = (\frac{v_f^2}{2}, \frac{v_f^2}{2}, 0, 0, 0, 0)$ with x_1 - x_2 plane being the fibres plane. It is possible practically to create such a specimen via compression moulding the sheet moulding compound with a very little deformation during the moulding process to prevent preferential orientation of the fibres [16]. The stiffness coefficients C_{pq}^2 are given by

$$\begin{aligned} C_{11}^2 &= 2 (b_1 + b_2) + v_f^2 (b_3 + b_4) + \frac{v_f^4}{2} (b_5 + b_6 + b_7 + 2b_8 + 2b_9) ; \\ C_{12}^2 &= 2b_1 + v_f^2 b_4 + \frac{v_f^4}{2} (b_5 + b_7 + 2b_9) ; \\ C_{13}^2 &= 2b_1 + \frac{v_f^2}{2} b_4 + \frac{v_f^4}{4} (b_7 + 4b_9) ; \\ C_{33}^2 &= 2 (b_1 + b_2) + v_f^4 (b_8 + b_9) ; \quad C_{44}^2 = b_2 + \frac{v_f^2}{4} b_3 + \frac{v_f^4}{8} (b_6 + 4b_8) . \end{aligned} \quad (19)$$

The remaining nonzero coefficients are

$$C_{22}^2 = C_{11}^2 ; \quad C_{23}^2 = C_{13}^2 ; \quad C_{55}^2 = C_{44}^2 ; \quad C_{66}^2 = 1/2 (C_{11}^2 - C_{12}^2) .$$

The composite is isotropic in the plane of fibres. The five independent constants for this composite, given by Eq. (19), can also be easily determined. The Eqs. (18) and (19) provide ten simultaneous equations to find nine unknowns b_1 - b_9 . The matrix of coefficients is nonsingular and hence any nine relations from the set of simultaneous equations may be used to calculate b_1 - b_9 . We select the first of the nine relations. The solution for nine unknown material constants is as follows:

$$\begin{aligned} b_1 &= \frac{-4C_{44}^1 + 4C_{33}^2 - 2C_{23}^1 - C_{22}^1 + 8C_{13}^2 - 4C_{12}^2 - 2C_{12}^1 + C_{11}^1}{8} ; \\ b_2 &= \frac{-4C_{44}^1 - 4C_{33}^2 - 2C_{23}^1 + 3C_{22}^1 + 8C_{13}^2 - 4C_{12}^2 - 2C_{12}^1 + C_{11}^1}{8} ; \\ b_3 &= -\frac{2C_{44}^1 + 2C_{12}^2 - 2C_{11}^2}{v_f^2} ; \quad b_4 = -\frac{-4C_{44}^1 + 4C_{33}^2 - C_{22}^1 - 4C_{12}^2 + C_{11}^1}{2v_f^2} ; \\ b_5 &= \frac{-4C_{44}^1 + 4C_{33}^2 + 2C_{23}^1 - C_{22}^1 - 8C_{13}^2 + 4C_{12}^2 - 2C_{12}^1 + C_{11}^1}{4v_f^4} ; \\ b_6 &= \frac{12C_{44}^1 - 4C_{33}^2 + 2C_{23}^1 - C_{22}^1 + 8C_{13}^2 + 4C_{12}^2 - 2C_{12}^1 - 8C_{11}^2 + C_{11}^1}{4v_f^4} ; \end{aligned}$$

$$b_7 = \frac{-4C_{44}^1 + 4C_{33}^2 - 2C_{23}^1 - C_{22}^1 - 4C_{12}^2 + 2C_{12}^1 + C_{11}^1}{2v_f^4};$$

$$b_8 = \frac{-4C_{44}^1 - 4C_{33}^2 - 6C_{23}^1 + 7C_{22}^1 + 8C_{13}^2 - 4C_{12}^2 - 2C_{12}^1 + C_{11}^1}{8v_f^4};$$

$$b_9 = \frac{-4C_{44}^1 + 4C_{33}^2 - 6C_{23}^1 - C_{22}^1 + 8C_{13}^2 - 4C_{12}^2 - 2C_{12}^1 + C_{11}^1}{8v_f^4}.$$

Conclusions. We have developed a method to calculate the elastic properties of a short fibre composite with an arbitrary orientation distribution of fibres. The approach utilizes a continuum concept in which a composite is regarded as a homogeneous body with an internal field consisting of distributed fibres. The fibre state, defined in terms of a second order tensorial descriptor of the fibre orientation state, is used as the internal state variable. Using concepts of thermodynamics with internal variables and invariant theory, the elastic stiffness tensor has been derived. The stiffness matrix, in general, has 21 constants and it is shown here that these constants are related to nine independent material constants. The material constants are the coefficients of the orthotropic stiffness matrix aligned with the coordinate axes formed by the principal directions of the fibre orientation tensor. A procedure has been developed by which, through familiar mechanical test on specimen materials of aligned and random in-plane fibre orientations, the material constants can be determined. Of course, it would be possible to estimate the material constants from micromechanics solutions for effective properties of aligned and random short fibre composites. However, the micromechanics solutions must be verified first by experimental data which may as well be used directly in calculating the material constants appearing in the formulation developed here.

Appendix. The coefficients of the elastic stiffness matrix C_{pq} are derived using (13) with f_2 given by (12). The individual functions in (12) are written in a compact notation as follows:

$$\begin{aligned} \text{tr} \boldsymbol{\varepsilon} &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3; \\ \text{tr} \boldsymbol{\varepsilon}^2 &= \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 1/2 (\varepsilon_4^2 + \varepsilon_5^2 + \varepsilon_6^2); \\ \text{tr}(\boldsymbol{\varepsilon} \mathbf{a}) &= a_1 \varepsilon_1 + a_2 \varepsilon_2 + a_3 \varepsilon_3 + a_4 \varepsilon_4 + a_5 \varepsilon_5 + a_6 \varepsilon_6; \\ \text{tr}(\boldsymbol{\varepsilon} \mathbf{a}^2) &= (a_1^2 + a_2^2 + a_6^2) \varepsilon_1 + (a_2^2 + a_4^2 + a_6^2) \varepsilon_2 + \\ &+ (a_3^2 + a_4^2 + a_5^2) \varepsilon_3 + (a_2 a_4 + a_5 a_6 + a_3 a_4) \varepsilon_4 + \\ &+ (a_1 a_5 + a_3 a_5 + a_4 a_6) \varepsilon_5 + (a_1 a_6 + a_4 a_5 + a_2 a_6) \varepsilon_6; \\ \text{tr}(\boldsymbol{\varepsilon}_2 \mathbf{a}) &+ a_1 \varepsilon_1^2 + a_2 \varepsilon_2^2 + a_3 \varepsilon_3^2 + 1/4 [(a_2 + a_3) \varepsilon_4^2 + (a_1 + a_3) \varepsilon_5^2 + \\ &+ (a_1 + a_2) \varepsilon_6^2] + 1/2 (a_4 \varepsilon_5 \varepsilon_6 + a_5 \varepsilon_4 \varepsilon_6 + a_6 \varepsilon_4 \varepsilon_5) + \varepsilon_1 (a_5 \varepsilon_5 + a_6 \varepsilon_6) + \\ &+ \varepsilon_2 (a_4 \varepsilon_4 + a_6 \varepsilon_6) + \varepsilon_3 (a_4 \varepsilon_4 + a_5 \varepsilon_5); \\ \text{tr}(\boldsymbol{\varepsilon}^2 \mathbf{a}^2) &= (a_1^2 + a_5^2 + a_6^2) \varepsilon_1^2 + (a_2^2 + a_4^2 + a_6^2) \varepsilon_2^2 + (a_3^2 + a_4^2 + a_5^2) \varepsilon_3^2 + \\ &+ 1/4 (a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2) \varepsilon_4^2 + 1/4 (a_1^2 + a_3^2 + a_4^2 + 2a_5^2 + a_6^2) \varepsilon_5^2 + \\ &+ 1/4 (a_1^2 + a_2^2 + a_4^2 + a_5^2 + 2a_6^2) \varepsilon_6^2 + (a_1 a_5 + a_4 a_6 + a_3 a_5) \varepsilon_1 \varepsilon_5 + \\ &+ (a_1 a_6 + a_2 a_6 + a_4 a_5) \varepsilon_1 \varepsilon_6 + (a_2 a_4 + a_5 a_6 + a_3 a_4) \varepsilon_2 \varepsilon_4 + \\ &+ (a_2 a_6 + a_4 a_5 + a_1 a_6) \varepsilon_2 \varepsilon_6 + (a_3 a_4 + a_5 a_6 + a_2 a_4) \varepsilon_3 \varepsilon_4 + \\ &+ (a_3 a_5 + a_4 a_6 + a_1 a_5) \varepsilon_3 \varepsilon_5 + 1/2 (a_1 a_6 + a_2 a_6 + a_4 a_5) \varepsilon_4 \varepsilon_5 + \\ &+ 1/2 (a_1 a_5 + a_4 a_6 + a_3 a_5) \varepsilon_4 \varepsilon_6 + 1/2 (a_5 a_6 + a_2 a_4 + a_3 a_4) \varepsilon_5 \varepsilon_6. \end{aligned}$$

The individual coefficients acquire the following form

$$\begin{aligned} C_{11} &= 2(b_1 + b_2) + 2\alpha_1(b_3 + b_4) + 2\alpha_1^2 b_5 + 2\{\alpha_1^2 + \alpha_5^2 + \alpha_6^2\}(b_6 + b_7) + \\ &+ 2\{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2\}(b_8 + b_9); \\ C_{12}=C_{21} &= 2b_1 + \{\alpha_1 + \alpha_3\} b_4 + 2\alpha_1 \alpha_2 b_5 + \{\alpha_1^2 + \alpha_2^2 + \alpha_4^2 + \alpha_5^2 + 2\alpha_6^2\} b_7 + \\ &+ 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2) b_9; \\ C_{13}=C_{31} &= 2b_1 + \{\alpha_1 + \alpha_3\} b_4 + 2\alpha_1 \alpha_3 b_5 + \{\alpha_1^2 + \alpha_3^2 + \alpha_4^2 + 2\alpha_5^2 + \alpha_6^2\} b_7 + \\ &+ 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2) b_9; \\ C_{14} &= C_{41} = \alpha_4 b_4 + 2\alpha_1 \alpha_4 b_5 + (\alpha_2 \alpha_4 + \alpha_3 \alpha_4 + \alpha_5 \alpha_6) b_7; \\ C_{15} &= C_{51} = \alpha_5 (b_3 + b_4) + 2\alpha_1 \alpha_5 b_5 + (\alpha_1 \alpha_5 + \alpha_4 \alpha_6 + \alpha_3 \alpha_5) (b_6 + b_7); \end{aligned}$$

$$\begin{aligned}
C_{16} &= C_{61} = \alpha_6 (b_3 + b_4) + 2\alpha_1\alpha_6b_5 + (\alpha_1\alpha_6 + \alpha_2\alpha_6 + \alpha_4\alpha_6) (b_6 + b_7) ; \\
C_{22} &= 2(b_1 + b_2) + 2\alpha_2(b_3 + b_4) + 2\alpha_3^2b_5 + 2(\alpha_2^2 + \alpha_4^2 + \alpha_6^2) (b_6 + b_7) + \\
&\quad + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2) (b_8 + b_9) ; \\
C_{23} &= C_{32} = 2b_1 + (\alpha_2 + \alpha_3) b_4 + 2\alpha_2\alpha_3b_5 + 2(\alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + \alpha_5^2 + \alpha_6^2) b_7 + \\
&\quad + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2) b_9 ; \\
C_{24} &= C_{42} = \alpha_4 (b_3 + b_4) + \alpha_2\alpha_6b_5 + (\alpha_2\alpha_4 + \alpha_3\alpha_4 + \alpha_5\alpha_6) (b_6 + b_7) ; \\
C_{25} &= C_{52} = \alpha_5b_4 + 2\alpha_2\alpha_6b_5 + (\alpha_1\alpha_5 + \alpha_4\alpha_6 + \alpha_3\alpha_5) b_7 ; \\
C_{26} &= C_{62} = \alpha_6 (b_3 + b_4) + 2\alpha_2\alpha_6b_5 + (\alpha_2\alpha_6 + \alpha_4\alpha_6 + \alpha_1\alpha_6) (b_6 + b_7) ; \\
C_{33} &= 2(b_1 + b_2) + 2\alpha_3(b_3 + b_4) + 2\alpha_3^2b_5 + 2(\alpha_3^2 + \alpha_4^2 + \alpha_5^2) (b_6 + b_7) + \\
&\quad + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2) (b_8 + b_9) ; \\
C_{34} &= C_{43} = \alpha_4 (b_3 + b_4) + 2\alpha_3\alpha_4b_5 + (\alpha_3\alpha_4 + \alpha_5\alpha_6 + \alpha_2\alpha_4) (b_6 + b_7) ; \\
C_{35} &= C_{53} = \alpha_5 (b_3 + b_4) + 2\alpha_3\alpha_5b_5 + (\alpha_1\alpha_5 + \alpha_4\alpha_6 + \alpha_3\alpha_5) (b_6 + b_7) ; \\
C_{36} &= C_{63} = \alpha_6b_4 + 2\alpha_3\alpha_6b_5 + (\alpha_1\alpha_6 + \alpha_4\alpha_6 + \alpha_2\alpha_6) b_7 ; \\
C_{45} &= C_{54} = 1/2\alpha_6b_3 + 2\alpha_4\alpha_5b_5 + 1/2(\alpha_1\alpha_6 + \alpha_2\alpha_6 + \alpha_4\alpha_6) b_8 ; \\
C_{46} &= C_{64} = 1/2\alpha_4b_3 + 2\alpha_5\alpha_6b_5 + 1/2(\alpha_2\alpha_4 + \alpha_5\alpha_6 + \alpha_3\alpha_4) b_8 ; \\
C_{55} &= b_2 + 1/2(\alpha_1 + \alpha_3) b_3 + 2\alpha_5^2b_5 + 1/2(\alpha_1^2 + \alpha_3^2 + 2\alpha_4^2 + \alpha_5^2 + \alpha_6^2) b_6 + \\
&\quad + (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2) b_8 ; \\
C_{56} &= C_{65} = 1/2\alpha_5b_3 + 2\alpha_4\alpha_6b_5 + 1/2(\alpha_4\alpha_6 + \alpha_1\alpha_5 + \alpha_3\alpha_5) b_6 ; \\
C_{66} &= b_2 + 1/2(\alpha_1 + \alpha_2) b_3 + 2\alpha_6^2b_5 + 1/2(\alpha_1^2 + \alpha_2^2 + \alpha_4^2 + \alpha_5^2 + 2\alpha_6^2) b_6 + \\
&\quad + (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2) b_8 .
\end{aligned}$$

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