Appl. Phys. B 45, 129–135 (1988)

Applied Physics B Photophysics and Laser Chemistry

Transient Effects in Four-Wave Mixing in Photorefractive Passive Phase Conjugate Mirrors

V. Kalinin* and L. Solymar

Holography Group, Department of Engineering Science, University of Oxford, Oxford, UK

Received 6 August 1987/Accepted 3 December 1987

Abstract. Partial differential equations are set up and solved partly analytically, partly numerically for the spatial and temporal variation of the amplitudes and phases of the four beams participating in four-wave mixing in a linear Passive Phase Conjugate Mirror. The instantaneous frequency of the resonating beam is also determined. It is shown that the duration and main features of the transient regime are determined not only by the properties of the photorefractive crystal but also by the external optical circuit, in particular by the quality factor and detuning of the resonator.

PACS: 42.65, 42.80

There has been considerable interest in the operation of Passive Phase Conjugate Mirrors (PPCM) because they do not require an additional source of pump beams. In particular, as it has been pointed out in [1, 2], the application of PPCMs to laser intercavity distortion correction appears to have good prospects.

The theory of linear PPCMs in a lossless photorefractive crystal in the transmission geometry under steady-state conditions has been formulated in [2, 3]. Also for the steady-state, numerical solutions which include the effect of absorption and are valid both for transmission and reflection gratings, have been reported [4].

The aim of this paper is to extend the calculations in [4] to the transient regime in a linear PPCM.

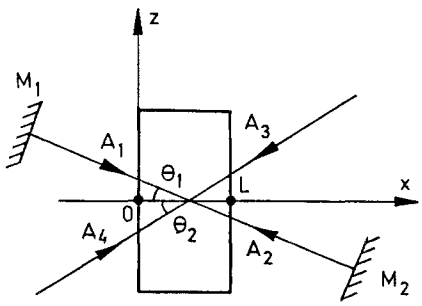

Fig. 1. Geometry of a linear PPCM

1. Theoretical Model

1.1. The Field Equations

Let us consider the phase conjugate mirror shown schematically in Fig. 1. It consists of a photorefractive crystal inside a Fabry-Perot cavity with (amplitude) mirror reflectivities of M_1 and M_2. We shall assume that the crystal is cut in such a way that only one transmission grating may exist. This grating will emerge as the consequence of the interaction between beam 4, the signal beam, and beam 1, a "noise" beam which is present in the resonator due to the scattering of the signal beam by various kinds of nonuniformities. As a result of photorefractive gain beams 1 and 2 will grow, the grating will become stronger and beam 3, the phase conjugate beam will appear. We shall denote the frequency of the input signal beam by ω_{in}, the frequency of the oscillating beam by ω_2 and then the frequency of the phase conjugate beam will be $2\omega_2 - \omega_{in}$.

* On leave of absence from Moscow Power Engineering Institute, SU-105835 Moscow, Krasnokazarmennaia, 14, USSR

Since the time constant associated with the emergence of the grating is much longer than the transit time of light through the crystal we can describe the dependence of the beam amplitudes upon x and t, the space and time coordinate, using conventional coupled wave equations [3, 4] which in our case can be written in the following form

$$\frac{\partial \bar{A}}{\partial \xi} = \hat{B}\bar{A}, \tag{1}$$

where

$$\bar{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} -\alpha & 0 & 0 & \gamma^*/\cos\theta \\ 0 & \alpha & \gamma/\cos\theta & 0 \\ 0 & -\gamma^*/\cos\theta & \alpha & 0 \\ -\gamma/\cos\theta & 0 & 0 & -\alpha \end{bmatrix}. \tag{2}$$

A_i is the complex, slowly varying (both in space and time) amplitude of the ith beam. The refractive index distribution may be written as

$$n(\bar{r}) = n_0\{1 - \mathrm{j}\gamma \exp j(\bar{K}_g \cdot \bar{r} + \Delta\omega t) + \text{c.c.}\}, \tag{3}$$

where n_0 is the average refractive index, $\gamma = \mathrm{j}\eta e^{\mathrm{j}\psi}$, η and ψ represent an amplitude and a phase, $\bar{K}_g$ is the grating vector, $\bar{r}$ is a radius vector, $\Delta\omega = \omega_2 - \omega_{in}$, $\xi = kx$, $\alpha = \tilde{\alpha}/k$, k is the wavenumber inside the crystal, $\tilde{\alpha}$ is the absorption coefficient and θ_1 and θ_2 are the angle between the x axis and beams 4 and 1 respectively (Fig. 1). Note that in deriving (1) and (2) we have used the usual approximations permissible for slowly varying amplitudes but neglected optical activity, birefringence and higher order diffracted beams.

The boundary conditions for the differential equations can be written as

$$\begin{aligned} &A_4(0,t) = h(t), \qquad A_3(L,t) = 0, \\ &A_1(0,t) = A_2(0,t)M_1, \quad A_2(L,t) = M_2 e^{\mathrm{j}2kl}, \end{aligned} \tag{4}$$

where $h(t)$ is the step function, L is the normalised length of the crystal and l is the optical length of the resonator.

We shall take into account only the longitudinal modes of the resonator in which case

$$2kl = 2\pi[m + (\omega_2 - \omega_m)/\Delta\omega_m], \tag{5}$$

where ω_m is the frequency of the mth mode of the resonator in the absence of the nonlinear interaction, and $\Delta\omega_m$ is the frequency spacing between the modes. It has been shown [2] that for photorefractive crystals the oscillation frequency detuning, $\Delta\omega \ll \Delta\omega_m$. We may therefore express (4) as

$$2kl \approx 2\pi(m + g_{in}), \tag{6}$$

where $g_{in} = (\omega_{in} - \omega_m)/\Delta\omega_m$. Thus the boundary conditions may be taken as fixed although the frequency of oscillations is not known in advance and varies with time. We may further assume ω_2 to be fixed which means that the variations of instantaneous frequency will be included in the phases of the beams.

1.2. The Materials Equations

Having described the effect of the grating upon the propagation of the electromagnetic waves we shall now consider the response of the crystal to the input of the waves. We shall use the equations presented in [5, 6] under the following approximation.

The total intensity of all four beams, $I_0 = \Sigma I_i = \Sigma|A_i|^2$, is small enough so that the average free electron concentration n_e is much less than the concentration of acceptors N_A, and the electron recombination time τ_e is much less than the dielectric relaxation time $\tau_d = \varepsilon_0\varepsilon_r/e\mu n_e$ (ε_0 and ε_r are the free space permittivity and dielectric constant respectively, e is the charge of the electron and μ is the mobility). In this case we can assume that n_e reaches its equilibrium value instantaneously.

With the aid of the above approximations and taking into account the fringe movement resulting from the fixed detuning we obtain the differential equation

$$\frac{\partial \gamma}{\partial t'} + (\mathrm{j}\delta + a)\gamma = ci, \tag{7}$$

where

$$i = (A_4A_1^* + A_2A_3^*)/I_0, \qquad t' = t/t_0,$$

$$t_0 = \tau_e(N_A I_0/I_{00}n_e),$$

I_{00} is the input intensity in the signal beam, $\delta = \Delta\omega t_0$.

$$a = I_0\frac{E_D + E_q + \mathrm{j}E_0}{E_M + E_D + \mathrm{j}E_0}, \quad c = \frac{r_{eff}n_0^2}{4} I_0 \frac{E_q(E_D + \mathrm{j}E_0)}{E_M + E_D + \mathrm{j}E_0},$$

$$E_q = \frac{eN_A\Lambda}{2\pi\varepsilon_r\varepsilon_0}, \quad E_M = \frac{\Lambda}{2\pi\mu\tau_e}, \quad E_D = \frac{2\pi k_B T}{e\Lambda}. \tag{8}$$

Here r_{eff} is the relevant electro-optic coefficient, k_B is Boltzmann's constant, T is the absolute temperature, E_0 is the electric field applied in the direction of $\bar{K}_g$, and Λ is the grating period. It is worth noting that t_0 is independent of space and time but I_0 may vary due to absorption and therefore n_e and τ_d may also be functions of space and time.

In order to have a complete set of equations describing the transient regime in a PPCM it is necessary to add some initial condition which will be discussed in the next section.

2. The Start of the Transients

The unknowns in our equations are the complex functions $A_i(x,t)$ ($i=1$ to 4) and ω_2 the variable frequency of oscillations. It is however not difficult to prove a property of (1–8) which enables us to find the frequency of the oscillations after the solution has been obtained.

Let us assume that $A_i(x,t)$ are the solution for a given δ (i.e. fixed ω_2). It may be shown then that

$$A'_{1,2}=A_{1,2}e^{-j\delta t'}, \quad A'_3=A_3e^{-j\delta t'}, \quad A'_4=A_4 \tag{9}$$

are also a solution of (1) and of the differential equation

$$\frac{\partial\gamma}{\partial t'}+a\gamma=c(A'_4A'^*_1+A'_2A'^*_2)/I_0\,. \tag{10}$$

It may now be seen by comparing (7) and (10) that we may choose arbitrary fixed detuning δ (i.e. ω_2) in (7) and the actual instantaneous frequency of oscillations may be found from the slowly varying phase of beam 1,

$$\omega(t)=\omega_2+\frac{\partial\psi_1}{\partial t}. \tag{11}$$

Looking at Eqs. (1–7) it seems fairly obvious that an exact analytic solution is not possible. We may however obtain analytic results for the start of the transient process when $|a|t'\ll 1$ and $A_3\ll A_{1,2}\ll A_4$. So we shall assume that $A_3=0$, $A_4=1$ (since the photorefractive effect depends only on the depth of modulation it is sufficient to use relative units). The amplitude of beam 1 (a noise beam initially) will grow due to two-wave interaction with the input signal beam 4. The amplitude of beam 2 will be determined by the amplitude of beam 1 at $\xi=L$. Beam 2 will propagate in the crystal without any interaction (there is no significant grating as yet) hence its amplitude will remain constant provided absorption in the crystal is neglected which, for simplicity, we do here. Equations (1, 4, 10) then reduce to

$$\frac{\partial\gamma}{\partial t'}=cA_1^*, \tag{12}$$

$$\frac{\partial A_1^*}{\partial\xi}=\frac{\gamma}{\cos\theta}, \tag{13}$$

$$A_1^*(0,t)=\tilde{M}A_1^*(L,t), \tag{14}$$

where $\tilde{M}=M_1M_2\exp(j2\pi g_{in})$. It is worth noting that (12–14) also describe the build-up of oscillations in a ring resonator with photorefractive gain [2]. The initial condition for A_1^*, compatible with (13) and (14) may be written as

$$A_1^*(\xi,0)=\beta_n\left(\xi+\frac{L\tilde{M}}{1-\tilde{M}}\right)\cos\theta, \tag{15}$$

where β_n is a small constant representing a small, uniform noise grating due to scattering of the input signal beam.

Since we are interested in the variation of A_1 in the vicinity of $t'=0$, we can assume that A_1 is constant in (12). The solution of the differential equations (12) and (13) satisfying the additional conditions represented by (14) and (15) may then be obtained in the relatively simple form

$$A_1^*(\xi,t)=\frac{\beta\cos\theta}{ct'}\left[\frac{(1-\tilde{M})\exp(ct'\xi/\cos\theta)}{1-\tilde{M}\exp(ct'L/\cos\theta}-1\right]. \tag{16}$$

For $ct'L/\cos\theta\ll 1$ and assuming that $E_0\gg E_D$, E_M and $\cos\theta\approx 1$ we can further simplify (16) and obtain for the intensity and phase of the oscillations

$$I_1(L,t)=|A_1(L,t)|^2\approx\frac{I_{10}}{1+(2\pi Qg_{in})^2}\times\left[1+\frac{2Q-(2\pi Qg_{in})^2}{1+(2\pi Qg_{in})^2}\,\tau\right], \tag{17}$$

$$\psi_1(L,t)=-2\pi Qg_{in}(1+Q\tau), \tag{18}$$

where $Q=M_1M_2/(1-M_1M_2)$ is, in the absence of absorption and diffraction losses, proportional to the quality factor of the resonator, and

$$\tau=\frac{r_{eff}n_0^2}{4}E_qLt'; \quad I_{10}=\left(\frac{\beta_nL}{1-M_1M_2}\right)^2. \tag{19}$$

We can now draw some conclusions with the aid of (17) and (18) about the character of the transient regime. First it may be noticed that if $g_{in}=0$ we have $\psi=0$ i.e. there is no change with time in the value of the instantaneous frequency. It may further be seen that a decrease in Q leads to (i) the slowing down of the growth of oscillations, (ii) less influence of g_{in} on the initial "noise" oscillation intensity and on the rate of growth, and (iii) a decrease in the initial oscillation frequency detuning whose sign and amplitude depends of course on g_{in}.

A further point of interest is that under our assumptions this initial stage of the transient process is independent of the applied electric field, E_0 and depends only on the electro-optic properties of the crystal and on its length.

The analytic solution presented in this section is valid only at the start of the transients. If we wish to investigate the emergence of the phase conjugate beam

and the transition to the equilibrium regime we have to resort to a numerical solution which will be done in the next section.

3. Numerical Algorithm

First we wish to note that the natural choice for the oscillation detuning δ is its value at equilibrium which can be determined by the method represented in [4]. In this case the phase of the oscillation ψ_1 will tend to a constant value as t' increases. If the equilibrium value of the detuning is chosen with some error then ψ_1 will vary linearly with time in the equilibrium regime.

The spatial distribution of the unknown functions $\bar{A}^{(n+1)}=\bar{A}(\xi, t'_{n+1})$ at $t'=t'_n+\Delta t'$ (where $\Delta t'$ is the time step) can be obtained from the solution of the linear differential equations (1) and the boundary conditions (4) where the coupling constant, $\gamma=\gamma^{(n)}$ is taken as its value at $t'=t'_n$. We solved this system numerically using standard finite difference method with deferred correction. The coupling constant for $t'=t'_{n+1}$ may then be obtained by solving (6) with the aid of the third order Runge-Kutta method giving

$$\gamma^{(n+1)}=\gamma^{(n)}+\Delta t\Big[C^{\left(n+\frac{1}{2}\right)}i^{\left(n+\frac{1}{2}\right)} -(a^{\left(n+\frac{1}{2}\right)}+\mathrm{j}\delta)(\gamma^{(n)}+\tfrac{1}{2}K^{(n)}\Delta t')\Big], \tag{20}$$

where

$$K^{(n)}=\gamma^{(n)}i^{(n)}-(a^{(n)}+\mathrm{j}\delta)\gamma^{(n)} \tag{21}$$

and the superscript $(n+\frac{1}{2})$ means that the variable is determined by solving (1–4) with

$$\gamma=\gamma^{(n)}+\tfrac{1}{2}K^{(n)}\Delta t'. \tag{22}$$

We found that $\Delta t'<[20\,\mathrm{Re}\{a\}]^{-1}$ provided sufficient (better than 1 %) accuracy for the calculations.

4. Results of the Numerical Calculations

As our example we chose a BSO crystal in the usual orientation [7] so that only one single transmission grating is significant for four-wave mixing applications. We took the following values for the crystal parameters:

$\tau_e=6.4\,\mu\mathrm{s}$, $\varepsilon_r=56$, $\mu=10^{-5}\,\mathrm{m^2/Vs}$,

$n_0=2.62$, $r_{\mathrm{eff}}=4\times10^{-12}\,\mathrm{m/V}$,

$N_A=0.95\times10^{22}\,\mathrm{m^{-3}}$. The calculations were performed for a wavelength of $\lambda=0.548\,\mu\mathrm{m}$, the length of the crystal was taken as 1 cm, and the external angle as 1°.

The rise of the transients, as in any self-oscillating system, may be expected to depend on the intensity of the "noise" oscillation I_{10}. The set of curves presented in Fig. 2 for different values of I_{10} clearly show that for sufficiently weak initial noise intensity the shape of the transient response does not depend on the exact value taken for the noise; it determines only the delay in the build-up. For our further calculations we chose the noise intensity to be 10^4 times below the input power in beam 4.

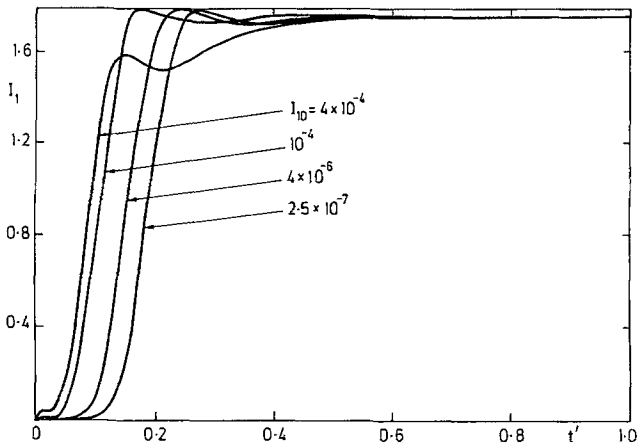

Fig. 2. Intensity of the oscillation versus normalised time ($E_0=2.5\,\mathrm{kV/cm}$, $\alpha=0$, $M_1=0.9$, $M_2=1$, $g_{\mathrm{in}}=-0.044$) for different values of noise intensity

The results of our calculations for $\alpha=0$ and $E_0=2.5\,\mathrm{kV/cm}$ are presented in Fig. 3 for different values of resonator detuning (the corresponding equilibrium oscillation detuning δ is given in the figure caption). One may immediately notice the following features of the transient regime. The oscillation beam, I_1, is the first one to appear, the reflected phase conjugate beam, I_3, emerges only a little later. Also, the intensity of the reflected beam changes more smoothly and more monotonically than the intensity of the oscillation. During the growth of oscillation, as may be expected, its phase changes rapidly but eventually tends to a constant, a proof that we guessed correctly the equilibrium value of δ.

It is interesting to note that the qualitative conclusions drawn on the basis of our analytical treatment of the initial stage of the transient regime turn out to be valid for the whole transient regime. In full accordance with (17) and (18) the rate of growth of the intensity drops significantly and the phase change becomes more rapid as $|g_{\mathrm{in}}|$ increases from zero. In particular, the growth of intensity at the very beginning of the transients is strongly delayed as $|g_{\mathrm{in}}|$ is increased. The delay becomes infinitely long when $|g_{\mathrm{in}}|$ is so large that according to the results of [2, 4] oscillation can no longer take place.

Let us discuss now the influence of the applied electric field on the transient regime in a PPCM. The

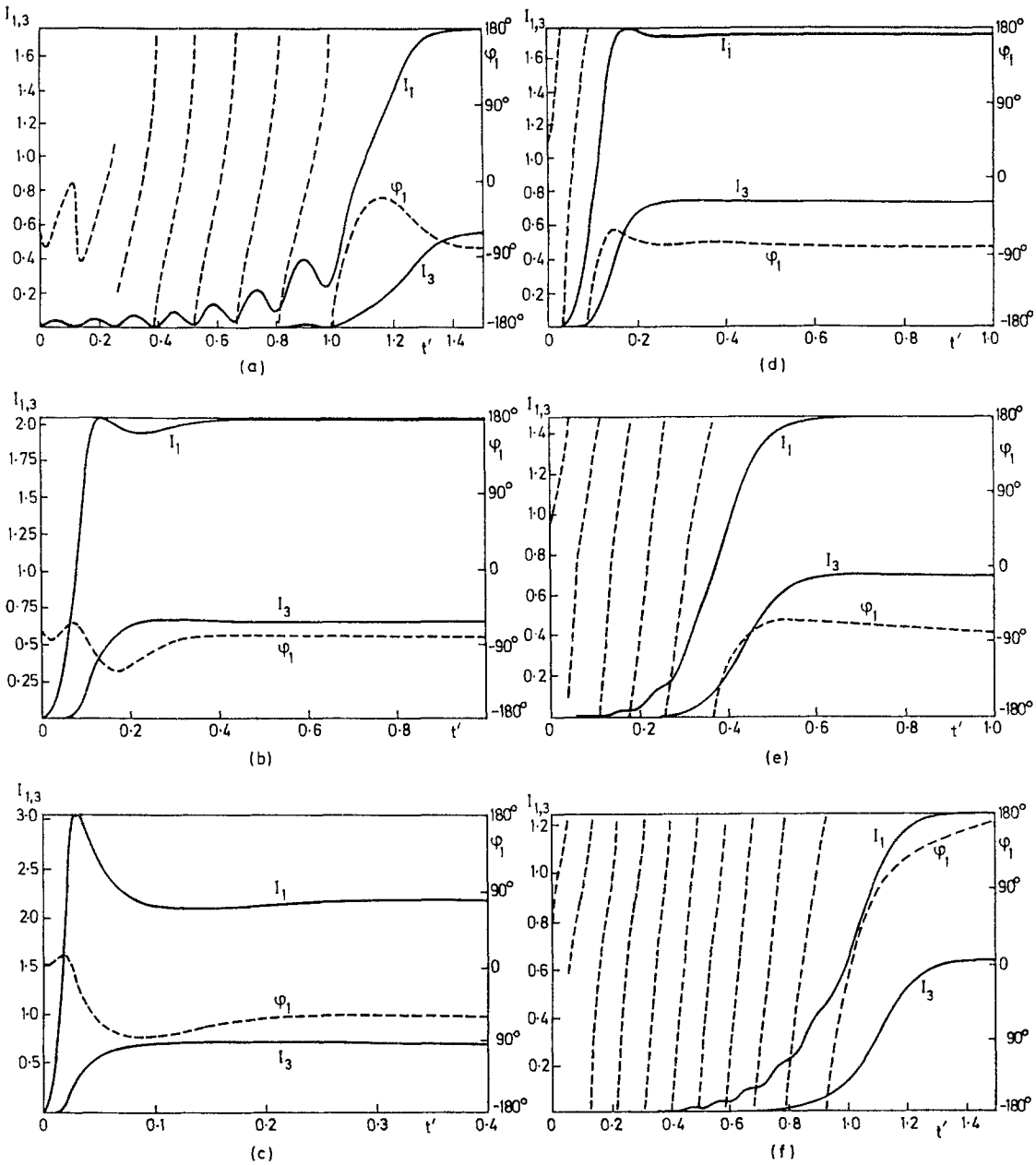

Fig. 3a–f. Intensity of oscillation, I_1, of the reflected beam, I_3 and the phase of oscillations φ_1 versus normalised time for $E_0=2.5$ kV/cm, $\alpha=0$, $M_1=0.9$, $M_2=1$, (a) $g_{in}=0.051$, $\delta=121.9$, (b) $g_{in}=0.03$, $\delta=127.5$, (c) $g_{in}=0$, $\delta=125$, (d) $g_{in}=-0.044$, $\delta=92.7$, (e) $g_{in}=-0.073$, $\delta=77.8$, (f) $g_{in}=-0.094$, $\delta=68.6$

grating time constant, characterized by $1/\mathrm{Re}\{a\}$ is known to increase with increasing electric field as shown in Fig. 4. But, according to Fig. 5, the increase of E_0 hardly affects at all the rise of I_1, a result predicted by our analytical solution for the initial stage of the transient regime. It may be further seen in Fig. 5a that an increase in E_0 causes a decrease in the equilibrium magnitude of oscillation. The explanation seems to follow from Fig. 5b namely that the stronger coupling to the resonator appears to favour the phase conjugate beam.

The effect of lower mirror reflectivity, shown in Fig. 6, is also in agreement with the predictions of (17) and (18). It may be seen by comparing Fig. 6 with Fig. 3 that the decrease in mirror reflectivity leads to slower variation in the transient regime and to a diminution of the dependence upon resonator detuning.

Absorption in the crystal may be expected to have a similar effect to that occurring with reduced mirror reflectivity, and this expectation is indeed born out by Fig. 7, plotted for $\alpha L=0.22$. In comparison with the $\alpha L=0$ case the curves become smoother, the ripples

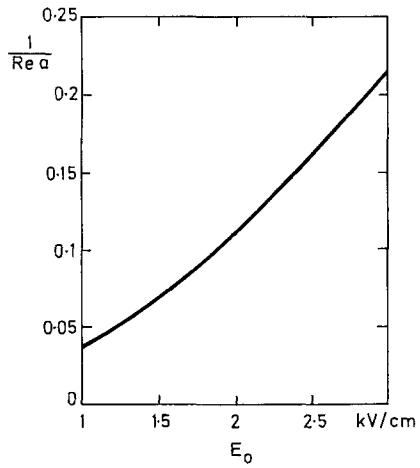

Fig. 4. Normalised grating time constant versus applied electric field

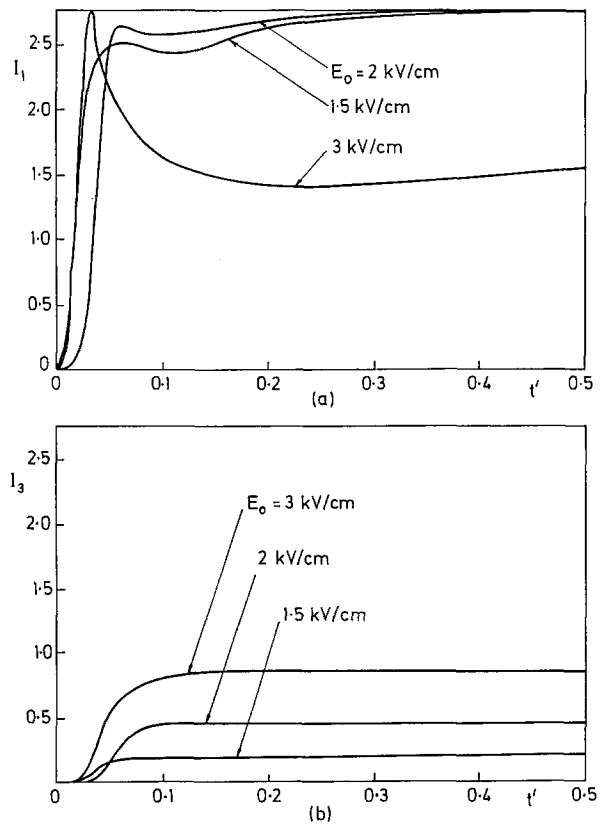

Fig. 5a, b. Intensity of the oscillation, I_1 and of the reflected beam, I_3 versus normalised time for $\alpha=0$, $M_1=0.9$, $M_2=1$, $g_{in}=0$. The curves shown are for $E_0=3\,\mathrm{kV/cm}$, $\delta=75.5$; $E_0=2\,\mathrm{kV/cm}$, $\delta=198.1$; $E_0=1.5\,\mathrm{kV/cm}$, $\delta=261.4$

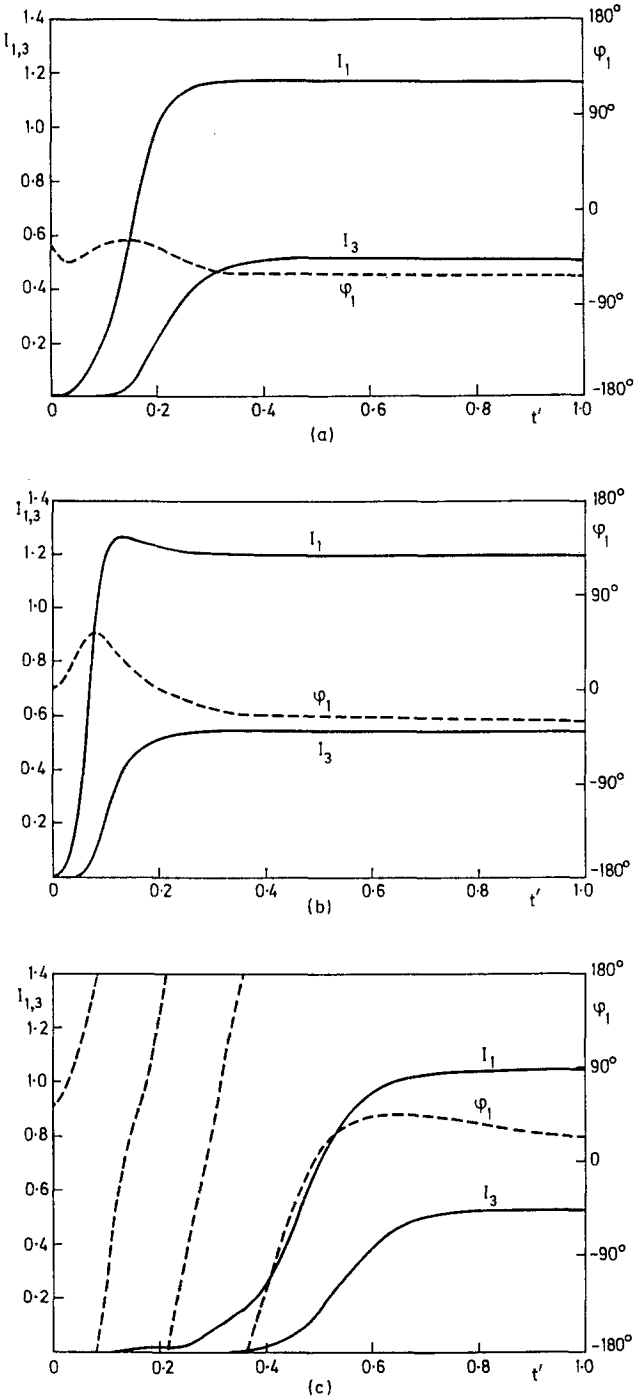

Fig. 6a–c. Intensity of the oscillation, I_1, of the reflected beam, I_3, and the phase of the oscillation φ_1 versus normalised time for $E_0=2.5\,\mathrm{kV/cm}$, $\alpha=0$, $M_1=0.8$, $M_2=0.9$, (a) $g_{in}=0.05$, $\delta=67.2$, (b) $g_{in}=0$, $\delta=64$, (c) $g_{in}=-0.095$, $\delta=53.3$

which occur for large $|g_{in}|$ disappear, the sensitivity to resonator detuning decreases, and the growth rate decreases as well. Thus absorption leads not only to a drop in phase conjugate reflectivity but also to a considerable slowing down of the response of a PPCM.

5. Conclusions

It has been shown that the duration and character of the transient regime in a PPCM is determined not only by the intensity of the input beam and the properties of

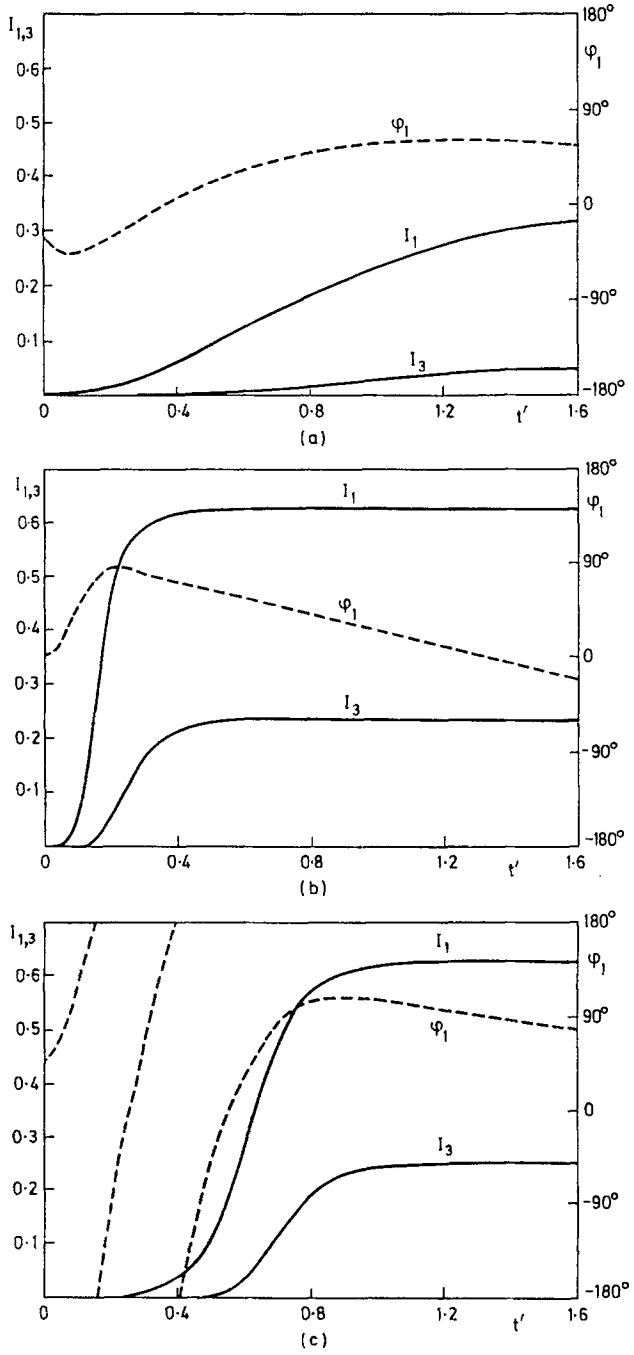

Fig. 7a–c. Intensity of the oscillation, I_1, of the reflected beam, I_3, and the phase of the oscillation φ_1 versus normalised time for $E_0 = 2.5$ kV/cm, $\alpha L = 0.22$, $M_1 = 0.9$, $M_2 = 1$, (a) $g_{in} = 0.1$, $\delta = 41.5$, (b) $g_{in} = 0$, $\delta = 42$, (c) $g_{in} = -0.1$, $\delta = 37$

the electro-optic crystal but also by the properties of the external optical circuit, in particular by the detuning and the quality factor of the resonator. The higher is the quality factor the faster is the response of the PPCM. The role of absorption has also been shown strongly to affect the character of the transient regime. We expect that an optimum value of absorption exists which will lead to a minimum duration for the transients. This is because the decrease in absorption constant leads to a decrease in photoexcited electron concentration which, in turn, leads to an increase in dielectric relaxation time.

We hope that the general conclusions drawn in the present paper for PPCMs are also valid for other kinds of geometries making use of photorefractive gain, e.g. the ring phase conjugate mirror described in [3].

Comparison with experimental results (which, as far as we know are not available as yet) may be expected to yield good qualitative agreement. The time scale will of course depend on the initial value of the "noise" which may vary considerably between crystals but it seems also likely that good quantitative agreement will require some refinement of the model by including optical activity and birefringence.

Acknowledgements. One of the authors (V.K.) wishes to acknowledge the support of the British Council and the hospitality of the Department of Engineering Science.

References

1. M. Cronin-Golomb, B. Fischer, J.O. White, A. Yariv: Appl. Phys. Lett. **41**, 219–220 (1982)
2. S.-K. Kwong, M. Cronin-Golomb, A. Yariv: IEEE J. QE-**22**, 1508–1523
3. M. Cronin-Golomb, B. Fischer, J.O. White, A. Yariv: IEEE J. QE-**20**, 12–30 (1984)
4. V.A. Kalinin, L. Solymar: IEEE J. QE (submitted)
5. V. Vinetskii, N. Kukhtarev, V. Markov, S. Odulov, M. Soskin: Izv. AN SSSR, Ser. Fiz. **41**, 811–820 (1977) (Bull. Acad. Sci. USSR, physical series)
6. N. Kukhtarev, V. Markov, S. Odulov: Opt. Commun. **23**, 338–343 (1977)
7. Ph. Refregier, L. Solymar, H. Rajbenbach, J.P. Huignard: J. Appl. Phys. **58**, 45–57 (1985)