Arch. Rational Mech. Anal. 123 (1993) 153–197. © Springer-Verlag 1993

Propagating Phase Boundaries: Formulation of the Problem and Existence via the Glimm Method

PHILIPPE LE FLOCH

Communicated by R. KOHN

Abstract

This paper treats the hyperbolic-elliptic system of two conservation laws which describes the dynamics of an elastic material having a non-monotone strain-stress function. Following ABEYARATNE & KNOWLES, we propose a notion of admissible weak solution for this system in the class of functions of bounded variation. The formulation includes an entropy inequality, a kinetic relation (imposed along any subsonic phase boundary) and an initiation criterion (for the appearance of new phase boundaries). We prove the L^1-continuous dependence of the solution to the Riemann problem. Our main result yields the existence and the stability of propagating phase boundaries. The proofs are based on GLIMM's scheme and in particular on the techniques of GLIMM and LAX. In order to deal with the kinetic relation, we prove a result of pointwise convergence of the phase boundary.

0. Introduction

This paper deals with the following system of two conservation laws which decribes the motion of an elastic material

$$(0.1) \qquad \partial_t w - \partial_x v = 0\,, \qquad \partial_t v - \partial_x \sigma(w) = 0\,.$$

Here $w > -1$ and v represent the displacement gradient and the velocity of the material, respectively. The stress $\sigma :]-1, \infty[\to \boldsymbol{R}$ is assumed to be monotonically increasing except in an interval $]w_M, w_m[$ (see Figure 0.1). Such a form of the stress is typical in the modeling of solid materials which admit different phases. A van der Waals gas also is described by a very similar system. System (0.1) is of mixed type, i.e., *hyperbolic* in the phase-1 region $\mathscr{H}_1 = \{w < w_M\}$ and in the phase-3 region $\mathscr{H}_3 = \{w > w_m\}$, but *elliptic* in the intermediate region of phase-2 states. The phase-2 states are known to be both mathematically and physically unstable (JAMES [23]). We consider here exclusively solutions which take their values in the *phase-1* or *phase-3* regions *only*.

Solutions to (0.1) in general are discontinuous, and so must be understood in the sense of distributions; see LAX [25, 26] for background on weak solutions. Such discontinuous solutions are in general non-unique and those having a physical meaning must be selected through an admissibility (or entropy) criterion. We refer to DAFERMOS [7] for a review of entropy conditions in the setting of hyperbolic problems. As was pointed out by JAMES [23], the mixed system (0.1) possesses a high degree of *non-uniqueness*, which a number of authors have attempted to resolve by means of suitable generalizations of entropy criteria from the theory of hyperbolic conservation laws. First of all, SHEARER [38] considered the LAX entropy criterion [25, 26]. The viscosity and viscosity-capillarity approaches have been analyzed by SLEMROD [41, 42]; cf. also HAGAN & SLEMROD [18], PEGO [35] and SHEARER [39, 40]. HATTORI [19, 20] has investigated the application to (0.1) of the entropy-rate admissibility criterion proposed by DAFERMOS [6]. HSIAO [22] has considered the Liu entropy criterion [32] which allows one to treat equations of state losing genuine nonlinearity in hyperbolic regions. Another approach to resolve the non-uniqueness can be found in a work by KEYFITZ [24]. Additional material on system (0.1) is found in [12, 13, 36].

All the above works treat the Riemann problem only, i.e., a Cauchy problem for (0.1) with initial condition that consists of two constant states. This problem can be solved explicitly (in a possibly non-unique way) by using simple waves (i.e., shock waves, rarefaction waves, and contact discontinuities). Adding an "admissibility criterion" allows one to reduce the class of (admissible) solutions and in most situations to select a unique solution. However, it must be emphasized that the solution of the Riemann problem (when it is unique) depends on the chosen admissibility criterion. It turns out that there is no preferred criterion for the selection of the "physically meaningful" solutions of (0.1).

A different approach was recently investigated by ABEYRATNE & KNOWLES in [2]. The main suggestion of these authors is that system (0.1) is not physically complete enough to describe the evolution of a phase boundary in an elastic material. It must be completed with a *kinetic relation* imposed along any subsonic phase boundary: This kinetic relation yields the rate of entropy dissipation across the phase discontinuity. Moreover, ABEYARATNE & KNOWLES add an *initiation criterion* which controls the possible appearance of a new phase. We refer to [1] and the references therein for the motivation of introducing a kinetic relation and an initiation criterion which are actually classical in the context of quasi-static problems. Cf. also GURTIN [17] and TRUSKINOVSKY [44] for related ideas.

ABEYARATNE & KNOWLES [2] proved that the Riemann problem for (0.1) always admits a unique admissible solution, i.e., a weak solution satisfying the kinetic relation and the initiation criterion, as well as the entropy inequality, which reads

$$\partial_t\left(W(w) + \tfrac{1}{2}\, v^2\right) - \partial_x\left(\sigma(w)\, v\right) \leqq 0, \tag{0.2}$$

where $W:]-1, \infty[\to \mathbf{R}$ is the internal energy function defined by

$$W(w) = \int_0^w \sigma(y)\, dy \quad \text{for } w \in]-1, \infty[. \tag{0.3}$$

Next they showed in [3] that the solution of the Riemann problem found by SLEMROD through the viscosity-capillarity approximation corresponds to a special choice of kinetic relation in their approach. It is not difficult to check also that the solution found by SHEARER using Lax entropy inequalities coincides with the maximally dissipative kinetic relation investigated in [4]. (I thank MICHAEL SHEARER for pointing this out to me.)

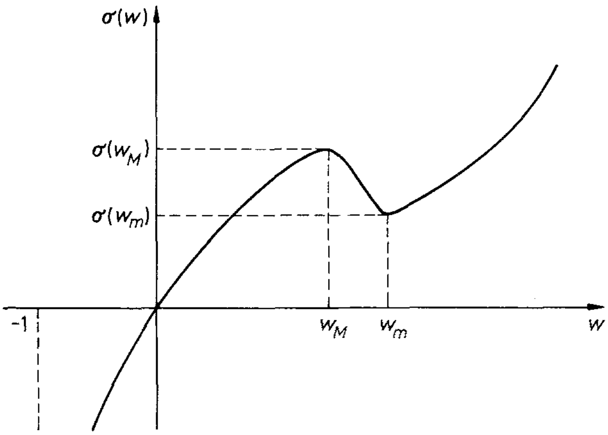

Figure 0.1

The present paper is devoted to continuing the analysis of system (0.1) through the approach of ABEYARATNE & KNOWLES. As in [2], we restrict our attention to the case of a piecewise linear stress. This assumption simplifies the calculations, but it is not a real restriction to the results of this paper.

Our purpose is first (Sections 1 and 2) to give a slightly different presentation of the ideas of [2], which I think clarifies the concepts of kinetic relation and initiation criterion introduced by ABEYARATNE & KNOWLES. Section 1 presents the mathematical formulation of a well-posed problem (at least for Riemann data) associated with system (0.1). As is usual for hyperbolic problems, we consider bounded solutions of bounded variation (*BV*). Our formulation follows [2] with however two main modifications. The kinetic relation is introduced from a completely *dynamical point of view* and not as a generalization of the quasi-static point of view as was done in [2]. That leads us to a *larger range* of admissible values for what we call below the entropy dissipation function in the kinetic relation. Furthermore, the initiation criterion at some point x is formulated here in two different ways, depending on whether x is in an interior point of the space interval $[a, b]$ where we set the problem, or x is a point of its boundary. For definiteness, we allow *spontaneous* *initiation of a new phase* only at the extremities of the bar $[a, b]$, which is consistent with the classical static theory.

Then Section 2 describes briefly the solution of the Riemann problem. We explain how to take into account the two changes mentioned above in the construction of [2]. The main result of this section establishes the *L^1-continuous dependence* of the Riemann solution with respect to its initial states. It must be emphasized that the two observations above are essential for the continuous-dependence property to hold, especially our condition that a new phase may occur spontaneously only at the extremities of the bar. The same results are also obtained for the Riemann problem in a half space. Note that, although uniqueness of the admissible solution holds for the Riemann problem, nothing is known for the general Cauchy problem. As a matter of fact, the issue of uniqueness for conservation laws is understood in a few situations only. (See, for hyperbolic problems, LE FLOCH & XIN [31] and the references therein.)

The second part of the paper (Sections 3 and 4) focuses on the solutions of the Cauchy problem for system (0.1), which are *BV* perturbations of a single propagating phase boundary separating a phase-1 state and a phase-3 state. We prove the existence of admissible weak solutions of this form, when the initial data on both sides of the phase discontinuity has small total variation. We treat the case of any non-characteristic phase boundary as well as the case of a characteristic phase boundary provided that no strong wave arises from perturbating the states on both sides of the phase boundary. The random-choice scheme due to GLIMM [15] is used to construct approximate solutions to the problem. Its stability in the *BV* norm is proved from an essentially linear estimate of wave interactions between two Riemann solutions. Such linear interaction terms were used in a different situation by CHERN [5] and SCHOCHET [37]. Note that the strength of the phase discontinuity is not (and cannot be) assumed to be small in any sense.

The stability of the scheme in the total variation norm is sufficient to extract a subsequence converging to a weak solution of the problem. This convergence result holds almost everywhere with respect to the Lebesgue measure. This is sufficient to show that the scheme converges to a weak solution of the problem. But, proving that this solution is *admissible* requires a result of *pointwise convergence* of the phase boundary. In Section 4, we establish this property by using the technique of analysis due to GLIMM & LAX [16]. We next prove that it is sufficient, at least for non-stationary phase boundaries, for the passage to the limit in the kinetic relation.

An extension of the results in this paper to arbitrary large initial data would require a better understanding of the complex phenomena of initiation of new phases.

Many ideas in this paper are related to those in the developing theory of nonlinear hyperbolic systems in non-conservative form for which we refer the reader to DAL MASO, LE FLOCH & MURAT [8] and LE FLOCH & LIU [30]; see also [27–29].

1. Mathematical formulation of the problem

This section describes the formulation of the Cauchy problem associated with the mixed system (0.1). The formulation includes the system of conservation laws (mass, momentum) (0.1) together with the (Clausius-Duhem) entropy inequality associated with the entropy $W(w) + \frac{1}{2} v^2$. It is made complete by adding to these both a kinetic relation along any subsonic phase boundary and an initiation criterion for the occurrence of possible new phase boundaries in the solution. We specify below the assumptions on the kinetic relation and the initiation criterion which will be essential to the results of Section 2. This section also introduces notation which will be of constant use throughout this paper.

$$\partial_t u + \partial_x f(u) = 0, \quad u = \begin{pmatrix} v \\ w \end{pmatrix}, \quad f(u) = \begin{pmatrix} -\sigma(w) \\ -v \end{pmatrix}. \tag{1.1}$$

For simplicity, we assume that the stress $\sigma :]-1, \infty[\to \boldsymbol{R}$ is a piecewise linear function of the following form (cf. Figure 1.1)

$$\sigma(w) = \begin{cases} k_1 w & \text{for } -1 \leqq w \leqq w_M, \\ k_3 w_m + (k_1 w_M - k_3 w_m)(w - w_m)/(w_M - w_m) & \text{for } w_M \leqq w \leqq w_m, \\ k_3 w & \text{for } w_m \leqq w. \end{cases} \tag{1.2}$$

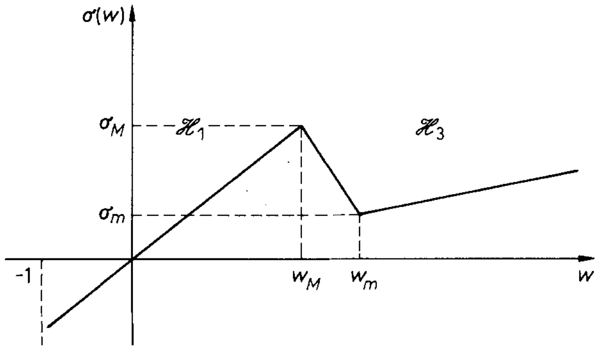

Figure 1.1

The constants k_1, k_3, w_m and w_M in (1.2) are assumed to satisfy the properties

$$0 < k_3 < k_1, \quad 0 < w_M < w_m. \tag{1.3}$$

We use the notation

$$\sigma_M = k_1 w_M, \quad \sigma_m = k_3 w_m.$$

The phase-1 region $\mathscr{H}_1 = \{-1 < w \leqq w_M\}$ and the phase-3 region $\mathscr{H}_3 = \{w \geqq w_m\}$ correspond to observable and stable states. In our formulation below, the solution *cannot enter* the unstable phase-2 region $\{w_M < w < w_m\}$ and so must jump from $\mathscr{H}_1$ to $\mathscr{H}_3$ or conversely. A discontinuity between two states in different phases is called a phase boundary.

System (1.1) is linear-hyperbolic in $\mathscr{H}_1$ and $\mathscr{H}_3$, and the corresponding characteristic speeds are

$$\pm c_1 = \pm\sqrt{k_1} \text{ in } \mathscr{H}_1, \qquad \pm c_3 = \pm\sqrt{k_3} \text{ in } \mathscr{H}_3.$$

In view of (1.3), the waves in phase 1 travel faster than those in phase 3, i.e., $c_1 > c_3$. We may also use the notation

$$c(w) = c_1 \text{ if } w \leqq w_M, \quad c_3 \text{ if } w \geqq w_m. \tag{1.4}$$

Note that $c(w)$ is not defined if w belongs to $]w_M, w_m[$. With some abuse of notation, $\mathscr{H}_1$ and $\mathscr{H}_3$ sometimes also denote $\{(v,w)/-1 < w \leqq w_M, v \in \boldsymbol{R}\}$ and $\{(v,w)/w_m \leqq w, v \in \boldsymbol{R}\}$, respectively. We also set $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_3$.

Since (1.1) is linear-hyperbolic in $\mathscr{H}_1$ and $\mathscr{H}_3$, possible discontinuities in the initial data for (1.1) are simply advected along the characteristic lines with slopes either $\pm c_1$ or $\pm c_3$. (This is true at least up to the time of appearance of a new phase.) The elementary waves in each of the regions $\mathscr{H}_1$ and $\mathscr{H}_3$ are contact discontinuities. Hence, the special choice (1.2) for the constitutive law is very convenient. It makes the analysis quite simple in the hyperbolic regions and allows us to focus on the phase boundaries between $\mathscr{H}_1$ and $\mathscr{H}_3$. We shall see that the description of the appearance and the evolution of the phase boundaries is far from trivial.

In the theory of hyperbolic conservation laws, it is standard to consider solutions $u = (v, w)$ to (1.1) in the function space $L^\infty_{\text{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H})$ (recall that $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_3$) which satisfy

(1.5a) system (1.1) in the sense of distributions,

(1.5b) the entropy inequality (0.2), (0.3) in the sense of distributions,

(1.5c) an initial condition u_0 at $t = 0$ in the L^1_{loc} sense.

Here u_0 is a given function in $L^\infty_{\text{loc}}(\boldsymbol{R}, \mathscr{H})$ and, for future reference, we rewrite the entropy inequality in the form

$$\partial_t U(u) + \partial_x F(u) \leqq 0 \tag{1.6a}$$

with

$$U(u) = W(w) + \tfrac{1}{2} v^2, \quad F(u) = -\sigma(w)\, v, \quad W(w) = \int_0^w \sigma(y)\, dy. \tag{1.6b}$$

We recall that solutions in the sense (1.5) are *unique* (at least for Riemann data) in the standard situation of a (genuinely nonlinear or linearly degenerate) *increasing* stress σ. This is no longer true in the case of the mixed system under consideration here: See for instance JAMES [23]. We also point out that the entropy function U is not a convex function.

To complete the formulation (1.5), we follow ABEYARATNE & KNOWLES [2]. Let us first give some motivation for their suggestion. Suppose $u^\varepsilon = (v^\varepsilon, w^\varepsilon)$

is the solution of a regularized version of system (1.1) obtained by adding high-order terms, depending on a (small) parameter ε, in the right-hand side of the equations (e.g., by using the viscosity-capillarity terms as was done by SLEMROD [41]). As was pointed out by LAX for general systems of conservation laws, the limit $u = \lim u^{\varepsilon}$, if it exists (and if the convergence holds in a suitable topology), must be a solution to (1.1) in the sense (1.5); in particular, the entropy inequality (1.6) must hold. Since (1.5) is incomplete, it seems natural to "keep more information" about the limiting function u from its regularization u^{ε}. Specifically ABEYARATNE & KNOWLES' suggestion is equivalent to replacing (1.6) with the stronger requirement that

$$\partial_t U(u) + \partial_x F(u) = \mu, \tag{1.7}$$

where μ is a given non-positive measure that clearly must satisfy certain restrictions. Note that in principle μ could be determined by the formula

$$\mu = \text{weak-star} \lim_{\varepsilon \to 0} \left(\partial_t U(u^{\varepsilon}) + \partial_x F(u^{\varepsilon})\right)$$

(*at least when* u^{ε} has uniformly bounded total variation in (t, x)). This formula may not give a very explicit expression for μ. Fortunately it turns out that (1.7) is needed (to achieve uniqueness) *only* for one kind of discontinuity: the subsonic phase boundaries. Moreover, in that case, we can allow a large range of measures μ. Here, we call subsonic and supersonic those phase boundaries that respectively travel with speed less than and greater than the contact discontinuities in phase $\mathscr{H}_3$.

The precise formulation of condition (1.7) given below requires that u be a bounded function of bounded variation. When u has bounded variation, we call *entropy dissipation* the value of the measure $\partial_t U(u) + \partial_x F(u)$ along a curve of (contact or phase) discontinuity of u. According to [2], the *kinetic relation* yields this entropy dissipation along any *subsonic phase boundary*, as an explicit function, say $\phi(V)$, of the speed V of propagation of this discontinuity. In applications, the actual kinetic relation, that is, the function ϕ, must be determined from the properties of the specific material under consideration. This kind of constitutive model is already in extensive use in the quasi-static setting for problems of phase transition in solids. We refer the reader to [1] as well as to TRUSKINOVSKY [44] and the references cited there. The speed V can also be interpreted as an internal variable and the kinetic relation indeed determines the evolution of this internal parameter.

Remark 1.1. 1) That subsonic and supersonic phase boundaries must be treated in a different way is clear, for instance, when solving Riemann problems. A wave structure with a supersonic phase boundary contains *two* waves, while one with a subsonic boundary is composed of *three* waves. This latter case suffers, without a kinetic relation, from a strong lack of uniqueness. Cf. JAMES [23] and Section 2.

2) The approach considered here has some similarity to the theory of nonlinear hyperbolic systems in non-conservative form; cf. DAL MASO, LE FLOCH & MURAT [8] and LE FLOCH & LIU [30]. Namely, as is the case for systems (1.1), the weak solutions to these systems are not uniquely determined

by the partial differential equations and an entropy inequality, but an additional constitutive relation must be added to ensure uniqueness. This fact was first pointed out in [27 – 29].

3) Conservation laws with measure source-terms like (1.7) have been useful in various contexts, cf. DIPERNA [9], DIPERNA & MAJDA [11], HOU & LE FLOCH [21].

Let us introduce some notation and recall some facts about functions of bounded variation, which can be found in VOLPERT [45] and FEDERER [14]. Let Ω be an open subset of $\boldsymbol{R}^m$. A function $u: \Omega \to \boldsymbol{R}^p$ belongs to the space $BV(\Omega, \boldsymbol{R}^p)$ (or $BV_{\text{loc}}(\Omega, \boldsymbol{R}^p)$) if $u \in L^1(\Omega, \boldsymbol{R}^p)$ (or $L^1_{\text{loc}}(\Omega, \boldsymbol{R}^p)$) and the distributional derivatives $\partial u / \partial y_j$ for $1 \leqq j \leqq m$ are bounded (or locally bounded) Borel measures on Ω. In what follows, we always consider functions in $L^\infty(\Omega, \boldsymbol{R}^p) \cap BV(\Omega, \boldsymbol{R}^p)$ or $L^\infty_{\text{loc}}(\Omega, \boldsymbol{R}^p) \cap BV_{\text{loc}}(\Omega, \boldsymbol{R}^p)$, often called for short BV functions or BV_{loc} functions. For each BV_{loc} function u, we have the decomposition

$$\Omega = C(u) \cup S(u) \cup E(u),$$

where

$C(u)$ is the set of all points of approximate continuity for u,
$S(u)$ is the set of all points of approximate jump for u,
E(u) is the set of exceptional points with the property $H_{m-1}(E(u)) = 0$.

Here H_{m-1} is the $(m-1)$-dimensional Hausdorff measure on $\boldsymbol{R}^m$. For each point y in $S(u)$, there exists a unit normal $\nu \in \boldsymbol{R}^m$ and approximate left and right limits for u that we denote by $u_\pm(y)$. The set $S(u)$ consists of the union of a countable number of rectifiable curves.

We denote the norm of u by $\|u\|_{BV(\Omega, \boldsymbol{R}^p)} = \|u\|_{L^1(\Omega, \boldsymbol{R}^p)} + |Du|(\Omega)$, where Du is the measure $\left(\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \ldots, \frac{\partial u}{\partial y_m}\right)$. When $u = u(t,x) \in L^\infty_{\text{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H})$ $\cap BV_{\text{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H})$, we use the notation

$$\nu(t,x) = \left(\nu_t(t,x), \nu_x(t,x)\right), \quad V(t,x) = -\frac{\nu_t(t,x)}{\nu_x(t,x)},$$

valid for all $(t,x) \in S(u)$. The ratio $V(t,x)$ represents the speed of propagation of the discontinuity in u at the point (t,x). Note that system (1.1) has the property of propagation with finite velocity (in regions $\mathscr{H}_1$ and $\mathscr{H}_3$). So $\nu_x(t,x)$ never vanishes, and for definiteness we always choose $\nu_x(t,x) > 0$. In the following, we shall always have $u(t) \in BV$ for all times t.

Let $\phi:]-c_3, c_3[\to \boldsymbol{R}$ be a function, called below the *entropy dissipation function*, satisfying the following properties:

(1.8a) ϕ belongs to $\mathscr{C}^2(]-c_3, 0[\cup]0, c_3])$, $\phi(0\pm)$ and $\phi'(0\pm)$ exist,

(1.8b) $$\lim_{V \to c_3^-} \phi = \overline{\psi}(c_3), \quad \phi'''(c_3-) \text{ exists},$$

(1.8c) $$\lim_{V \to -c_3^+} \phi = -\infty,$$

(1.8d) $$\phi \text{ is increasing on }]-c_3, c_3],$$

(1.8e) $$\underline{\psi}(V) \leqq \phi(V) \leqq 0 \quad \text{for } V \in]-c_3, 0],$$
$$0 \leqq \phi(V) \leqq \overline{\psi}(V) \quad \text{for } V \in [0, c_3].$$

In (1.8b) and (1.8e), the *minimal* and *maximal entropy dissipation functions* $\underline{\psi}$: $]-c_3, 0] \to \boldsymbol{R}_-$ and $\overline{\psi}$: $[0, c_3] \to \boldsymbol{R}_+$ are defined by

(1.9a) $$\underline{\psi}(V) = \frac{(k_1 - k_3)}{2} w_M \left(w_m - \frac{k_1 - V^2}{k_3 - V^2} w_M \right) \quad \text{for } V \in]-c_3, 0],$$

(1.9b) $$\overline{\psi}(V) = \frac{(k_1 - k_3)}{2} w_m \left(w_M - \frac{k_3 - V^2}{k_1 - V^2} w_m \right) \quad \text{for } V \in [0, c_3].$$

Cf. Figure 1.2 for a graphical representation of ϕ, $\underline{\psi}$ and $\overline{\psi}$.

Remark 1.2. 1) Inequalities (1.8e) give the range of values taken by the entropy dissipation rate $\mathscr{E}(u)$ (see below) when varying the left and right values at a discontinuity satisfying the Rankine-Hugoniot relations and the entropy condition.

2) In [2], instead of (1.8e), Abeyaratne & Knowles assume the (more restrictive) condition:

(1.8e)′ $$\underline{\psi}(0) \leqq \phi(V) \leqq 0 \quad \text{for } V \in]-c_3, 0],$$
$$0 \leqq \phi(V) \leqq \overline{\psi}(0) \quad \text{for } V \in [0, c_3].$$

3) Assumptions (1.8) made in this paper are indeed satisfied in the examples considered by [3] and [4]. For instance, they are fulfilled by the *maximally dissipative function* $\phi_{\max}$ defined by

$$\phi_{\max}(V) = \underline{\psi}(V) \text{ for } V \in]-c_3, 0], \ \overline{\psi}(V) \text{ for } V \in [0, c_3].$$

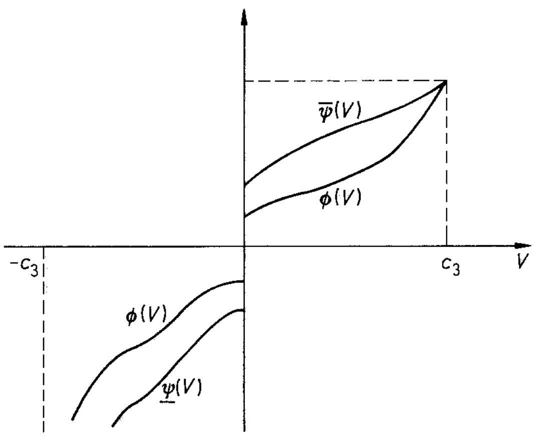

Figure 1.2

We next define the *entropy dissipation rate* $\mathscr{E}(u)$ associated with any function $u \in L^\infty_{\rm loc}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H}) \cap BV_{\rm loc}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H})$ by the formula

$$\mathscr{E}(u) = -\left(U(u_+) - U(u_-)\right) - \frac{v_x}{v_t}\left(F(u_+) - F(u_-)\right), \tag{1.10}$$

which defines $\mathscr{E}(u)\,(t,x)$ at H_1-almost every point (t,x), where $v_t(t,x) \neq 0$ (i.e., $V(t,x) \neq 0$). $\mathscr{E}(u)$ is the product of $-1/v_t$ with the jump of the measure $\partial_t U(u) + \partial_x F(u)$ along the curve of approximate jump of u. Formula (1.10) makes sense only if $v_t(t,x) \neq 0$. However, it is a simple observation that if u is assumed to be a *weak solution* to system (1.1), then the above jump (i.e., the entropy dissipation) vanishes at the points where v_t vanishes. This fact allows us to define $\mathscr{E}(u)\,(t,x)$ H_1-almost everywhere, as shown by the following lemma.

Lemma 1.1. *If* $u \in L^\infty_{\rm loc}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H}) \cap BV_{\rm loc}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H})$ *is a weak solution to* (1.1), *then*

$$\mathscr{E}(u) = -\int_{w_-}^{w_+} \{\sigma(y) - \tfrac{1}{2}\left(\sigma(w_+) + \sigma(w_-)\right)\}\,dy, \tag{1.10$'$}$$

at H_1-almost every (t,x) such that $v_t(t,x) \neq 0$.

From now on, we use (1.10)′ to define $\mathscr{E}(u)\,(t,x)$.

Proof of Lemma 1.1. At a point of approximate discontinuity (t,x) of the solution u, the following Rankine-Hugoniot relations hold:

$$v_t(w_+ - w_-) - v_x(v_+ - v_-) = 0,$$

$$v_t(v_+ - v_-) - v_x\left(\sigma(w_+) - \sigma(w_-)\right) = 0.$$

These relations when used in (1.10) yield

$$\begin{aligned} -\mathscr{E}(u) &= \int_{w_-}^{w_+} \sigma(y)\,dy + \tfrac{1}{2}\,(v_+^2 - v_-^2) - \frac{v_x}{v_t}\left(\sigma(w_+)\,v_+ - \sigma(w_-)\,v_-\right) \\ &= \int_{w_-}^{w_+} \sigma(y)\,dy + \tfrac{1}{2}\,(v_+ + v_-)\,\frac{v_x}{v_t}\left(\sigma(w_+) - \sigma(w_-)\right) \\ &\quad - \frac{v_x}{v_t}\left(\sigma(w_+)\,v_+ - \sigma(w_-)\,v_-\right). \end{aligned}$$

We thus get

$$\begin{aligned} -\mathscr{E}(u) = \int_{w_-}^{w_+} \sigma(y)\,dy + \frac{v_x}{2v_t}\{&v_+\sigma(w_+) + v_-\sigma(w_+) - v_+\sigma(w_-) - v_-\sigma(w_-) \\ &- 2\sigma(w_+)\,v_+ + 2\sigma(w_-)\,v_-\}, \end{aligned}$$

so that

$$-\mathscr{E}(u) = \int_{w_-}^{w_+} \sigma(y)\,dy - \frac{1}{2}\frac{v_x}{v_t}(v_+ - v_-)\,(\sigma(w_+) + \sigma(w_-)),$$

which, in view of the Rankine-Hugoniot relations above, gives the desired result (1.10)′. □

Let us denote by $B_{\text{sub}}(u)$ the set of all points of approximate discontinuity in a weak solution u that correspond to a *subsonic* phase boundary. This means that

$$B_{\text{sub}}(u) = \{(t,x) \in S(u) \mid \text{either } u_-(t,x) \in \mathscr{H}_1, u_+(t,x) \in \mathscr{H}_3, \ |V| \leqq c_3, \\ \text{or } u_-(t,x) \in \mathscr{H}_3, \ u_+(t,x) \in \mathscr{H}_1, \ |V| \leqq c_3\}.$$

In view of (1.6), the Borel measure $\partial_t U(u) + \partial_x F(u)$ is globally non-positive. The *kinetic relation* now specifies the value itself (and not only the sign) of this measured along any subsonic phase boundary. In other words, for H_1-almost all $(t,x) \in \mathscr{B}_{\text{sub}}(u)$, one must have

$$\mathscr{E}(u)\,(t,x) = \begin{cases} -\phi(V(t,x)) & \text{if } u_-(t,x) \in \mathscr{H}_1, \\ \phi(-V(t,x)) & \text{if } u_-(t,x) \in \mathscr{H}_3. \end{cases} \tag{1.11}$$

Remark 1.3. As a matter of fact, the traveling waves obtained through the viscosity-capillarity regularization to system (1.1) converge to weak solutions of (1.1) that satisfy the kinetic relation (1.1) with a specific choice of function ϕ. This function can be determined explicitly and depends only on the viscosity and capillarity coefficients introduced in the regularization (cf. [3]).

Finally, we have to formulate the *initiation criterion*, which together with the above kinetic relation will allow us to rule out all non-physical solutions to our problem. Let $]a,b[$ be a space interval in which we are going to set the problem, with $a < b$ and possibly $a = -\infty$ and/or $b = +\infty$. The initiation criterion will reflect the following facts:

(1.12) No new phase occurs from any point x in $]a,b[$ except if no solution exists without creation of a new phase.

(1.13) A new phase state may occur at the boundary point $x = a$, even if a solution with no new phase exists; a criterion is required to make the choice.

(1.14) A new phase state may occur at $x = b$, even if a solution with no new phase exists; a criterion is required to make the choice.

From the mathematical point of view, condition (1.12) is essential: It ensures that spontaneous initiation of a new phase inside $]a,b[$ cannot occur from two nearby initial states in the same phase (cf. Section 2). This does not exclude the possibility (which really happens) that an initial discontinuity with large jump gives rise to, for instance, a phase-1 state although the states on both sides of the initial discontinuity are in phase 3. However, by condition (1.12), a single *constant state* is always a (trivial) *admissible solution*. (This property was not satisfied in the construction of [2].) This is also essential to get the L^1-continuous dependence property for Riemann solutions, proved below in Section 2.

Conditions (1.13) and (1.14) follow the quasi-static theory [1]. They allow "spontaneous nucleation" of a new phase only at the end points of $[a, b]$. Note that, more generally, we could as well allow nucleation at some arbitrary given points of $[a, b]$. Our actual restriction is that the points of spontaneous nucleation are known a priori and follow a selection criterion of the form specified below. However, while this formulation is fully satisfactory from the mathematical point of view, it does not reproduce what is really observed in practical experiments with elastic bars. Namely, in experiments, when pulling out an elastic bar uniformly in phase 1, initiation of phase-3 regions in the bar occurs successively and (apparently) randomly at various places in the bar. Physicists assert that initiation occurs at microscopic inhomogeneities of the material. A complete treatment of the initiation mechanism is beyond the scope of this paper and would probably require a statistical description. (As a matter of fact, this might quite easily be included in the random choice scheme, studied in Sections 3 and 4 below.)

It remains to provide an analytic version of the conditions (1.12)–(1.14). For convenience, we use here an averaged strain in our formulation. (In [1] and [2], the stress and the entropy dissipation rate, respectively, are used instead.) Given any function $u = (v, w)$ in $L^\infty_{\mathrm{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H}) \cap BV_{\mathrm{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R}, \mathscr{H})$, we set

$$(1.15) \qquad h_u = \frac{c(w_-)\, w_- + c(w_+)\, w_+ + v_+ - v_-}{c(w_-) + c(w_+)},$$

which defines $h_u(t,x)$ for H_1-almost every (t,x) in $\boldsymbol{R}_+ \times \boldsymbol{R}$. We note that $h_u(t,x) = w(t,x)$ when (t,x) is a point of approximate continuity of u. So $h_u(t,x)$ represents an *averaged strain* at the point (t,x) and determines the dynamics at this point. For instance, if u_- and u_+ are in the same phase, then h_u is the intermediate value between the 1-wave and the 2-wave in the solution of the Riemann problem with initial data u_- and u_+ (cf. Section 2).

For each interior point $x \in \,]a, b[$ and for each time $t \geqq 0$, the initiation criterion, by definition, is

(1.16) If $u_-(t,x)$ and $u_+(t,x)$ belong to $\mathscr{H}_1$ or to $\mathscr{H}_3$, then $h_u(t,x) \in \mathscr{H}_1$ or $\mathscr{H}_3$ respectively, if and only if there exists $\varepsilon > 0$ such that $u(s,y) \in \mathscr{H}_1$ or $\mathscr{H}_3$ respectively, for $(s,y) \in [t, t+\varepsilon[\, \times \,]x-\varepsilon, x+\varepsilon[$.

According to (1.12), condition (1.16) ensures that, locally in time, the solution remains in the same phase whenever this is possible. Cf. Section 2.

We are now concerned with the boundary points $x = a$ and $x = b$. We assume that $u(t)$ is defined for all times and has bounded variation in x. (This is the regularity of the solutions found in Section 4.) The material is assumed to be fixed at the end points, i.e., when $a \neq -\infty$ and/or $b \neq +\infty$, we have

(1.17a) $$v_+(t,a) = 0 \quad \text{for } L^1\text{-almost every } t > 0,$$

(1.17b) $$v_-(t,b) = 0 \quad \text{for } L^1\text{-almost every } t > 0,$$

where L^1 denotes the one-dimensional Lebesgue measure. Since v has bounded variation, it admits an L^1 trace at $x = a$ and $x = b$. Let w_M^{cr} and w_m^{cr} be two constants, called *critical values for the initiation*, that must satisfy the inequalities

(1.18) $$\frac{\sigma_0}{k_1} \leqq w_M^{\mathrm{cr}} \leqq w_M, \qquad w_m \leqq w_m^{\mathrm{cr}} \leqq \frac{\sigma_0}{k_3},$$

where σ_0 is the so-called *Maxwell stress* given by

(1.19) $$\sigma_0 = \sqrt{\sigma_m \sigma_M} = c_1 c_3 \sqrt{w_m w_M}\,.$$

Note that, as pointed out to me by ABEYARATNE, the critical values for initiation should in principle depend on the speed of propagation of the phase discontinuity. At the point $x = a$ (when $a \neq -\infty$), we impose for *all* times $t \geqq 0$ the following two conditions:

$(1.20)_i$ If $u_+(t,a)$ belongs to $\mathscr{H}_3$, then $h_u(t,a) \geqq w_m^{\mathrm{cr}}$ if and only if there exists $\varepsilon > 0$ such that $u_+(s,a) \in \mathscr{H}_3$ for $s \in [t, t+\varepsilon[$.

$(1.20)_{ii}$ If $u_+(t,a)$ belongs to $\mathscr{H}_1$ then $h_u(t,a) \in \mathscr{H}_1$ if and only if there exists $\varepsilon > 0$ such that $u_+(s,t) \in \mathscr{H}_1$ for all $s \in [t, t+\varepsilon[$.

In order to satisfy the boundary condition (1.17a), the term $h_u(a,t)$ in $(1.20)_i$ is defined by formula (1.15) with

(1.21) $$v_-(t,a) = -v_+(t,a)\,, \quad w_-(t,a) = w_+(t,a)\,.$$

Similarly, at the point $x = b$ (when $b \neq +\infty$), we impose for *all* times $t > 0$ the requirements

$(1.22)_i$ If $u_-(t,b)$ belongs to $\mathscr{H}_1$, then $h_u(t,b) < w_M^{\mathrm{cr}}$ if and only if there exists $\varepsilon > 0$ such that $u_-(s,b) \in \mathscr{H}_1$ for $s \in [t, t+\varepsilon[$.

$(1.22)_{ii}$ If $u_-(t,b)$ belongs to $\mathscr{H}_3$ then $h_u(t,b) \in \mathscr{H}_3$ if and only if there exists $\varepsilon > 0$ such that $u_-(s,b) \in \mathscr{H}_3$ for all $s \in [t, t+\varepsilon[$.

As previously, we set

(1.23) $$v_+(t,b) = -v_-(t,b)\,, \quad w_+(t,b) = w_-(t,b)\,.$$

We call an *admissible weak solution* to system (1.1) a function $u = (v, w)$ which satisfies the *conservation laws* (1.1), the *entropy inequality* (1.6), the *kinetic relation* (1.11), the *boundary condition* (1.17) (if instead of $\boldsymbol{R}$ an interval $]a, b[$ is considered) and the *initiation criterion* (1.16), (1.20) and (1.22).

In this paper, we prove the existence of such an admissible weak solution for two kinds of Cauchy data: the Riemann problem (in the whole space and in a half space) and a perturbation of a single propagating phase boundary.

These results provide a strong justification for our formulation here. It would be interesting to address the general question of existence and uniqueness for system (1.1) in the setting introduced in this section.

Remark 1.4. 1) A phase boundary necessarily is a wave with a *large* strength (at least $|w_M - w_m|$). This implies that, in *BV* solutions, phase boundaries cannot accumulate in a bounded region of the (t, x)-plane. Phase boundaries are thus isolated, and this justifies the formulations (1.16), (1.20) and (1.22).

2) Sections 3 and 4 provide an existence result for *small BV* perturbations of phase boundaries. I believe this result to be true for any *finite* number of phase boundaries. However, for *arbitrary large* data, the appearance of an infinite number of phase boundaries is not excluded a priori. In such a case, the solution would not have bounded total variation. A challenging issue is to extend the present formulation to the framework of L_∞ solutions.

3) The formulation of this section can be extended to the case that the stress is not a piecewise affine function but an arbitrary piecewise monotone function.

4) Shearer's solution [38] corresponds to the choice $w_M^{cr} = w_M$, $w_m^{cr} = w_m$ and $\phi = \phi_{max}$ (see Remark 1.2 for the definition of ϕ_{max}).

2. The Riemann Problem and Continuous Dependence

This section gives an explicit description of the admissible weak solution of the problem formulated in Section 1, in two cases: the Riemann problem in the whole space and the Riemann problem in a half space. Our main result in this section in the L^1-continuous dependence property of the solution of these problems. Note that the formulation of Section 1 and the assumptions made there are essential for this property to hold.

We consider the following two problems:

1) *The Riemann problem in the whole space* $]a, b[=]-\infty, +\infty[$, which corresponds to initial data of the form

$$u_0(x) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0. \end{cases} \tag{2.1}$$

Here $u_L = (v_L, w_L) \in \mathscr{H}$ and $u_R = (v_R, w_R) \in \mathscr{H}$ are two given constant states.

2) *The Riemann problem in the half space* $]a, b[=]0, +\infty[$, which corresponds to the initial data

$$u_0(x) = u_0 \quad \text{for all } x > 0 \tag{2.2}$$

where $u_0 = (v_0, w_0) \in \mathscr{H}$ is a constant state.

We shall describe successively the admissible solutions to problems 1) and 2), by following closely the work by Abeyaratne & Knowles. However our construction is slightly different from that in [2], due to our formulation. We shall not address the question of uniqueness of the solution here, since it is an easy matter from the results in [2] (which yield for their construction uniqueness in the class of solutions composed of simple waves).

To begin with, we deal with problem 1) and distinguish between several cases:

Case 1-a: $u_L \in \mathscr{H}_1$ and $u_R \in \mathscr{H}_3$,

Case 1-b: $u_L \in \mathscr{H}_1$ and $u_R \in \mathscr{H}_1$,

Case 1-c: $u_L \in \mathscr{H}_3$ and $u_R \in \mathscr{H}_3$,

Case 1-d: $u_L \in \mathscr{H}_3$ and $u_R \in \mathscr{H}_1$.

Cases 1-c and 1-d are very similar to cases 1-b and 1-a, respectively. (Use the transformation $x \to -x$ and the fact that the equations (1.1) and more generally all the requirements in the formulation of Section 1 are invariant under this transformation.) So we omit cases 1-c and 1-d and focus on cases 1-a and 1-b.

Case 1-a. Suppose that $u_L \in \mathscr{H}_1$ and $u_R \in \mathscr{H}_3$.

We must construct a solution to (1.1), (2.1) which is admissible in the sense of Section 1. The solution necessarily contains a phase boundary (and only one as was checked in [2]) with phase-1 states at the left and phase-3 states at the right. Two different wave structures are possible, depending on whether the phase boundary is *subsonic* or *supersonic*. Let V be the speed of the phase boundary and set

$$h_{LR} = h_u(0,x) = \frac{1}{c_1 + c_3}(c_1 w_L + c_3 w_R + v_R - v_L). \tag{2.3}$$

We distinguish between two cases depending on the sign of h_{LR}.

Case 1-a1. Suppose moreover that $h_{LR} > 0$.

In this case, we seek the solution u in the form

$$u(t,x) = \begin{cases} u_L & \text{for } x < -c_1 t, \\ u_- & \text{for } -c_1 t < x < Vt, \\ u_+ & \text{for } Vt < x < c_3 t, \\ u_R & \text{for } x > c_3 t, \end{cases} \tag{2.4}$$

where the constants $u_- = (v_-, w_-)$ and $u_+ = (v_+, w_+)$ belong to $\mathscr{H}_1$ and $\mathscr{H}_3$ respectively (cf. Figure 2.1). The solution contains a contact discontinuity of speed $-c_1$, the phase boundary with subsonic speed $|V| < c_3$ and a con-

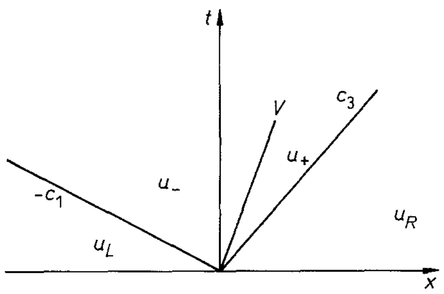

Figure 2.1

tact discontinuity with speed c_3. We are going to prove that such a solution indeed exists by determining explicitly the values of the constants u_-, u_+ and V.

First of all, u given by (2.4) must be a weak solution to (1.1), and so must satisfy the four Rankine-Hugoniot relations

$$\begin{aligned} &c_1(w_- - w_L) - (v_- - v_L) = 0, \quad V(w_+ - w_-) + v_+ - v_- = 0, \\ &V(v_+ - v_-) + c_3^2 w_+ - c_1^2 w_- = 0, \quad c_3(w_+ - w_R) + v_+ - v_R = 0. \end{aligned} \tag{2.5}$$

If V is chosen as a parameter, then (2.5) yields explicit expressions for v_-, v_+, w_-, w_+ as functions of V:

$$\begin{aligned} v_- &= v_L - c_1 w_L + \frac{c_3+V}{c_1+V} c_1 h_{LR}, \quad w_- = \frac{c_3+V}{c_1+V} h_{LR}, \\ v_+ &= v_R + c_3 w_R - \frac{c_1-V}{c_3-V} c_3 h_{LR}, \quad w_+ = \frac{c_1-V}{c_3-V} h_{LR}. \end{aligned} \tag{2.6}$$

Formulas (2.6) define a *one-parameter family of solutions* to problem (1.1), (2.1). Note that w_- and w_+ are always non-negative.

Next we take into account the kinetic relation that states (cf. (1.10) and (1.11)):

$$U(u_+) - U(u_-) + \frac{1}{V}(F(u_+) - F(u_-)) = \phi(V),$$

or using the more general form (1.10)′:

$$\int_{w_-}^{w_+} \{\sigma(y) - \tfrac{1}{2}(\sigma(w_+) + \sigma(w_-))\} dy = \phi(V).$$

Using the expression (1.2) for the function σ, we convert this to

$$\tfrac{1}{2}(k_1 - k_3)(w_M w_m - w_+ w_-) = \phi(V). \tag{2.7}$$

If we use in (2.7) the expressions for w_+ and w_- given by (2.6), it follows that

(2.8)

$$\theta(V) = \phi(V), \quad \text{where } \theta(V) = \frac{(k_1 - k_3)}{2}\left\{ w_M w_m - \frac{(c_3+V)(c_1-V)}{(c_1+V)(c_3-V)} h_{LR}^2 \right\}.$$

Note that the function θ *depends only on the averaged strain* h_{LR}. In view of our set of assumptions (1.8) and

$$\theta' < 0, \quad \theta(-c_3) = \overline{\psi}(c_3), \quad \lim_{V \to c_3} \theta(V) = -\infty,$$

one easily checks that equation (2.8) admits a unique root V (cf. Figure 2.2). Moreover, if this specific value of V is used in (2.6) to get w_-, w_+, v_- and

v_+, then our construction is consistent in the sense that

(2.9) $$w_- \in \mathscr{H}_1, \quad w_+ \in \mathscr{H}_3.$$

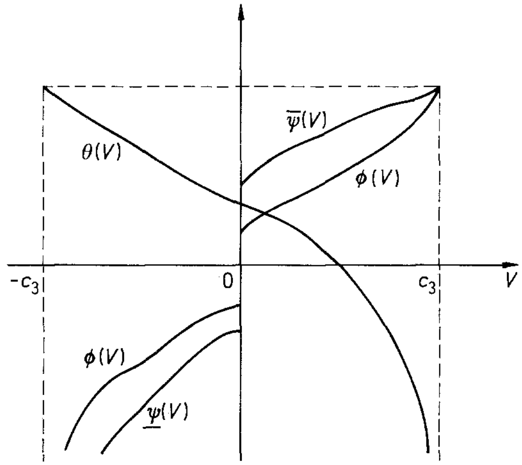

Figure 2.2

We now prove (2.9) by using the assumption (1.8e) on the function ϕ (a stronger assumption was made in [2] to derive (2.9)).

Let us for instance check that $w_- \in \mathscr{H}_1$, in other words, that $w_- \leqq w_M$. In view of (1.8e) and (2.8), one has $\underline{\psi}(V) < \theta(V)$, so that by (1.9a) and (2.8)

$$w_M w_m - \frac{k_1 - V^2}{k_3 - V^2} w_M^2 < w_M w_m - \frac{(c_3 + V)(c_1 - V)}{(c_1 + V)(c_3 - V)} h_{LR}^2.$$

Since $|V| < c_3$, we obtain

$$w_M^2 > h_{LR}^2 \frac{(c_1 + V)^2}{(c_3 + V)^2}.$$

But $w_M > 0$ and $h_{LR} > 0$ by assumption, so

$$w_M > h_{LR} \frac{c_1 + V}{c_3 + V} = w_-,$$

in view of the expressions (2.6). The proof of (2.9) is complete.

Finally, we note that the entropy inequality (1.6) is trivially satisfied along the contact waves, while it is a consequence of the kinetic relation (1.10), (1.11) along the phase boundary. Thus, in the present case, (1.6) yields no additional constraint.

The above construction yields the admissible weak solution of the problem. Based on the explicit expressions (2.6) and the implicit equation (2.8), it is elementary to prove the following regularity result for the Riemann solution.

Lemma 2.1. *Consider the Riemann problem* (1.1), (2.1) *in case 1-a1, i.e., with* $u_L \in \mathscr{H}_1$, $u_R \in \mathscr{H}_3$ *and* $h_{LR} > 0$. *Then the admissible weak solution to this problem is given by formulas* (2.4), (2.6) *and* (2.8). *One can consider the states* u_- *and* u_+ *and the speed* V *in* (2.4) *as functions of the initial states* u_L *and* u_R, *or more precisely,*

$$v_- = v_-(u_L, h_{LR}), \quad w_- = w_-(h_{LR}),$$

$$v_+ = v_+(u_R, h_{LR}), \quad w_+ = w_+(h_{LR}), \quad V = V(h_{LR}).$$

The functions u_-, u_+ *and* V *are Lipschitz continuous in the range of values* $\{u_L \in \mathscr{H}_1, u_R \in \mathscr{H}_3 \mid h_{LR} > 0\}$. *They are of class* $\mathscr{C}^1$ (*with Lipschitz continuous derivatives*) *away from* $V = 0$. *The behavior of* u_-, u_+ *and* V *when* $h_{LR} \to 0^+$ *is given by*

$$\lim_{h_{LR}\to 0+} v_- = v_L - c_1 w_L, \quad \lim_{h_{LR}\to 0+} w_- = 0, \tag{2.10a}$$

$$\lim_{h_{LR}\to 0+} v_+ = v_R + c_3\left\{w_R - \sqrt{\frac{\phi'(c_3)}{c_3}}\right\}, \quad \lim_{h_{LR}\to 0+} w_+ = \sqrt{\frac{\phi'(c_3)}{c_3}}, \tag{2.10b}$$

$$\lim_{h_{LR}\to 0+} V = c_3, \quad \lim_{h_{LR}\to 0+} \frac{dV}{dh_{LR}} = (c_3 - c_1)\sqrt{\frac{c_3}{\phi'(c_3)}}, \tag{2.10c}$$

$$\lim_{h_{LR}\to 0+} \frac{\partial v_-}{\partial h_{LR}} = \frac{2c_1c_3}{c_1 + c_3}, \quad \lim_{h_{LR}\to 0+} \frac{dw_-}{dh_{LR}} = \frac{2c_3}{c_1 + c_3}. \tag{2.10d}$$

Remark 2.1. 1) Assumptions (1.8a) and (1.8d) imply that ϕ^{-1} exists and is a Lipschitz continuous function. Away from $V = 0$, ϕ^{-1} is of class $\mathscr{C}^2$, and so is the function $V(h_{LR})$ in view of (2.8).

2) If the function $\phi \in \mathscr{C}^2(]-c_3, c_3])$, then all the functions in Lemma 2.1 are globally of class $\mathscr{C}^2$. (This is *not* the case of the maximally dissipative function quoted in Remark 1.2.)

3) In the special case that $\phi = \phi_{\max}$, we find $\sqrt{\phi'(c_3)/c_3} = w_m$.

Case 1-a2. Suppose now that $h_{LR} \leq 0$.

In this case, the solution is composed of a contact discontinuity with speed $-c_1$ and a phase boundary with *supersonic* speed $V > c_3$. There is no c_3 contact wave. We use the notation

$$u(t,x) = \begin{cases} u_L & \text{for } x < -c_1 t, \\ u_- & \text{for } -c_1 t < x < Vt, \\ u_R & \text{for } x > Vt. \end{cases} \tag{2.11}$$

The state $u_- = (v_-, w_-) \in \mathscr{H}_1$ and the speed V must satisfy the jump relations

$$c_1(w_- - w_L) - (v_- - v_L) = 0,$$

$$V(w_R - w_-) + v_R - v_- = 0, \quad V(v_R - v_-) + c_3^2 w_R - c_1^2 w_- = 0.$$

We thus get v_- and w_- explicitly as functions of V:

(2.12a) $$v_- = v_L - c_1 w_L + \frac{c_3 + V}{c_1 + V} c_1 h_{LR}, \quad w_- = \frac{c_3 + V}{c_1 + V} h_{LR},$$

the speed V being given by the implicit algebraic equation

(2.12b)

$$V^2\{-c_3 w_R + (c_1 + c_3) h_{LR}\} + (c_3^2 - c_1^2) w_R V + (c_1 + c_3) c_1 (c_3 w_R - c_1 h_{LR}) = 0.$$

Note that w_- given by (2.12a) is always non-positive.

One can check [2] that (2.12b) has a unique solution, which indeed belongs to the (physically interesting) interval $[c_3, c_1[$, if and only if h_{LR} satisfies the restriction

(2.13)

$$h_\infty < h_{LR} \leqq 0 \quad \text{with } h_\infty = \frac{1}{c_1 + c_3}\{c_3 w_R - c_1 - (c_3^2 w_R^2 + (c_1^2 + c_3^2) w_R + c_1^2)^{1/2}\}.$$

In other words, the Riemann problem can be solved when $h_{LR} \leqq 0$ if and only if $h_{LR} > h_\infty$. We emphasize that the kinetic relation was not used here and V is found to be supersonic; this is in complete agreement with the fact that (1.11) is imposed only for subsonic phase boundaries.

Lemma 2.2. *Consider the Riemann problem* (1.1), (2.1) *in the case 1-a2 that* $u_L \in \mathscr{H}_1$, $u_R \in \mathscr{H}_3$ *and* $h_{LR} \leqq 0$ (*with the restriction* (2.13)). *Then the admissible weak solution to this problem is given by* (2.11), (2.12). *One can consider the state* u_- *and the speed* V *as functions of the initial states* u_L *and* u_R, *or more precisely,*

$$v_- = v_-(u_L, h_{LR}, w_R), \quad w_- = w_-(h_{LR}, w_R), \quad V = V(h_{LR}, w_R).$$

Then the functions u_- *and* V *are* $\mathscr{C}^\infty$ *functions of their arguments, and when* $h_{LR} \to 0-$, *they satisfy*

(2.14a) $$\lim_{h_{LR} \to 0-} v_- = v_L - c_1 w_L, \quad \lim_{h_{LR} \to 0-} w_- = 0,$$

(2.14b) $$\lim_{h_{LR} \to 0-} V = c_3, \quad \lim_{h_{LR} \to 0-} \frac{\partial V}{\partial h_{LR}} = \frac{(c_1 + c_3)^2 (c_1 - c_3)}{(c_1^2 + c_3^2) w_R},$$

(2.14c) $$\lim_{h_{LR} \to 0-} \frac{\partial v_-}{\partial h_{LR}} = \frac{2 c_1 c_3}{c_1 + c_3}, \quad \lim_{h_{LR} \to 0-} \frac{dw_-}{dh_{LR}} = \frac{2 c_3}{c_1 + c_3}.$$

Remark 2.2. The limits found in (2.14a) and (2.14c) coincide with those in (2.10a) and (2.10d) respectively. This implies that, *except* at those points where $V = 0$, the function $u_- = (v_-, w_-)$ is of class $\mathscr{C}^1$ (with Lipschitz continuous derivatives) in the *whole* domain $\{u_L \in \mathscr{H}_1, u_R \in \mathscr{H}_3 \mid h_{LR} > h_\infty\}$.

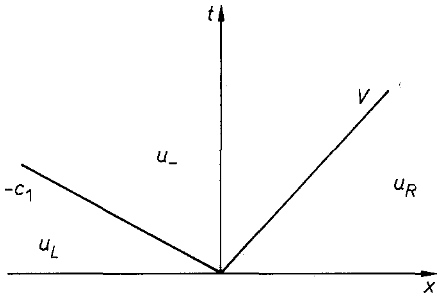

Figure 2.3. Case 1-a2

Case 1-b. Suppose that $u_L \in \mathscr{H}_1$ and $u_R \in \mathscr{H}_1$. Here, by definition,

$$h_{LR} = \frac{1}{2c_1}\,(c_1 w_L + c_1 w_R + v_R - v_L) = \frac{w_L + w_R}{2} + \frac{v_R - v_L}{2c_1}. \tag{2.15}$$

According to our formulation (1.16) of the initiation criterion, the solution takes its values in the phase-1 region only if $h_{LR} \leqq w_M$, while a phase-3 state appears in the solution if h_{LR} exceeds w_M. We distinguish between these two situations.

Case 1-b1. Suppose moreover that $-1 < h_{LR} \leqq w_M$.

We seek the solution in the form of three contact states separated by a $-c_1$ constant wave and a c_1 contact wave (cf. Figure 2.4):

$$u(t,x) = \begin{cases} u_L & \text{for } x < -c_1 t, \\ u_* & \text{for } -c_1 t < x < c_1 t, \\ u_R & \text{for } x > c_1 t. \end{cases} \tag{2.16}$$

The intermediate state $u_* = (v_*, w_*) \in \mathscr{H}_1$ must satisfy the jump conditions

$$-c_1(w_* - w_L) + v_* - v_L = 0, \quad c_1(w_R - w_*) + v_R - v_* = 0,$$

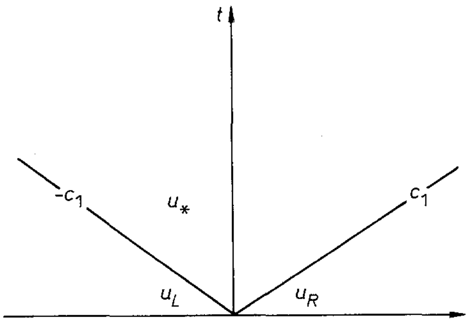

Figure 2.4. Case 1-b1

which lead us to explicit expressions

$$v_* = v_L + c_1(h_{LR} - w_L), \quad w_* = h_{LR}. \tag{2.17}$$

Because of the assumption that $-1 < h_{LR} \leqq w_M$, it is immediate that w_* belongs to $\mathscr{H}_1$. Note that no solution taking its values in $\mathscr{H}$ exists when $h_{LR} < -1$.

For further reference, we state

Lemma 2.3. *Consider the Riemann problem* (1.1), (2.1) *in case 1-b1, i.e., when* $u_L \in \mathscr{H}_1$, $u_R \in \mathscr{H}_3$ *and* $-1 < h_{LR} \leqq w_M$. *Then the admissible weak solution to these problems is given by* (2.16), (2.17). *The state* $u_* = (v_*, w_*)$ *is a* $\mathscr{C}^\infty$ *function of the initial states* u_L *and* u_R. *Moreover, when* h_{LR} *tends to* w_M,

$$\lim_{h_{LR}\to w_M^-} v_* = v_L + c_1(w_M - w_L), \quad \lim_{h_{LR}\to w_M^-} w_* = w_M, \tag{2.18a}$$

$$\lim_{h_{LR}\to w_M^-} \frac{\partial v_*}{\partial h_{LR}}(u_L, h_{LR}) = c_1, \quad \lim_{h_{LR}\to w_M^-} \frac{dw_*}{dh_{LR}} = 1. \tag{2.18b}$$

Remark 2.3. 1) It is of interest to note that the assumption that $-1 < h_{LR} \leqq w_M$ in Lemma 2.3 is always fulfilled if both u_L and u_R belong to $\mathscr{H}_1$ and $|u_R - u_L|$ is small enough. This is clear in view of (2.15).

2) In [2], an initiation criterion was introduced in the case 1-b1. Indeed, instead of the solution (2.16) containing no phase boundary, the criterion in [2] selects in some cases a solution containing two phase boundaries (cf. (2.19) below).

Case 1-b2. Suppose now that $h_{LR} > w_M$.

According to our initiation criterion (1.16), the solution must contain (at least) one phase-3 state. We seek the solution in the form (Figure 2.5)

$$u(t,x) = \begin{cases} u_L & \text{for } x < -c_1 t, \\ u_1 & \text{for } -c_1 t < x < V', \\ u_2 & \text{for } V' < x < V, \\ u_3 & \text{for } V < x < c_1 t, \\ u_R & \text{for } x > c_1 t, \end{cases} \tag{2.19}$$

where u_1, $u_3 \in \mathscr{H}_1$ and $u_2 \in \mathscr{H}_3$ and $-c_3 < V' < 0 < V < c_3$. The jump conditions read

$$\begin{aligned} -c_1(w_1 - w_L) + v_1 - v_L = 0, &\qquad c_1(w_R - w_3) + v_R - v_3 = 0, \\ V'(w_2 - w_1) + v_2 - v_1 = 0, &\qquad V'(v_2 - v_1) + c_3^2 w_2 - c_1^2 w_1 = 0, \\ V(w_R - w_3) + v_R - v_3 = 0, &\qquad V(v_R - v_3) + c_3^2 w_R - c_1^2 w_3 = 0. \end{aligned} \tag{2.20a}$$

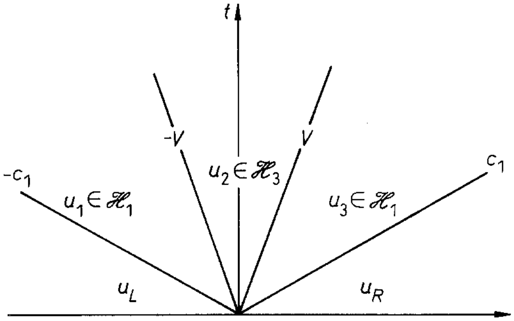

Figure 2.5. Case 1-b2

They must be completed by the kinetic relation along the lines $x/t = V'$ and $x/t = V$:

(2.20b) $$\frac{1}{2}\,(k_1 - k_3)\,(w_M w_m - w_1 w_2) = -\phi(-V')\,,$$
$$\frac{1}{2}\,(k_1 - k_3)\,(w_M w_m - w_2 w_3) = \phi(V)\,.$$

It can be shown that, in fact, $V' = V$ (cf. our previous calculations, (2.7)). As for case 1-a1, it can be checked that (2.20) determine uniquely the admissible solution. We omit the details and simply state our result of continuous dependence.

Lemma 2.4. *Consider the Riemann problem* (1.1), (2.1) *in case 1-b2, i.e., when* $u_L \in \mathscr{H}_1$, $u_R \in \mathscr{H}_1$ *and* $h_{LR} > w_M$. *Then the admissible weak solution of this problem is given by* (2.19), (2.20). *One can consider the states* u_1, u_2, u_3 *and the speed* V *as functions of* u_L *and* u_R. *Then the functions* u_j *and* V *are in* $\mathscr{C}^1$ (*with Lipschitz continuous derivatives*) *and*

(2.21a) $$\lim_{h_{LR}\to w_M} V = 0\,, \qquad \lim_{h_{LR}\to w_M} v_j = v_L + c_1(w_M - w_L) \text{ for } j = 1, 2 \text{ or } 3,$$

(2.21b) $$\lim_{h_{LR}\to w_M} w_1 = \lim_{h_{LR}\to w_M} w_3 = w_M\,, \qquad \lim_{h_{LR}\to w_M} w_2 = \frac{k_1}{k_3}\, w_M\,,$$

(2.21c) $$\lim_{h_{LR}\to w_M} \frac{\partial v_j}{\partial h_{LR}} = c_1\,, \qquad \lim_{h_{LR}\to w_M} \frac{\partial w_j}{\partial h_{LR}} = 1 \text{ for } j = 1 \text{ or } 3\,.$$

Note that the limits found in (2.18) and (2.21) for the functions v_* and w_* and v_j and w_j (for $j = 1$ or 3) coincide. Hence, if in case 1-b1 we set

(2.22) $$v_j = v_*\,, \qquad w_j = w_* \text{ for } j = 1 \text{ or } 3\,,$$

then the functions v_j and w_j are globally of class $\mathscr{C}^1$ with Lipschitz continuous derivatives in the whole range of values $\{u_L \in \mathscr{H}_1, u_R \in \mathscr{H}_3\}$.

From Lemmas 2.1 to 2.4, we deduce the following property of *continuous dependence* of the solution of the Riemann problem.

Theorem 2.1. *Consider the admissible weak solution to the Riemann problem* (1.1), (2.1) *described in Lemmas 2.1 to 2.4. Then the states and the wave speeds in the solution are locally Lipschitz continuous functions of the initial constant states u_L and u_R. As a consequence, if $u_1(\cdot, 0)$ and $u_2(\cdot, 0)$ are two Riemann initial data for system* (1.1), *the corresponding admissible solutions u_1 and u_2 satisfy the L^1-continuity property*:

$$\int_A^B |u_2(t,x) - u_1(t,x)|\, dx \leqq O(1) \int_{A-c_1 t}^{A+c_1 t} |u_2(0,x) - u_1(0,x)|\, dx \tag{2.23}$$

for all $A < B$ and $t \geqq 0$.

We now turn to the Riemann problem in the half space $]0, \infty[$, i.e., problem (1.1), (2.2). Two cases must be distinguished:
case 2-a: $u_0 \in \mathscr{H}_3$,
case 2-b: $u_0 \in \mathscr{H}_1$.

Case 2-a. Suppose that $u_0 \in \mathscr{H}_3$.

According to condition $(1.20)_i$, the solution must contain a phase boundary if and only if $h_0 < w_m^{\text{cr}}$ where

$$h_0 = w_0 + \frac{v_0}{c_3} \tag{2.24}$$

and w_m^{cr} is the critical value for initiation introduced in Section 1. We recall that $w_0 \geqq w_m$ and $w_m^{\text{cr}} \geqq w_m$.

Case 2-a1. Suppose moreover that $h_0 \geqq w_m^{\text{cr}}$.

Then the solution u to (1.1), (2.2) must stay entirely in phase 3, so we seek u in the form

$$u(t,x) = \begin{cases} u_* & \text{for } x < c_3 t, \\ u_0 & \text{for } x > c_3 t. \end{cases} \tag{2.25}$$

(Cf. Figure 2.6.) To satisfy the boundary condition (1.17a), we must have

$$v_* = 0. \tag{2.26a}$$

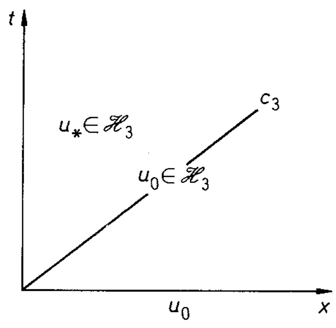

Figure 2.6. Case 2-a1

From the Rankine-Hugoniot relation $c_3(w_0 - w_*) + v_0 - v_* = 0$, we obtain w_*:

(2.26 b) $$w_* = w_0 + \frac{v_0}{c_3} = h_0.$$

The solution is completely determined by (2.25), (2.26).

Case 2-a2. Suppose now that $h_0 < w_m^{\text{cr}}$.

In this case, the solution must contain a phase boundary, so we set

(2.27) $$u(t,x) = \begin{cases} u_- & \text{for } x < Vt, \\ u_+ & \text{for } Vt < x < c_3 t, \\ u_0 & \text{for } x > c_3 t \end{cases}$$

(cf. Figure 2.7). In view of the boundary condition (1.17 a), one has

$$v_- = 0.$$

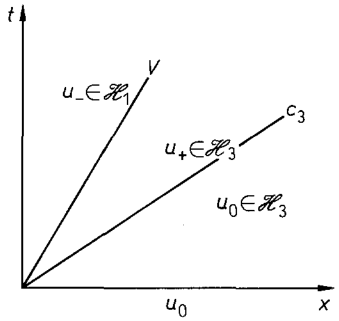

Figure 2.7. Case 2-a2

We determine v, w_- and $u_+ = (v_+, w_+)$ by writing the Rankine-Hugoniot relations satisfied along the lines $x/t = V$ and $x/t = c_1$, as well as the kinetic relation along $x/t = V$. By calculations similar to those made in case 1-a1, we obtain the formulas

(2.28 a) $$v_- = 0, \quad w_- = \frac{c_3 + V}{c_1 + V} h_0,$$

(2.28 b) $$v_+ = v_0 + c_3 w_0 - \frac{c_1 - V}{c_3 - V} c_3 h_0 = -\frac{c_1 - c_3}{c_3 - V} c_3 h_0, \quad w_+ = \frac{c_1 - V}{c_3 - V} h_0,$$

where V is given by the implicit equation

(2.28 c) $$\frac{k_1 - k_3}{2} \left\{ w_M w_m - \frac{(c_3 + V)\,(c_1 - V)}{(c_1 + V)\,(c_3 - V)} h_0^2 \right\} = \phi(V).$$

These formulas determine the solution in this case.

Case 2-b. Suppose that $u_0 \in \mathscr{H}_1$.

According to condition $(1.20)_{ii}$, we have to distinguish between two cases. Here $h_0 = w_0 + (v_0/c_1)$.

Case 2-b1. Suppose moreover that $h_0 \leqq w_M^{\text{cr}}$.

Then the solution stays entirely in phase 1:

$$u(t,x) = \begin{cases} u_* & \text{for } x < c_1 t, \\ u_0 & \text{for } x > c_1 t \end{cases} \tag{2.29}$$

(cf. Figure 2.8), where

$$v_* = 0, \quad w_* = \frac{v_0}{c_1} + w_0 = h_0 \in \mathscr{H}_1. \tag{2.30}$$

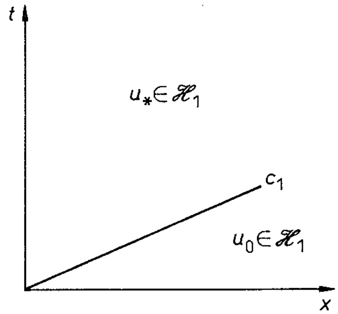

Figure 2.8. Case 2-b1

Case 2-b2. Suppose moreover that $h_0 > w_M^{\text{cr}}$.

Then the solution contains a phase boundary, i.e.,

$$u(t,x) = \begin{cases} u_- & \text{for } x < Vt, \\ u_+ & \text{for } Vt < x < c_1 t, \\ u_0 & \text{for } x > c_1 t, \end{cases} \tag{2.31}$$

where $u_- \in \mathscr{H}_3$, $u_+ \in \mathscr{H}_1$ and $V \in]0, c_1[$. (Cf. Figure 2.9.) The states u_-, u_+ and the speed V are uniquely determined by (1.17a), the Rankine-Hugoniot relations and the kinetic relation. We omit the details.

Finally, we conclude with the result of L^1-continuous dependence for the Riemann problem in a half space.

Theorem 2.2. *Consider the admissible weak solution of the Riemann problem* (1.1), (2.2) *described by cases 2. The states and the wave speeds in the solution are Lipschitz continuous functions of the initial state* u_0. *As a consequence, if* u_0' *and* u_0'' *are two Riemann data for the system* (1.1) *in the half space, then the cor-*

responding admissible solutions u' and u'' satisfy

$$\int_A^B |u'(t,x) - u''(t,x)|\, dx \leqq O(1)\ (B + c_1 t - \max(0, A - c_1 t))\, |u_0' - u_0''|$$

for all $0 < A < B$ *and* $t \geqq 0$.

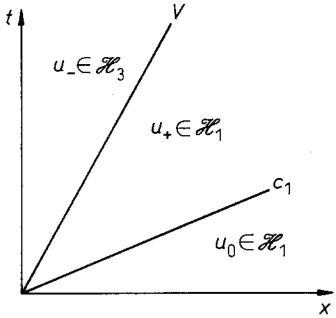

Figure 2.9. Case 2-b2

3. Existence via Glimm's Scheme: Stability

This section and the following one deal with the application of the random choice method, introduced by Glimm [15] for hyperbolic problems, to the system of mixed type (1.1). Our main result establishes existence of a class of admissible weak solutions to (1.1). This serves to justify the formulation of the Cauchy problem proposed in Section 1.

Based on successive solutions of Riemann problems, Glimm's method yields a sequence of approximate solutions for the Cauchy problem associated with (1.1). Our goal is to prove the convergence of these approximate solutions to an admissible weak solution to the problem, in case the initial data form a small *BV* perturbation of a single propagating phase boundary. The main result of this section, Theorem 3.1, yields the stability of the scheme in the *BV* norm. This guarantees its convergence in the L^1 norm to a function of bounded variation, which indeed is a weak solution of (1.1). Showing that this function is an admissible solution requires a more detailed analysis, which is performed in the next section.

It is emphasized that a phase boundary is a wave with (necessarily) *large* strength. Our result of stability here is related to the ones obtained by Chern [5] and Schochet [37] who treated Glimm's scheme with large data for strictly hyperbolic systems.

We consider the system (1.1) on the whole line ($x \in \boldsymbol{R}$) with the Cauchy data

$$\text{(3.1a)} \qquad u(0,x) = u_0(x) = \begin{cases} u_L^0(x) = (v_L^0(x), w_L^0(x)) & \text{for } x < 0, \\ u_R^0(x) = (v_R^0(x), w_R^0(x)) & \text{for } x > 0. \end{cases}$$

The functions $u_L^0 \in BV_{\rm loc}(\boldsymbol{R}_-, \mathscr{H})$ and $u_R^0 \in BV_{\rm loc}(\boldsymbol{R}_+, \mathscr{H})$ are assumed to be close to two given constant states $u_L^* = (v_L^*, w_L^*)$ and $u_R^* = (v_R^*, w_R^*)$ respectively, i.e.,

$$\text{(3.1 b)} \qquad \|u_L^0 - u_L^*\|_{BV(\boldsymbol{R}_-)} + \|u_R^0 - u_R^*\|_{BV(\boldsymbol{R}_+)} \ll 1 .$$

For definiteness, we consider the case that $u_L^* \in \mathscr{H}_1$ and $u_R^* \in \mathscr{H}_3$. We are assuming that the Riemann problem with data u_L^* and u_R^* is solved by a unique wave, with a single-phase boundary but no contact discontinuity. This assumption allows us to focus our attention on phase boundaries which cause the main difficulty for system (1.1). Let u^* be the solution of this Riemann problem; for some speed V^*, one has

$$\text{(3.2 a)} \qquad u^*(t,x) = \begin{cases} u_L^* & \text{for } x < V^*t, \\ u_R^* & \text{for } x > V^*t. \end{cases}$$

In the case of a characteristic phase boundary, i.e., when $V^* = c_3$, we restrict our attention to the case that *no strong wave* arises from a perturbation of the initial states u_L^* and u_R^*. According to Lemma 2.1 of Section 2 (cf. formulas (2.10a) and (2.10b)), this holds under the following condition:

$$\text{(3.2 b)} \qquad \text{If } V^* = c_3, \text{ then } w_L^* = 0 \text{ and } w_R^* = \sqrt{\frac{\phi'(c_3)}{c_3}} .$$

Note that (3.2b) implies that $h_{LR} = 0$ and $v_R = v_L - \sqrt{c_3 \phi'(c_3)}$. So a Riemann problem with initial data in a neighborhood of u_L and u_R takes its values in the same neighborhood.

We shall prove that problem (1.1), (3.1) admits an admissible weak solution, which has the following structure (cf. Figure 3.1):

$$\text{(3.3 a)} \qquad u(t,x) = \begin{cases} u_L(t,x) & \text{for } x < \chi(t), \\ u_R(t,x) & \text{for } x > \chi(t), \end{cases}$$

where

$$\text{(3.3 b)} \qquad u_L \in L^\infty_{\rm loc}([0,\infty[, BV(\boldsymbol{R}, \mathscr{H}_1)), \qquad u_R \in L^\infty_{\rm loc}([0,\infty[, BV(\boldsymbol{R}, \mathscr{H}_3)),$$

$$\text{(3.3 c)} \qquad \chi \in W^{1,\infty}_{\rm loc}([0,\infty[, \boldsymbol{R}), \qquad \frac{d\chi}{dt} \in BV_{\rm loc}([0,\infty[, \boldsymbol{R}).$$

Setting

$$\tilde{u}^*(t,x) = \begin{cases} u_L^* & \text{for } x < \chi(t), \\ u_R^* & \text{for } x > \chi(t), \end{cases}$$

we shall also show that for all times $T > 0$,

$$\text{(3.4)} \qquad \|u(T) - \tilde{u}^*(T)\|_{L^\infty(\boldsymbol{R},\mathscr{H})} + TV_{\boldsymbol{R}}\big(u(T) - \tilde{u}^*(T)\big) + TV_o^T\left(\frac{d\chi_0}{dt} - V^*\right) \ll 1 .$$

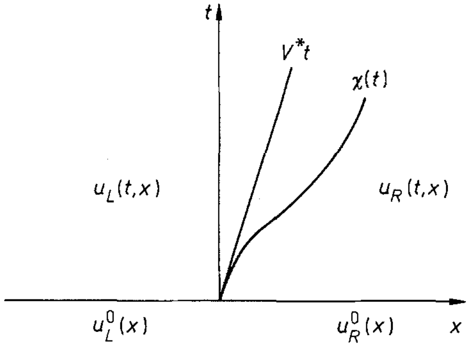

Figure 3.1

The solution will be obtained as the limit of approximate solutions to (1.1), (3.1), having a structure similar to that described by (3.3), (3.4). These approximate solutions are given by Glimm's scheme, which we now describe.

Let $\tau > 0$ and $h > 0$ be time and space mesh sizes satisfying the Courant-Friedrichs-Lewy (CFL) condition $\tau c_1 < h$. The ratio $\lambda = h/\tau$ is taken to be a constant. Let $\{a_n\}_{n \geqq 1}$ be an equidistributed sequence with values in the interval $]-1, 1[$. We define $u^h(0,x)$ by L^2-projection from the data u_0:

$$(3.5\text{a}) \qquad u^h(0,x) = \frac{1}{2h} \int_{mh}^{(m+2)h} u_0(y)\, dy \quad \text{for } x \in [mh, (m+2)h[\text{ with } m \text{ even}.$$

Note here that $|u^h(0,x) - u_L^*| \ll 1$ for $x < 0$ and $|u^h(0,x) - u_R^*| \ll 1$ for $x > 0$; also, u^h satisfies $u^h(0,x) \in \mathscr{H}_1 \cup \mathscr{H}_3$. If u^h is known up to the time $t = n\tau - 0$, we define $u^h(n\tau + 0, x)$ by a random choice projection using a_n:

$$(3.5\text{b}) \qquad u^h(n\tau + 0, x) = u^h(n\tau - 0, (m + 1 + a_n)\, h - 0)$$

for $x \in [mh, (m+2)\,h[$ with $m + n$ even. Then the approximate solution u^h in the strip $\{n\tau \leqq t < (n+1)\,\tau\}$ is computed by solving the Riemann problems for system (1.1) at each center $x = mh$ with $m + n$ even.

As a consequence of our result of stability below, this construction indeed makes sense and yields $u^h(t,x)$ for all times $t \geqq 0$. In particular, because of the assumption (3.2b), the values $u^h(t,x)$ stay in the neighborhoods of u_L^* or u_R^*. This implies that case 1-b2 of Section 2 never occurs here. The possible wave structures of the Riemann problem used in the construction of u^h are listed in Figure 3.2.

Remark 3.1. As a very first step toward a general proof of convergence of u^h, we may consider the case when $_L^0$ and u_R^0 are constant, equal to u_L^* and u_R^* respectively. In this case, u^h can be computed explicitly and consist for each time t of a single-phase discontinuity connecting u_L^* at the left to u_R^* at the right. The position of the phase discontinuity, say $\chi^h(t)$, is shifted to the left or to the right (depending on a_n and the speed V^*) at each time $n\tau$. This is typical behavior for Glimm's scheme, which, as is well known, does not produce any numerical diffusion of the discontinuities. Using only the equidistribution of $\{a_n\}$, we easily show that $\chi^h(t)$ converges to $V^*t \equiv \chi(t)$ for each time $t \geqq 0$. (Cf. Figure 3.3.)

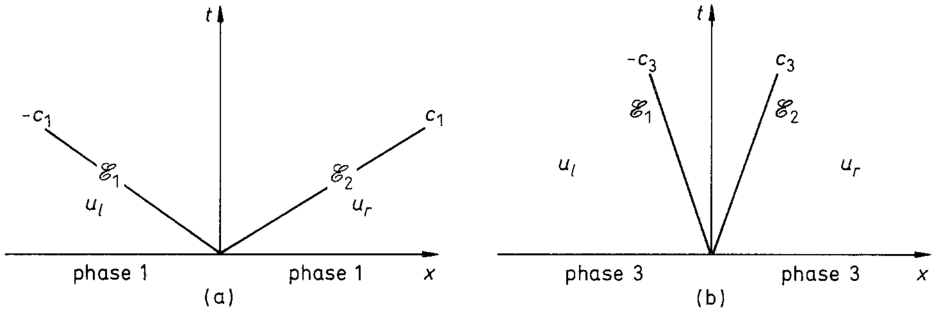

Figure 3.2

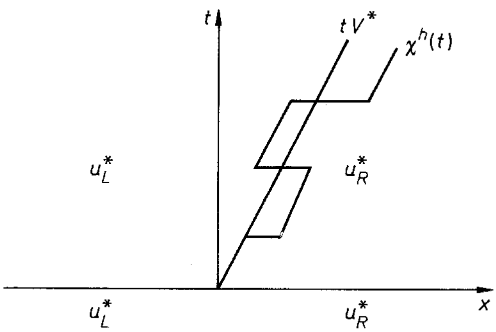

Figure 3.3

According to Glimm's technique, the first step toward a proof of uniform BV stability for the scheme consists in studying wave interactions between Riemann wave patterns. Let $R(u_l, u_r)$ be the solution of the Riemann problem with initial data u_l at the left and u_r at the right (cf. Section 2 for the explicit construction). The wave strengths are defined first in case $V^* \neq c_3$. We denote by $\mathscr{E}_1(u_l, u_r)$ and $\mathscr{E}_2(u_l, u_r)$ the strengths of the left and right contact waves in $R(u_l, u_r)$ respectively. By convention, $\mathscr{E}_2(u_l, u_r) = 0$ when $R(u_l, u_r)$ contains a supersonic phase boundary so that no right contact discontinuity is present. We denote by $\mathscr{E}_0(u_l, u_r)$ the strength of the phase boundary in $R(u_l, u_r)$ when u_l and u_r are in different phases. By convention, the strengths are always measured in terms of the jump of the variable w across the wave under consideration. When a phase boundary is present, we denote its speed by $\mathscr{V}(u_l, u_r)$.

Consider now the case that $V^* = c_3$. We define $\mathscr{E}_0$, $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{V}$ in the same way as above, except when the Riemann problem $R(u_l, u_r)$ admits a supersonic phase boundary. In this latter case, we virtually split the phase discontinuity into two distinct waves and set

$$\mathscr{E}_2(u_l, u_r) = w_r - w_+|_{h_{lr}=0} - h_{lr}\frac{\partial w_+}{\partial h_{lr}}|_{h_{lr}=0} = w_r - \sqrt{\frac{\phi'(c_3)}{c_3}} - 2c_3 h_{lr}/(c_1 + c_3),$$

$$\mathscr{E}_2(u_l, u_r) = w_+|_{h_{lr}=0} + h_{lr}\frac{\partial w_+}{\partial h_{lr}}|_{h_{lr}=0} - w_- - \sqrt{\frac{\phi'(c_3)}{c_3}} + 2c_3 h_{lr}/(c_1 + c_3) - w_-,$$

where w_- and w_+ are the values taken by the solution of $R(u_l, u_r)$ at the left and at the right of the phase boundary, respectively, and h_{lr} is given by (2.3). We recall that $(\partial w_+/\partial h_{lr})|_{h_{lr}=0}$ is given by Lemma 2.1. In other words, we extend the definition of $\mathscr{E}_0$ and $\mathscr{E}_2$, known for $h_{lr} > 0$, to negative values of h_{lr} so that their extensions are of class $\mathscr{C}^1$.

From the results in Lemmas 2.1 and 2.2, one easily checks that

(3.6) The functions $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_0$ and $\mathscr{V}$ are Lipschitz continuous functions of their arguments (u_l, u_r). The $\mathscr{E}_i$'s are of class $\mathscr{C}^1$ (with Lipschitz continuous derivatives) away from $\mathscr{V} = 0$.

In (3.6) and in the sequel, only states which are close to either u_L^* or u_R^* are considered.

The wave interaction estimates are derived in the following lemma (cf. Figures 3.4 and 3.5).

Lemma 3.1. *Consider states u_l, u_p and u_r which are close to either u_L^* or u_R^*.*
1) *If $u_l \in \mathscr{H}_1$, $u_p \in \mathscr{H}_1$ and $u_r \in \mathscr{H}_3$, then for $j = 0, 1, 2$,*

$$\mathscr{E}_j(u_l, u_r) = \mathscr{E}_j(u_l, u_p) + \mathscr{E}_j(u_p, u_r) + O(1)\,|\,\mathscr{E}_2(u_l, u_p)\,|, \tag{3.7a}$$

$$\mathscr{V}(u_l, u_r) = \mathscr{V}(u_p, u_r) + O(1)\,|\,\mathscr{E}_2(u_l, u_p)\,|. \tag{3.7b}$$

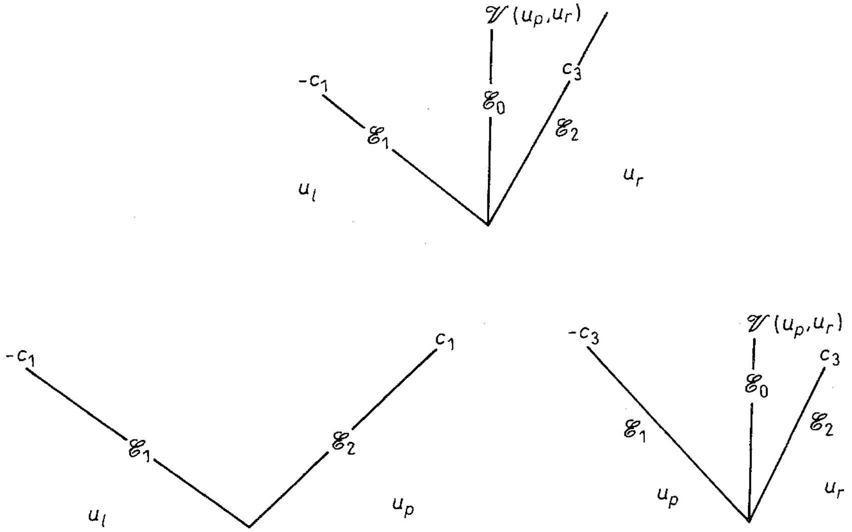

Figure 3.4

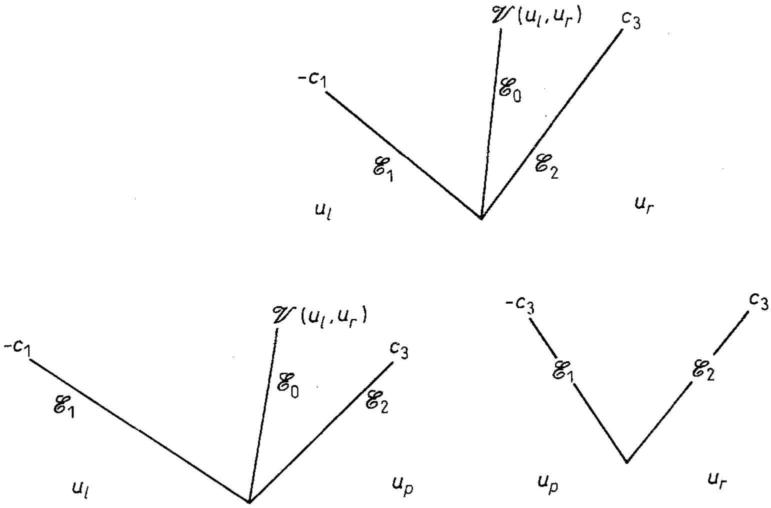

Figure 3.5

2) *If* $u_l \in \mathscr{H}_1$, $u_p \in \mathscr{H}_3$ *and* $u_r \in \mathscr{H}_3$, *then for* $j = 0, 1, 2$,

$$\mathscr{E}_j(u_l, u_r) = \mathscr{E}_j(u_l, u_p) + \mathscr{E}_j(u_p, u_r) + O(1)\, D(u_l, u_p, u_r)\,, \tag{3.8a}$$

$$\mathscr{V}(u_l, u_r) = \mathscr{V}(u_l, u_p) + O(1)\{|\mathscr{E}_1(u_p, u_r)| + |\mathscr{E}_2(u_p, u_r)|\}, \tag{3.8b}$$

where

(3.9)

$$D(u_l,u_p,u_r) = \begin{cases} |\mathscr{E}_1(u_p,u_r)| & \text{if } \mathscr{V}(u_l,u_p) \leqq c_3, \\ |\mathscr{E}_1(u_p,u_r)| + (\mathscr{V}(u_l,u_p) - c_3)|\mathscr{E}_2(u_p,u_r)| & \text{if } \mathscr{V}(u_l,u_p) \geqq c_3. \end{cases}$$

3) *If either* u_l, u_p, $u_r \in \mathscr{H}_1$ *or* u_l, u_p, $u_r \in \mathscr{H}_3$, *then for* $j = 1, 2$,

$$\mathscr{E}_j(u_l,u_r) = \mathscr{E}_j(u_l,u_p) + \mathscr{E}_j(u_p,u_r). \tag{3.10}$$

Remark 3.2. 1) Estimates in Lemma 3.1 mainly contain *linear* interaction terms instead of quadratic ones as is the case in [15]. Linear error terms were previously found useful by [5] and [37] to treat *strictly hyperbolic* systems with *large* data. In (3.9), the interaction term is proportional to the *angle* between the c_3 contact discontinuity and the phase boundary. Such a term was used by LIU [34] to analyze *non-genuinely nonlinear* systems of conservation laws.

2) When $V^* \neq c_3$, the derivation of the estimates in Lemma 3.1 requires only the Lipschitz continuity of the functions $\mathscr{E}_j$ and $\mathscr{V}$, which is exactly the regularity available in general.

3) The smallness condition (on $|u_l - u_L^*|$, etc.) in Lemma 3.1 is necessary only to prevent initiation of a new phase when solving a Riemann problem with data in a single phase.

Proof of Lemma 3.1. We first give the proof of (3.7), and then that of (3.8). The proof of (3.10) is trivial.

In view of the results of Section 2, the functions $\mathscr{E}_0$, $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{V}$ are (at least) Lipschitz continuous functions of their arguments. Hence the formulas (3.7) follow easily if we check that

$$\mathscr{E}_j(u_l,u_r) = \mathscr{E}_j(u_l,u_p) + \mathscr{E}_j(u_p,u_r), \tag{3.11 a}$$

$$\mathscr{V}(u_l,u_r) = \mathscr{V}(u_p,u_r) \tag{3.11 b}$$

hold whenever $\mathscr{E}_2(u_l,u_p) = 0$, i.e., when there is no right wave in the left wave packet $R(u_l,u_p)$. But this last statement is obvious because the left waves in $R(u_l,u_p)$ and $R(u_p,u_r)$ are associated with a linearly degenerate characteristic field. Such waves can be superimposed without any interaction and the wave strengths are simply summed up, cf. (3.11 a). The speed of the phase boundary remains unchanged, cf. (3.11 b). (These facts can be checked directly from the analytical expressions in Section 2.) The proof of (3.7) is complete.

We now prove (3.8). We notice first that (3.11 a) as well as

$$\mathscr{V}(u_l,u_r) = \mathscr{V}(u_l,u_p) \tag{3.11 c}$$

do hold provided that D given by (3.9) vanishes. Specifically, if $\mathscr{V}(u_l,u_p) \leqq c_3$ and if $D(u_l,u_p,u_r) = 0$, then the right wave packet does not contain a $-c_3$ contact wave. In this situation, the two wave patterns can be superimposed, without any interaction. If $\mathscr{V}(u_l,u_p) \geqq c_3$ and if

$D(u_l, u_p, u_r) = 0$, then the right wave packet does not contain a $-c_3$ contact wave and, moreover, either it also has no c_3 contact wave or the speed $\mathscr{V}(u_l, u_p)$ equals c_3. In both situations, the left and right wave patterns can be superimposed. Again, there is no interaction. This proves (3.11 a) and (3.11 c).

When $\mathscr{V}(u_l, u_p) \leqq c_3$, estimates (3.8) follow from (3.11 a), (3.11 c) and the Lipschitz continuity of $\mathscr{E}_j$ and $\mathscr{V}$. When $\mathscr{V}(u_l, u_p) > c_3$, $\mathscr{V}$ is bounded away from $\mathscr{V} = 0$. It then follows from the results in Section 2 that the functions $\mathscr{E}_j$ and $\mathscr{V}$ are of class $W^{2,\infty}$. This fact allows us to apply the classical lemma of division (e.g., [21]) and again to deduce (3.87) from (3.11 a), (3,11 c). The proof of (3.8) is complete. □

We use Glimm's technique to deduce from Lemma 3.1 the result of *BV* stability of the scheme. We refer to [15, 16] for the terminology we use here. At this stage, we have to define functionals to control the total variation of the solutions. The choice we propose is motivated by the form of the terms of interaction found in Lemma 3.1. Note that the phase boundary which is a "strong wave" are treated separately from the "small waves".

The (t, x)-plane is divided into a set of diamonds $\Delta_{m,n}$ with centers $(n\tau, mh)$ $(n + m$ even) and with vertices

$$N = \big((n+1)\,\tau,\, (m + a_{n+1})\,h\big), \quad E = \big(n\tau,\, (m + 1 + a_n)\,h\big),$$

$$W = \big(n\tau,\, (m - 1 + a_n)\,h\big), \quad S = \big((n-1)\,\tau,\, (m + a_{n-1})\,h\big).$$

Given a diamond Δ_{mn}, we denote by u_N, u_E, u_W and u_S the values taken by u^h at the vertices N, E, W and S respectively.

We give now the definition of the *approximate phase boundary* in u^h, which we denote by $\chi^h: \boldsymbol{R}_+ \to \boldsymbol{R}$. First of all, it is a simple (but useful) observation that the phase boundary in u^h is actually located at a single space position for each time $t = n\tau$. In other words, there is no spreading of the phase boundary. Let χ^h be the piecewise linear curve which is discontinuous at each $t = n\tau$ and coincides with the phase boundary in u^h inside each slab $[n\tau, (n+1)\,\tau[$. Let $\mathscr{D}$ be the set of all diamonds that are crossed out by the phase boundary χ^h.

We then introduce several functionals defined on space-like curves, say J, passing through vertices of diamonds. Define

(3.12 a) $$L(J) = \sum \left(|\mathscr{E}_1| + |\mathscr{E}_2|\right),$$

the summation being on all small waves crossing the curve J, and

(3.12 b) $$B(J) = |\mathscr{E}_0|$$

where $\mathscr{E}_0$ is the strength of the phase boundary when crossing the curve J. The functional $L(J)$ bounds the total variation of u^h along the curve J on both sides of the phase boundary. $B(J)$ measures the jump of u^h across the phase boundary. Next we define the potential interaction $Q(\Delta)$ in a diamond

Δ by

$$(3.13)\qquad Q(\Delta) = \begin{cases} 0 & \text{if } \Delta \notin \mathscr{D}, \\ |\mathscr{E}_2(u_W, u_S)| & \text{if } \Delta = \Delta_{m,n} \in \mathscr{D} \text{ and } mh < \chi^h(n\tau), \\ |\mathscr{E}_1(u_S, u_E)| + \theta(\mathscr{V}(u_W, u_S) - c_3)\,|\mathscr{E}_2(u_S, u_E)| & \\ & \text{if } \Delta = \Delta_{m,n} \in \mathscr{D} \text{ and } mh \geqq \chi^h(n\tau), \end{cases}$$

where θ: $\boldsymbol{R} \to \boldsymbol{R}$ is the function defined by

$$\theta(y) = \begin{cases} 0 & \text{for } y < 0, \\ y & \text{for } y \geqq 0. \end{cases}$$

Finally the potential wave interaction $Q(J)$ associated with a curve J is

$$(3.14)\qquad Q(J) = \sum_{\substack{\text{waves at the} \\ \text{left of } \chi^h}} |\mathscr{E}_2| + \sum_{\substack{\text{waves at the} \\ \text{right of } \chi^h}} \{|\mathscr{E}_1| + \theta(V - c_3)\,|\mathscr{E}_2|\},$$

where V is the speed of the phase boundary when crossing J and the summation is taken over all waves crossing J. Note that $Q(J)$ is a linear functional in terms of wave strengths.

Lemma 3.2. *Let K be a sufficiently large constant. Let J_1 and J_2 be two space-like curves, J_2 being a successor of J_1. Then*

$$(3.15\text{a})\qquad L(J_2) + KQ(J_2) \leqq L(J_1) + KQ(J_1),$$

$$(3.15\text{b})\qquad B(J_2) + KQ(J_2) \leqq B(J_2) + KQ(J_2).$$

Proof of Lemma 3.2. We need only prove (3.15) when J_2 is an immediate successor of J_1; the general case follows by induction. We first check the formula

$$(3.16)\qquad Q(J_2) - Q(J_1) \leqq \tfrac{1}{2}\,Q(\Delta),$$

where Δ is the diamond bounded by J_1 and J_2. If $\Delta \in \mathscr{D}$ and the right wave packet contains the phase discontinuity, then in view of (3.14), (3.13),

$$Q(J_2) - Q(J_1) = -|\mathscr{E}_2(u_W, u_W)| = -Q(\Delta) \leqq -\tfrac{1}{2}\,Q(\Delta).$$

If $\Delta \in \mathscr{D}$ and the left wave packet contains the phase discontinuity, then in view of (3.14),

$$\begin{aligned} Q(J_2) - Q(J_1) = {} & -|\mathscr{E}_1(u_S, u_E)| - \theta(\mathscr{V}(u_W, u_S) - c_3)\,|\mathscr{E}_2(u_S, u_E)| \\ & + \{\theta(\mathscr{V}(u_W, u_E) - c_3) - \theta(\mathscr{V}(u_W, u_S) - c_3)\} \sum_{\substack{\text{waves on the right} \\ \text{side of } \Delta}} |\mathscr{E}_2|. \end{aligned}$$

Since θ is Lipschitz continuous, it follows from definition (3.13) and (3.8b) that

$$\begin{aligned} Q(J_2) - Q(J_1) &= -Q(\Delta) + O(1)\,Q(\Delta) \sum_{\substack{\text{waves on the right} \\ \text{side of } \Delta}} |\mathscr{E}_2| \\ &= Q(\Delta)\{-1 + O(1)\,L(J_1)\} \leqq -\tfrac{1}{2}\,Q(\Delta) \end{aligned}$$

where in the last inequality we have assumed that $O(1)L(J_1) \leqq \frac{1}{2}$. The condition $L(J_1) \ll 1$ is indeed ensured by induction if the initial total variation is small enough. Let us content ourselves with checking that $L(J_0) \ll 1$ where J_0 is the curve connecting centers of diamonds on the lines $t=0$ and $t=t_1=\tau$: Using the definition of the wave strengths and (3.2b), one gets

$$L(J_0) = O(1)\{TV^0_{-\infty}(u^0_L) + TV_0^{+\infty}(u^0_R) + \|u^0_L - u^*_L\|_{L^\infty(R_-)} + \|u^0_R - u^*_R\|_{L^\infty(R_+)}\}.$$

The right-hand side of this formula is small in view of (3.1b).

If now $\Delta \notin \mathscr{D}$, then from (3.10) one has trivially

$$Q(J_2) - Q(J_1) \leqq 0, \qquad Q(\Delta) = 0.$$

Thus the proof of (3.16) is complete.

We next consider $L(J_2) - L(J_1)$. If $\Delta \in \mathscr{D}$, then by (3.7a), (3.8a), (3.12a), (3.13),

$$\begin{aligned} L(J_2) - L(J_1) &= |\mathscr{E}_1(u_W, u_E)| + |\mathscr{E}_2(u_W, u_E)| \\ &\quad - |\mathscr{E}_1(u_W, u_S)| - |\mathscr{E}_2(u_W, u_S)| - |\mathscr{E}_1(u_S, u_E)| - |\mathscr{E}_2(u_S, u_E)| \\ &= O(1)\,Q(\Delta). \end{aligned}$$

If $\Delta \notin \mathscr{D}$, then

$$L(J_2) = L(J_1), \qquad Q(\Delta) = 0.$$

This proves the formula

$$L(J_2) = L(J_1) + O(1)\,Q(\Delta). \tag{3.17}$$

From (3.16) and (3.17), we easily deduce (3.15a) provided that the constant K in (3.15a) is large enough.

Finally, it can be proved similarly that

$$B(J_2) = B(J_1) + O(1)\,Q(\Delta), \tag{3.18}$$

which implies (3.15b) in view of (3.16). □

Lemma 3.2 provides a uniform bound for the total variation of $u^h(t)$ at times $t = t_n$. Since

$$TV(u^h(t)) \leqq O(1)\,L(J), \qquad \text{for all times } t \in [t_n, t_{n+1}[,$$

where J is the curve lying between the lines $t=t_n$ and $t=t_{n+1}$, we obtain a uniform control of the total variation of u^h for all times. Let us define the function $\tilde{u}^h$: $\boldsymbol{R}_+ \times \boldsymbol{R} \to \mathscr{H}$ by

$$\tilde{u}^h(t,x) = \begin{cases} u^*_L & \text{if } x < \chi^h(t), \\ u^*_R & \text{if } x > \chi^h(t), \end{cases} \tag{3.19}$$

where χ^h is the approximate phase boundary associated with u^h. From Lemma 3.2, one deduces the following result of stability.

Theorem 3.1. *The functions u^h given by Glimm's scheme applied to the mixed system* (1.1) *and the data* (3.1), (3.2) *satisfy the stability estimates*:

(3.20a) $$TV_{-\infty}^{+\infty}(u^h - \tilde{u}^h)(t) \leqq O(1)\, N_1,$$

(3.20b) $$\|u^h(t) - \tilde{u}^h(t)\|_{L^\infty(\boldsymbol{R},\mathscr{H})} \leqq O(1)\;(N_1 + N_2),$$

(3.20c) $$\|u^h(t) - u^h(t')\|_{L^1(\boldsymbol{R},\mathscr{H})} \leqq O(1)\, N_1(|t - t'| + h)$$

for all times $t \geqq 0$ and $t' \geqq 0$, with N_1 and N_2 defined by

(3.21a) $$N_1 = \begin{cases} TV_{-\infty}^{0}(u_L^0) + TV_0^{\infty}(u_R^0) & \text{if } V^* \neq c_3, \\ TV_{-\infty}^{0}(u_L^0) + TV_0^{\infty}(u_R^0) + N_1 & \text{if } V^* = c_3. \end{cases}$$

(3.21b) $$N_2 = \|u_L^0 - u_L^*\|_{L^\infty(\boldsymbol{R}_-,\mathscr{H}_1)} + \|u_R^0 - u_R^*\|_{L^\infty(\boldsymbol{R}_+,\mathscr{H}_3)}.$$

By Helly's theorem, the estimates (3.20) imply that (a subsequence of) $\{u^h\}$ converges in L^1_{loc} strongly to a function u as $h \to 0$. This function has bounded variation in space and satisfies the same bounds as u^h in (3.20). It is a classical matter (GLIMM [15], LIU [33, 34]) to check that u indeed is a *weak solution* to the system of conservation laws (1.1). It also satisfies the entropy inequality as well as the initial condition. It remains to show that u is admissible, i.e., satisfies the kinetic relation (cf. Section 4).

Remark 3.3. If condition (3.2b) is violated, then perturbation of a characteristic phase boundary produces a c_3-contact wave with strong strength. Then one would have to deal with interaction between two strong waves traveling with arbitrarily close speeds. Initiation of new phases is possible. It is not clear whether the total variation of u^h would remain uniformly bounded in that case.

4. Existence via Glimm's Scheme: Admissibility

In this section, we prove that the weak solution $u = \lim u^h$ found in Section 3 by Glimm's scheme does satisfy the kinetic relation (1.11). This establishes that u is an admissible weak solution to our problem and leads to the desired result of existence and stability.

First of all, we notice that the kinetic relation (1.11) is formulated in a pointwise sense, more precisely, (1.11) must hold almost everywhere with respect to the *Hausdorff measure* H_1. However from the results in Section 3, we only have that u^h converges to u at almost every point with respect to the *Lebesgue measure* on $\boldsymbol{R}_+ \times \boldsymbol{R}$. This latter property is thus not sufficient to pass to the limit in the kinetic relation.

In this section we prove a result of pointwise convergence for the approximate phase boundary χ^h, cf. Theorem 4.1. This result is derived by using the techniques introduced by GLIMM & LAX [16]. The focus of [16] was on a strictly hyperbolic system of two conservation laws with small data. Extensions of the results in [16] can also be found in the papers of DIPERNA [10] and LIU

[34]. In our situation, we have (a special case of) a system of mixed type with large data.

Next, in Theorem 4.2, we prove that the above result is sufficient for the passage to the limit in the kinetic relation, assuming that the speed of the phase boundary V^* does not vanish.

Let us consider the phase boundary χ^h: $\boldsymbol{R}_+ \to \boldsymbol{R}$ for the approximate solution u^h. The function χ^h is discontinuous and piecewise linear. It jumps up to a distance of $\pm 2h$ at each time step. It is easy to verify the following lemma.

Lemma 4.1. *The function χ^h: $\boldsymbol{R}_+ \to \boldsymbol{R}$ satisfies the uniform estimate*

$$(4.1) \qquad |\chi^h(t) - \chi^h(t')| \leqq \frac{1}{\lambda}|t - t'| + 2h \quad \textit{for } 0 \leqq t \leqq t'.$$

By Ascoli's theorem, the sequence $\{\chi^h\}$ must converge on each compact set in the uniform topology to a function $\chi \in W^{1,\infty}_{\mathrm{loc}}([0, \infty[, \boldsymbol{R})$. The next lemma gives a bound for the total variation of the functions $\dot{\chi}^h$: $\boldsymbol{R}_+ \to \boldsymbol{R}$ defined by

$$(4.2) \qquad \dot{\chi}^h(t) = \frac{d\chi^h(t)}{dt} \quad \text{(constant) on each interval } [n\tau, (n+1)\tau[.$$

From an analysis of the waves crossing the phase boundary, we prove the following result. (The proof is given after the statement of Theorem 4.1.)

Lemma 4.2. *For all times $T > 0$, one has the uniform estimate*

$$(4.3) \quad TV_0^T(\dot{\chi}^h) \leqq O(1)\{TV^0_{-(T/\lambda)-2h}(v_L^0 - c_1 w_L^0) + TV_0^{(T/\lambda)+2h}(u_R^0) + N\},$$

where $N = 0$ if $V^* \neq c_3$ and $N = \|u_R^0 - u_L^*\|_{L^\infty(-(T/\lambda)-2h)} + \|u_R^0 - u_R^*\|_{L^\infty(0, T/\lambda+2h)}$ if $V^* = c_3$.

Hence, from Lemmas 4.1 and 4.2, the equidistribution of the sequence $\{a_n\}$ and the arguments in [16], we deduce the following pointwise convergence property.

Theorem 4.1. *The functions χ^h and $\dot{\chi}^h$ converge to the functions χ and $d\chi/dt$ respectively in the sense that*

$$(4.4) \qquad \|\chi - \chi^h\|_{L^\infty(]0,T[,\boldsymbol{R})} \to 0 \textit{ when } h \to 0, \quad \textit{for all } T > 0,$$

$$(4.5) \qquad \dot{\chi}^h(t) \to \frac{d\chi(t)}{dt} \quad \textit{for all times } t \in \boldsymbol{R}_+ \setminus E,$$

where $E \subset \boldsymbol{R}_+$ is an at most countable set.

We give first the proof of Lemma 4.2 and then that of Theorem 4.1.

Proof of Lemma 4.2. Let T be fixed and let N be such that $N\tau \leqq T < (N+1)\tau$. Recall that $\mathscr{D}$ is the set of all diamonds which contain a part of the phase boundary. Let J be the space-like curve which bounds the domain of dependence of the diamonds in $\mathscr{D}$ with centers below the line $t = N\tau$. For each time $t = n\tau$, $n = 0, 1, \ldots, N$, J encloses a finite number of diamonds, which we denote by Δ_n^m for $m = 1, 2, \ldots, N + 1 - n$. They are ordered increasingly. We define $m(n)$ to be such that $\Delta_n^{m(n)} \in \mathscr{D}$. (Cf. Figure 4.1.)

By Lemma 3.1, the speed of χ^h at the time $n\tau$ is estimated from its value at time $(n-1)\tau$:

$$\dot{\chi}(n\tau + 0) = \dot{\chi}^h((n-1)\tau + 0) + O(1) \, |\mathscr{E}(\Delta_{n,m(n)})| , \tag{4.6}$$

where $|\mathscr{E}(\Delta_{n,m(n)})|$ represent the strength of the waves entering the diamond. By summation with respect to $n = 1, \ldots, N$, we obtain

$$\begin{aligned} TV_0^{(N+1)\tau}(\dot{\chi}^h) &= \sum_{n=1}^{N} |\dot{\chi}^h(n\tau + 0) - \dot{\chi}((n-1)\tau + 0)| \\ &= O(1) \sum_{n=0}^{N-1} |\mathscr{E}(\Delta_{n,m(n)})| , \end{aligned}$$

which is bounded by the total variation of u^h measured along both sides of the phase boundary. By using conservation laws for wave strengths, as in [16], one could check that the total variation of u^h along this curve is bounded by the initial total variation. Thus we have

$$TV_0^{(N+1)\tau}(\dot{\chi}^h) = O(1) \sum_{m=1}^{N+1} |\mathscr{E}(\Delta_{0,m})| , \tag{4.7}$$

with

$$\sum_{m=1}^{m(0)} |\mathscr{E}(\Delta_{0,m})| = O(1) \, TV^0_{-(T/\lambda)-2h}(v_L^0 - c_1 w_L^0) , \tag{4.8a}$$

$$\sum_{m=m(0)}^{N+1} |\mathscr{E}(\Delta_{0,m})| = \begin{cases} O(1) \, TV_0^{(T/\lambda)+2h}(u_R^0) & \text{if } V^* \neq c_3 , \\ O(1) \, TV_0^{(T/\lambda)+2h}(u_R^0) + O(1) N & \text{if } V^* = c_3 . \end{cases} \tag{4.8b}$$

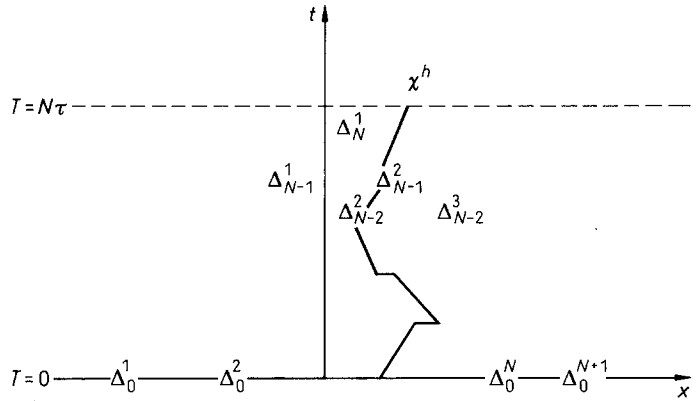

Figure 4.1

Combining (4.7) and (4.8) gives (4.3). The proof of the lemma is complete. □

Proof of Theorem 4.1. In view of Lemma 4.1 and Ascoli's theorem, we have the convergence result (4.4). In view of Lemma 4.2, the total variation of $\dot{\chi}^h$ on a compact set $[0, T]$ is uniformly bounded. By again extracting a subsequence, we use Helly's theorem to obtain

$$\dot{\chi}^h(t) \to k(t) \quad \text{for all times } t \geqq 0, \tag{4.9a}$$

$$TV_0^t(\dot{\chi}^h) \to l(t) \quad \text{for all times } t \geqq 0, \tag{4.9b}$$

where $k \in BV_{\text{loc}}([0, \infty[, \boldsymbol{R})$ and $l \in L^\infty_{\text{loc}}([0, \infty[, \boldsymbol{R}_+)$. We define $E \subset \boldsymbol{R}_+$ as the set of all points of discontinuity of the function l. This set is at most countable since the function l is non-decreasing (and so has bounded variation).

We are going to prove that (4.4) holds with this choice of set E. Let t be in $\boldsymbol{R}_+ \setminus E$, and let $\varepsilon > 0$ be so small that

$$TV_{t-\varepsilon}^{t+\varepsilon}(\dot{\chi}^h) < \varepsilon. \tag{4.10}$$

This is possible because $t \notin E$. Then, in view of (4.9a) and (4.10), we have

$$k(t) - \varepsilon < \dot{\chi}^h(s) < k(t) + \varepsilon \quad \text{for } s \in]t - \varepsilon, t + \varepsilon[. \tag{4.11}$$

On the other hand, we know that the curve χ^h has the slope $\dot{\chi}^h(t)$ on the interval $[n\tau, (n+1)\tau[\ni t$ and jumps by $\pm 2h$ at times $(n+1)\tau$. The slope $\dot{\chi}^h$ of χ^h is "controlled" by inequalities (4.11), while the jumps of χ^h are determined by the given sequence $\{a_n\}$.

Let n' and n'' be two integers such that $(n'-1)\tau \leqq t' < n'\tau$ and $n''\tau \leqq t'' < (n''+1)\tau$, where $t - \varepsilon < t' < t'' < t + \varepsilon$ are given. We set

$$\Omega_+ = \left\{ m \,\middle|\, m \text{ integer}, \ n' \leqq m \leqq n'' \text{ and } a_m < (k(t) - \varepsilon)\,\frac{\tau}{h} \right\},$$

$$\Omega^* = \left\{ m \,\middle|\, m \text{ integer}, \ n' \leqq m \leqq n'' \text{ and } a_m > (k(t) - \varepsilon)\,\frac{\tau}{h} \right\}.$$

In view of (4.11), between times $n'\tau$ and $n''\tau$ the curve χ^h has at least $\#\Omega_*$ jumps to the right and at most $n'' - n' - \#\Omega_*$ to the left; thus we have

$$\chi^h(n''\tau) - \chi^h(n'\tau) \geqq (2\#\Omega_* - n' + n'')\,h. \tag{4.12a}$$

Similarly, for Ω^* we get

$$\chi^h(n''\tau) - \chi^h(n'\tau) \leqq (2\#\Omega_* - n' + n'')\,h. \tag{4.12b}$$

But the equidistribution of $\{a_n\}$ means that

$$\frac{\#\Omega_*}{n'' - n'} \to \frac{1}{2} + (k(t) - \varepsilon)\,\frac{\tau}{2h}, \qquad \frac{\#\Omega^*}{n'' - n'} \to \frac{1}{2} + (k(t) - \varepsilon)\,\frac{\tau}{2h}, \tag{4.13}$$

when $h \to 0$.

Combining (4.12) and (4.13) and letting $h \to 0$ yield the inequalities

$$\chi(t'') - \chi(t') \geqq (k(t) - \varepsilon)(t'' - t'),$$

$$\chi(t'') - \chi(t') \leqq (k(t) - \varepsilon)(t'' - t'),$$

which are valid for all $t - \varepsilon < t' < t'' < t + \varepsilon$, and thus in particular, imply

$$k(t) - \varepsilon \leqq \frac{d\chi(t')}{dt} \leqq k(t) + \varepsilon \quad \text{for } t - \varepsilon < t' < t + \varepsilon. \tag{4.14}$$

Letting ε go to zero in (4.14) yields

$$\frac{d\chi(t)}{dt} = k(t).$$

The proof is complete. □

Remark 4.1. Estimate (4.3) of Lemma 4.2 makes clear that only the c_1 waves located at the left of the initial phase discontinuity and the $\pm c_3$ waves located at the right of the initial phase discontinuity contribute to the change in speed of the phase boundary.

We finally prove that the result in Theorem 4.1 is sufficient for the passage to the limit in the kinetic relation.

Theorem 4.2. *Suppose that $V^* \neq 0$. Then the limit function u given by Glimm's scheme satisfies*

$$\partial_t U(u) + \partial_x F(u) = -v_t \phi \left(- \frac{v_t}{v_x} \right) \delta_{x=\chi(t)} \tag{4.15}$$

H_1-almost everywhere on the set $\mathscr{B}_{\text{sub}}(u)$.

Equality (4.15) is understood as equality between Borel measures on $\boldsymbol{R}_+ \times \boldsymbol{R}$. Here $\mathscr{B}_{\text{sub}}(u)$ (according to the definition of Section 1) is the set of all points of approximate jump of u associated with subsonic phase discontinuities. In view of the formula of Section 1, it is clear that, when $V^* \neq 0$, (4.15) is equivalent to the formulation (1.11) of the kinetic relation. The case $V^* = 0$ could in principle be treated by the same technique but this would require further analysis.

Remark 4.2. 1) The pointwise convergence property of Glimm's scheme was already used in LE FLOCH & LIU [30] to derive an existence result for nonlinear hyperbolic systems in nonconservative form.

2) If $V^* = 0$, $\dot{\chi}$ may vanish, and then relation (4.15) is not sufficient to characterize uniquely the solution (e.g., of the Riemann problem).

Proof of Theorem 4.2. For all times $t \geqq 0$, we introduce an approximate normal $v^h(t) = (v_t^h(t), v_x^h(t))$ by

$$v_t^h(t)^2 + v_x^h(t)^2 = 1, \quad \dot{\chi}^h(t) = -\frac{v_t^h(t)}{v_x^h(t)}, \quad v_x^h(t) > 0.$$

Similarly, from $\dot{\chi}(t)$, we define $v(t) = (v_t(t), v_x(t))$. According to the notation of Section 1, we in fact have $v(t) = v(t, \chi(t))$. First of all, we assert that

$$v_t^h \phi(\dot{\chi}^h)\, \delta_{x=\chi^h} \to v_t \phi\left(\frac{d\chi}{dt}\right) \delta_{x=\chi} \tag{4.16}$$

in the weak-star topology of bounded Borel measures $\boldsymbol{R}_+ \times \boldsymbol{R}$.

Since $\dot{\chi}^h$ satisfies (4.3) and since the right-hand side of (4.3) is small by the assumption (3.1b), the function $\dot{\chi}^h$ has small total variation. When $V^* \neq 0$, we can ensure that $\dot{\chi}^h$ is bounded away from zero uniformly with respect to h. In view of properties (1.8), the function ϕ is (at least) continuous in the range of values taken by $\dot{\chi}^h$. This fact combined with the convergence result (4.5) gives

$$v_t^h(t)\, \phi(\dot{\chi}^h(t)) \to v_t(t)\, \phi\left(\frac{d\chi}{dt}(t)\right) \quad \text{for all } t \in \boldsymbol{R}_+ \setminus E, \tag{4.17}$$

where E is an at most countable set. From (4.17) and the uniform convergence of χ^h to χ, cf. (4.4), we deduce that

$$\int v_t^h(t)\, \phi(\dot{\chi}^h(t))\, \theta(\chi^h(t))\, dt \to \int v_t(t)\, \phi\left(\frac{d\chi}{dt}(t)\right) \theta(\chi(t))\, dt$$

for each continuous function θ: $\boldsymbol{R} \to \boldsymbol{R}$ with compact support. This proves (4.16).

By construction, the approximate solutions u^h satisfy the kinetic relation

$$\partial_t U(u^h) + \partial_x F(u^h) = -v_t^h \phi\left(-\frac{v_t^h}{v_x^h}\right) \delta_{x=\chi^h} \tag{4.18}$$

H_1-almost everywhere on the set $\mathscr{B}_{\text{sub}}(u^h)$. We assert that using (4.16) we can pass to the limit in (4.18) and obtain

$$\partial_t U(u) + \partial_x F(u) = -v_t \phi\left(-\frac{v_t}{v_x}\right) \delta_{x=\chi} \tag{4.19}$$

H_1-almost everywhere on the set $\mathscr{B}_{\text{sub}}(u)$.

The left-hand sides of (4.18) and (4.19) are treated easily since they have a (divergence-like) conservation form. In particular, we have

$$\partial_t U(u^h) + \partial_x F(u^h) \to \partial_t U(u) + \partial_x F(u) \tag{4.20}$$

in the weak-star topology of bounded Borel measures on $\boldsymbol{R}_+ \times \boldsymbol{R}$.

In case $V^* < c_3$, (4.18) is satisfied on the whole space $\boldsymbol{R}_+ \times \boldsymbol{R}$ and so the desired result (4.19) is an immediate consequence of (4.18), (4.16) and (4.20).

When $V^* > c_3$, nothing has to be proved since no kinetic relation is imposed then.

The final case $V^* = c_3$ is treated as follows. We note that one can find two Lipschitz continuous functions $\tilde{\phi}(V)$ and $\tilde{\phi}_+(V)$ defined for V in a neighborhood of c_3 such that the kinetic relation (e.g., for u^h) is equivalent to the two inequalities

$$\partial_t U(u^h) + \partial_x F(u^h) \leqq -v_t^h \tilde{\phi}_+ \left(-\frac{v_t^h}{v_x^h}\right) \delta_{x=\chi^h}, \tag{4.21 a}$$

$$\partial_t U(u^h) + \partial_x F(u^h) \geqq -v_t^h \tilde{\phi}_- \left(-\frac{v_t^h}{v_x^h}\right) \delta_{x=\chi^h}, \tag{4.21 b}$$

where $\tilde{\phi}_\pm$ are chosen so that

$$\tilde{\phi}_+(V) = \tilde{\phi}_-(V) = \phi(V) \quad \text{for } V < c_3,$$

$$\tilde{\phi}_+ \text{ and } \tilde{\phi}_- \text{ are Lipschitz continuous with: } \tilde{\phi}_-(V) < \tilde{\phi}_+(V).$$

This is possible since (4.21 a), (4.21 b) when $V \leqq c_3$ give back the kinetic relation; while for $V > c_3$ (4.21 a), (4.21 b) are trivially satisfied provided that the entropy dissipation in the supersonic case remains in the interval $[\tilde{\phi}_-(V), \tilde{\phi}_+(V)]$. In this latter case, the entropy dissipation across the phase boundary, say $\tilde{\phi}(V)$, is the following (cf. the notation of Section 2):

$$\tilde{\phi}(V) = \frac{1}{2}(k_1 - k_3)(w_M w_m - w_R w_-) = \frac{1}{2}(k_1 - k_3)\left(w_M w_m - \frac{c_3 + V}{c_1 + V} w_R h_{LR}\right),$$

where $V = V(h_{LR}) > c_3$ is a root of the equation (2.12b). By (1.8b) and (1.9b), we have

$$\lim_{\substack{u \to c_3 \\ V > c_3}} \tilde{\phi}(V) = \tfrac{1}{2}(k_1 - k_3) w_M w_m = \bar{\psi}(c_3) = \lim_{\substack{V \to c_3 \\ V < c_3}} \phi(V).$$

This proves the continuity of the entropy dissipation at $V = c_3$. Moreover $\tilde{\phi}$ is clearly Lipschitz continuous in view of Lemma 2.2.

Hence, for $\phi_\pm(V)$ suitably chosen and $V - c_3$ sufficiently small, the entropy dissipation $\tilde{\phi}(V)$ remains in the interval $[\tilde{\phi}_-(V), \tilde{\phi}_+(V)]$.

It is clear that (4.16) still holds if ϕ is replaced by $\tilde{\phi}_-$ or $\tilde{\phi}_+$, i.e., in the weak-star topology we have

$$v_t^h \tilde{\phi}_\pm(\dot{\chi}^h)\, \delta_{x=\chi^h} \xrightarrow{\text{weak*}} v_t \tilde{\phi}_\pm \left(\frac{d\chi}{dt}\right) \delta_{x=\chi}. \tag{4.22}$$

Then (4.20) and (4.22) used in (4.21) yield

$$\partial_t U(u) + \partial_x F(u) \leqq -v_t \tilde{\phi}_+ \left(-\frac{v_t}{v_x}\right) \delta_{x=\chi^h},$$

$$\partial_t U(u) + \partial_x F(u) \geqq -v_t \tilde{\phi}_- \left(-\frac{v_t}{v_x}\right) \delta_{x=\chi^h},$$

which give (4.15). The proof is complete. □

We summarize in the following theorem the results obtained in Section 3 and in the present section.

Theorem 4.3. *Consider the mixed system* (1.1) *with an initial condition which is a small perturbation in the BV norm of a single propagating phase boundary with speed* V^*. *Suppose that* $V^* \neq 0$ and condition (3.2b) *is satisfied if* $V^* = c_3$. *Then Glimm's scheme for this problem converges to an admissible weak solution which has the structure described in* (3.3), (3.4).

Remark 4.3. 1) Note that the proof of Theorem 4.2 uses the property that the entropy dissipation across a contact discontinuity is identically zero.

2) I believe that Theorem 4.3 could be extended to a finite number of propagating phase boundaries. Also the restriction $V^* \neq 0$ is only a technical assumption and could be removed by using other techniques from [16].

3) However, there is a main obstacle to a general result of existence of *BV* solutions for (1.1). Indeed, for arbitrary large data, the phenomenon of initiation of new phases arises, and it is an open problem to derive a uniform bound on the total variation in that case.

Acknowledgements. This work was done while I was an assistant professor at the Courant Institute and on leave from a CNRS research position at the Ecole Polytechnique (France). This work was supported in part by a NSF grant DMS-88-06731 and by the CNRS. It is a great pleasure for me to thank Bob Kohn who encouraged me to study this problem and accepted the job of reading the first version of this paper. The hospitality of Peter Lax at Courant Institute was highly appreciated. I am also grateful to Rohan Abeyaratne and Michael Shearer for fruitful discussions.

References

1. Abeyaratne, R. & Knowles, J. K., On the dissipative response due to discontinuous strains in bars of unstable elastic materials, *Int. J. Solids Structures* **24**, 1988, 1021–1044.
2. Abeyaratne, R. & Knowles, J. K., Kinetic relations and the propagation of phase boundaries in solids, *Arch. Rational Mech. Anal.* **114**, 1991, 119–154.
3. Abeyaratne, R. & Knowles, J. K., Implications of viscosity and strain gradient effects for the kinetics of propagating phase boundaries in solids, *SIAM J. Appl. Math.* **51**, 1991, 1205–1221.
4. Abeyaratne, R. & Knowles, J. K., On the propagation of maximally dissipative phase boundaries in solids, *Technical Report No. 1*, California Inst. of Technology, Pasadena, CA, August 1990; to appear in *Quart. Appl. Math.*
5. Chern, I. L., Stability theorem and truncation error analysis for the Glimm scheme and for a front tracking method for flows with strong discontinuities, *Comm. Pure Appl. Math.* **42**, 1989, 815–844.
6. Dafermos, C. M., The entropy rate admissibility criterion for solutions of hyperbolic conservation laws, *J. Diff. Eqs.* **14**, 1973, 202–212.
7. Dafermos, C. M., Hyperbolic systems of conservation laws, in *Systems of Nonlinear Partial Differential Equations*, ed. J. M. Ball, Reidel, 1983, 25–70.

8. DAL MASO, G., LE FLOCH, P. & MURAT, F., Definition and weak stability of non-conservative products, preprint, Ecole Polytechnique (Palaiseau), to be submitted.
9. DIPERNA, R. J., Decay and asymptotic behavior of solutions to nonlinear hyperbolic systems of conservation laws, *Indiana Univ. Math. J.* **24**, 1975, 1041–1071.
10. DIPERNA, R., Singularities of solutions of nonlinear hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal.* **60**, 1975, 75–100.
11. DIPERNA, R. J. & MAJDA, A., The validity of nonlinear geometrical optics for weak solutions of conservation laws, *Comm. Math. Phys.* **98**, 1985, 313–347.
12. FAN, H., A vanishing viscosity approach on the dynamics of phase transitions in van der Waals fluids, preprint, Inst. Math. Appls., Univ. Minnesota, 1991.
13. FAN, H. & SLEMROD, M., The Riemann problem for systems of conservation laws of mixed type, preprint, Inst. Math. Appls., Univ. Minnesota, 1991.
14. FEDERER, H., *Geometric Measure Theory*, Springer-Verlag, 1969.
15. GLIMM, J., Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18**, 1965, 697–715.
16. GLIMM, J. & LAX, P. D., Decay of solutions of systems of nonlinear hyperbolic conservation laws, *Memoirs of the Amer. Math. Soc.*, **101**, 1970.
17. GURTIN, M. E., On a theory of phase transitions with interfacial energy, *Arch. Rational Mech. Anal.* **87**, 1984, 187–212.
18. HAGAN, R. & SLEMROD, M., The viscosity-capillarity criterion for shocks and phase transitions, *Arch. Rational Mech. Anal.* **83**, 1983, 333–361.
19. HATTORI, H., The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion: Isothermal case, *Arch. Rational Mech. Anal.* **92**, 1986, 246–263.
20. HATTORI, H., The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion: Non-isothermal case, *J. Diff. Eqs.* **65**, 1986, 158–174.
21. HOU, L. Y. & LE FLOCH, P., Why non-conservative schemes converge to wrong solutions: error analysis, to appear in *Math. of Comp.*, 1993.
22. HSIAO, L., Uniqueness of admissible solutions of the Riemann problem for a system of conservation laws of mixed type, *J. Diff. Eqs.* **86**, 1990, 197–233.
23. JAMES, R. D., The propagation of phase boundaries in elastic bars, *Arch. Rational Mech. Anal.* **73**, 1980, 125–158.
24. KEYFITZ, B. L., The Riemann problem for non-monotone stress-strain functions: a "hysteresis" approach, *Lec. in Appl. Math.* **23**, 1986, 379–395.
25. LAX, P. D., Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.* **10**, 1957, 537–566.
26. LAX, P. D., *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Regional Conf. Series in Appl. Math., **11**, SIAM, Philadelphia, 1973.
27. LE FLOCH, P., Entropy weak solutions to nonlinear hyperbolic systems in non-conservative form, *Comm. Part. Diff. Eqs.* **13**, 1988, 669–727.
28. LE FLOCH, P., Shock waves for nonlinear hyperbolic systems in non-conservative form, preprint, Inst. Math. Appl., Univ. Minnesota, 1989.
29. LE FLOCH, P., Entropy weak solutions to nonlinear hyperbolic systems in conservative form, Proc. Intern. Conf. on Hyperbolic problems, ed. J. BALLMANN & R. JELTSCH, *Note on Num. Fluid Mech.*, Vol. 24, Vieweg, Braunschweig, 1989, 362–373.
30. LE FLOCH, P., & LIU, T.-P., Existence theory for nonlinear hyperbolic systems in non-conservative form, to appear in *Forum Mathematicum*, 1993.
31. LE FLOCH, P. & XIN, Z. P., Uniqueness via the adjoint problem for some systems of conservation laws, to appear in *Comm. Pure Appl. Math.*, 1993.

32. LIU, T.-P., The Riemann problem for general systems of conservation laws, *J. Diff. Equa.* **18**, 1975, 218–234.
33. LIU, T.-P., The deterministic version of the Glimm scheme, *Comm. Math. Phys.* **57**, 1977, 135–148.
34. LIU, T.-P., Admissible solutions of hyperbolic systems of conservation laws, *Memoirs of the Amer. Math. Soc.* **240**, 1981.
35. PEGO, R. L., Phase transitions in one-dimensional nonlinear viscoelasticity: admissibility and stability, *Arch. Rational Mech. Anal.* **97**, 1987, 353–394.
36. PEGO, R. L. & SERRE, D., Instability in Glimm's scheme for two systems of mixed type, *SIAM J. Num. Anal.* **25**, 1988, 965–988.
37. SCHOCHET, S., Sufficient conditions for local existence via Glimm's scheme for large BV data, *J. Diff. Eqs.* **89**, 1991, 317–354.
38. SHEARER, M., The Riemann problem for a class of conservation laws of mixed type, *J. Diff. Eqs.* **46**, 1982, 426–443.
39. SHEARER, M., Nonuniqueness of admissible solutions of the Riemann initial value problem for a system of conservation laws of mixed type, *Arch. Rational Mech. Anal.* **93**, 1986, 45–59.
40. SHEARER, M., Dynamic phase transitions in a van der Waals gas, *Quart. Appl. Math.* **46**, 1988, 631–636.
41. SLEMROD, M., Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Arch. Rational Mech. Anal.* **81**, 1983, 301–315.
42. SLEMROD, M., A limiting viscosity approach to the Riemann problem for materials exhibiting change of phase, *Arch. Rational Mech. Anal.* **105**, 1989, 327–365.
43. TEMPLE, B., Global solution of the Cauchy problem for a class of 2×2 non-strictly hyperbolic conservation laws, *Adv. in Appl. Math.* **3**, 1982, 335–375.
44. TRUSKINOVSKY, L., Kinks versus shocks, in *Shock induced transitions and phase structures in general media*, ed. R. FOSDICK, E. DUNN & H. SLEMROD, Springer-Verlag, 1991.
45. VOLPERT, A. I., The space BV and quasilinear equations, *USSR Sb.* **2**, 1967, 225–267.

Center for Applied Mathematical Sciences
University of Southern California
Los Angeles, California 90089

(Accepted October 23, 1992)