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The Hele-Shaw Problem and Area-Preserving Curve-Shortening Motions

XINFU CHEN

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Abstract

We prove existence, locally in time, of a solution of the following Hele-Shaw problem: Given a simply connected curve contained in a smooth bounded domain Ω, find the motion of the curve such that its normal velocity equals the jump of the normal derivatives of a function which is harmonic in the complement of the curve in Ω and whose boundary value on the curve equals its curvature. We show that this motion is a curve-shortening motion which does not change the area of the region enclosed by the curve. In case Ω is the whole plane $\mathscr{R}^2$, we also show that if the initial curve is close to an equilibrium curve, i.e., to a circle, then there exists a global solution and the global solution tends to some circle exponentially fast as time tends to infinity.

1. Introduction

Let Ω be a bounded and simply connected domain in $\mathscr{R}^2$ and let Γ_0 be a smooth simply-connected curve in Ω. Consider the free-boundary problem of finding a function $u(x, t)$, $x \in \Omega$, $t \geqq 0$, and a free boundary $\Gamma = \bigcup_{t\geqq 0}(\Gamma_t \times \{t\})$ satisfying

$$\begin{aligned}
\triangle u(\cdot, t) &= 0 && \text{in } \Omega\backslash\Gamma_t,\ t \geqq 0,\\
u_n &= 0 && \text{on } \partial\Omega \times [0, \infty),\\
u &= \kappa && \text{on } \Gamma_t,\ t \geqq 0, \qquad (1.1)\\
[u_n]_{\Gamma_t} &= V && \text{on } \Gamma_t,\ t \geqq 0,\\
\Gamma \cap \{t = 0\} &= \Gamma_0 && \text{on } \{t = 0\}
\end{aligned}$$

where n is the unit outward normal to $\partial\Omega$ or to Γ_t, u_n denotes the normal derivative, $[u_n]_{\Gamma_t}$ is the sum of the *outward* normal derivatives of u from each side of Γ_t (which is also equal to the jump of the normal derivatives across Γ_t), κ and V are respectively the curvature and the normal velocity of Γ_t, with the sign convention that the curvature of a circle is positive and the normal velocity of a shrinking curve is positive.

Problem (1.1) is often referred to as a Hele-Shaw problem in studying equations that approximate the pressure in a system containing two immiscible fluids [6]. The case when Γ_0 is a graph $y = f(x)$ ($f(x) \in H^{5/2}(\mathscr{R}^1)$ and $\Omega = \Omega_t$ is the upper half space bounded by Γ_t) was studied by DUCHON & ROBERT [5].

When (1.1a) is replaced by the heat equation $u_t - \Delta u = 0$, the system (1.1) is known as the Stefan problem with the Gibbs-Thomson relation for the equilibrium of the solid-liquid interface. Its global weak solutions were recently established by LUCKHAUS [16] and ALMGREN & WANG [2]. The existence of local classical solutions was recently proved by RADKEVITCH [19].

Solutions of (1.1) are closely related to the asymptotic limit, as $\varepsilon \to 0$, of the Cahn-Hilliard equation $\varepsilon w_t^\varepsilon + \Delta(\varepsilon^2 \Delta w^\varepsilon - f(w^\varepsilon)) = 0$ for appropriate initial and boundary conditions, where $f(w) = w^3 - w$; formal analysis shows that the chemical potential $u^\varepsilon \equiv f(w^\varepsilon) - \varepsilon^2 \Delta w^\varepsilon$ tends to the solution of (1.1); see PEGO [18]. Recently, ALIKAKOS & FUSCO [1] showed that if initially $u^\varepsilon(x, 0)$ is close to an equilibrium of (1.1) (i.e., Γ_0 is a circle and $u(x, 0)$ is a constant equal to the curvature of the circle), then u^ε will stay close to this equilibrium for a very long time (of order $e^{-c/\varepsilon}$).

For connections of the Hele-Shaw problem with the asymptotic behavior of a phase-field model, see CAGINALP [3].

First of all, we point out two important features of the solution of (1.1). Denote by Ω_t the region enclosed by Γ_t and by $\mathscr{A}(t)$ and $S(t)$ the area of Ω_t and the arc length of Γ_t, respectively. Then we can calculate

$$\frac{d}{dt}\mathscr{A}(t) = -\int_{\Gamma_t} V = -\int_{\Gamma_t} \left[\frac{\partial u}{\partial n}\right] = \int_{\partial\Omega} \frac{\partial u}{\partial n} - \int_{\Omega\setminus\Gamma_t} \Delta u = 0;$$

that is, the Hele-Shaw motion preserves the area of the region enclosed by Γ_t, for all $t > 0$. Also, we can compute

$$\frac{d}{dt} S(t) = -\int_{\Gamma_t} \kappa V = -\int_{\Gamma_t} u\left[\frac{\partial u}{\partial n}\right] = -\int_{\Omega} |\nabla u|^2 \leqq 0;$$

that is, the Hele-Shaw motion is a curve-shortening motion.

In this paper, we shall establish the local existence of a solution of (1.1) for an arbitrary (smooth) initial curve, and global existence of a solution when the initial curve is close to a circle. We shall also prove that when the initial curve is close to a circle, the global solution tends to some circle exponentially fast as $t \to \infty$.

The method used by DUCHON & ROBERT [5] is based on the observation that the operator $\partial_n : u|_\gamma \to u_n|_\gamma$ can be written as $\partial_n = HD_s + R$ where γ is a curve given by $y = f(x)$, $f \in H^{5/2}(\mathscr{R}^1)$, D_s is the derivative with respect to the arc length, H is the Hilbert transformation on the real line, and R is a compact operator. The operator $\Lambda \equiv HD_s$ is the square root of $-\Delta$ in the sense that Λ is positive-definite and $\Lambda^2 = -D_{ss}$. When we write the unit tangent of Γ_t in the form $e^{i\varphi(s,t)}$, $s \in \mathscr{R}$, $t > 0$, problem (1.1) is equivalent to finding φ satisfying $L\varphi \equiv \varphi_t - \Lambda D_{ss}\varphi = F(\varphi)$ where F is a functional having some kind of compactness. Therefore, the existence of a solution to (1.1) is equivalent to the existence of a fixed point for $L^{-1}F$. Their method, however, is very difficult to carry out here since the arc length of Γ_t is finite and changing and if one tries to decompose the operator ∂_n as $HD_s + R$, then the time derivative of the arc length appears in R and is not easily controlled.

In studying the Stefan problem with the Gibbs-Thomson law, RADKEVITCH [19] used a Newton method of successive approximation to show the short-time existence of a classical solution. The key step is to show the invertibility of a Fréchet derivative of a linearized operator. His method requires some technical assumptions on the initial data and may not be applied here since without the term u_t in (1.1a), the continuity of the solution in time is very hard to obtain. LUCKHAUS [16] and ALMGREN & WANG [2] used another approach. They discretize the time and implement numerical schemes to construct a sequence of approximate solutions and then show that a subsequence of the approximate solutions converges to a weak solution global in time. In their schemes, finding the solution in the next time step involves finding a *global* minimizer of a functional; thus, as pointed out in [16], an interface may suddenly disappear, and therefore a strong solution (if it exists) may not be a weak solution (defined in [16]). The main difficulty in carrying out their schemes to our case is similar to that of RADKEVITCH, i.e., lack of control in the t direction.

Here we shall instead use a new approach totally different from those of [2, 5, 16, 19]. We shall consider the regularized equation

$$V + \varepsilon\kappa_{ss} = [\kappa_n]_{\Gamma_t}, \quad t > 0, \tag{1.2}$$

for small positive ε and then let ε go to zero. Here κ_{ss} is the second derivative of the curvature κ with respect to the arc length and $[\kappa_n]_{\Gamma_t}$ is the jump of the normal derivatives of u satisfying the first three equations in (1.1). (In the sequel, we shall mix the usage of $[u_n]_{\Gamma_t}$ and $[\kappa_n]_{\Gamma_t}$.) By means of a priori estimates, we show that (1.2) has a solution in a time interval independent of ε, and by extracting a convergent subsequence we obtain a local solution. In case the initial curve is close to a circle, the a priori estimates can lead to the existence of a global solution.

To study the asymptotic behavior of the global solution, notice that a curve is an equilibrium if and only if it is a circle. Therefore all equilibria consist of a three-dimensional manifold parametrized by the radius, and the x and y coordinates of the center of the circle. This manifold contributes to the existence of three zero-growth modes for the linearized Hele-Shaw problem at any equilibrium. By using this observation and its corresponding implications in the nonlinear case, we show that the global solution tends to a circle exponentially fast, even actually without introducing the linearized Hele-Shaw problem.

We prove the local existence of solutions of (1.2) in § 2. After establishing several energy-type identities for the solution of (1.2) in §3, we present some a priori estimates in §4. In §5 we define solutions to (1.1) in some Sobolev space and show the existence of a solution to (1.1) by letting $\varepsilon \to 0$. Finally, in §§6, 7, we establish the global existence and the long-time behavior of the solution that is initially close to a circle.

Remark 1.1. (1) The Hele-Shaw problem is closely related to the equation $V = \kappa$ for motion by mean curvature. For the one-dimensional motion by curvature flow, the arc length of the curve monotonically decreases, whereas the area enclosed by the curve decreases with a constant speed 2π. It is known that any imbedded curve in $\mathscr{R}^2$ or imbedded convex hypersurfaces in $\mathscr{R}^N (N > 2)$ shrinks to a point in a finite time T^* and that if one scales the curve (or hypersurface) such that its enclosed region has the same area or volume as its initial region, then the curve or hypersurface tends to a circle or a sphere as $t \to T^*$; see [8, 10, 11, 12].

(2) The difficulty for the global existence of a solution to the Hele-Shaw problem arises from the possible topological changes of the curve. For the motion-by-mean-curvature flow, the surface of a barbell with long and thin handle in higher dimension splits into two pieces at a time before the surface shrinks into a point. However, in the one-dimensional case, the solution of any imbedded curve remains as an imbedded curve until it shrinks to a point; that is, a 'global' solution exists; see [10]. Here we do not know if the corresponding conclusion holds for the Hele-Shaw problem, i.e., if a solution global in time exists.

(3) To get global (weak) solutions for the motion-by-mean-curvature equation (for higher dimensions), people invented viscosity or generalized solutions and have been extremely successful [4, 7, 20]. However, we do not know if one can generalize these ideas to the Hele-Shaw problem or to the equation (1.2) with $\varepsilon \geqq 0$.

2. The Regularized Problem

Let $T > 0$ and $\varepsilon \in (0, 1)$ be fixed constants. Consider the evolution problem

$$\begin{aligned} V + \varepsilon\kappa_{ss} &= [\kappa_n]_{\Gamma_t} \quad \text{for } t \in (0, T), \\ \Gamma \cap \{t = 0\} &= \Gamma_0 \quad \text{for } t = 0 \end{aligned} \tag{2.1}$$

where V, κ, $[\kappa_n]_{\Gamma_t}$, and Γ_0 are as in §1.

It is convenient to write (2.1) in terms of local coordinates. Let $\mathscr{M}_0$ (close to Γ_0) be a one-dimensional closed C^4 manifold embedded in Ω. By scaling the space if necessary, we may assume that the length of $\mathscr{M}_0$ is 2π so that $\mathscr{M}_0$ is diffeomorphic to S^1, the unit circle. Let $x = X^0(\eta)$ be the diffeomorphism from S^1 to $\mathscr{M}_0$ such that η is the arc length and that the positive direction of $\mathscr{M}_0$ is counter-clockwise. Then the unit tangent τ^0, the unit outward normal n^0, and the curvature κ^0 of $\mathscr{M}_0$ at $X^0(\eta)$ satisfy the relations

$$\tau^0(\eta) = X^0_\eta(\eta), \quad \tau^0_\eta(\eta) = -\kappa^0(\eta)\, n^0(\eta), \quad n^0_\eta(\eta) = \kappa^0(\eta)\, \tau^0(\eta), \quad \eta \in S^1.$$

Set

$$\delta_0 = \tfrac{1}{2} \min\left\{1,\ \operatorname{dist}(\mathscr{M}_0, \partial\Omega),\ \frac{1}{\|\kappa^0\|_{L^\infty(S^1)}}\right\} \tag{2.2}$$

and define $Y: S^1 \times (-\delta_0, \delta_0) \to \mathscr{M}_0^{\delta_0} \equiv \{x \in \mathscr{R}^2 \mid \operatorname{dist}(x, \mathscr{M}_0) < \delta_0\}$ by

$$Y(\eta, h) = X^0(\eta) + hn^0(\eta). \tag{2.3}$$

Clearly, Y is a diffeomorphism from $S^1 \times (-\delta_0, \delta_0)$ to $\mathscr{M}_0^{\delta_0}$; see, for example, [9]. Hence, we can use (η, h) to represent points in $\mathscr{M}_0^{\delta_0}$, the δ_0-neighborhood of $\mathscr{M}_0$.

In the sequel, we shall assume that Γ_0 is in a $(\delta_0/2)$-neighborhood of $\mathscr{M}_0$ and that in the local coordinates it has the representation

$$\Gamma_0 = \{Y(\eta, h) \mid h = d^0(x),\ \eta \in S^1\} \tag{2.4}$$

where $d^0(\cdot) : S^1 \to (-\delta_0/2, \delta_0/2)$ is a C^2 function.

Let $\Gamma = \bigcup_{t \in [0,T)} (\Gamma_t \times \{t\})$ be a family of curves having the representation

$$\Gamma_t = \{X(\eta, t) \mid X(\eta, t) = Y(\eta, d(\eta, t)),\ \eta \in S^1\}, \quad t \in [0, T),$$

where $d(\eta, t) : S^1 \times [0, T) \to (-\delta_0, \delta_0)$ is a smooth function. Then the unit tangent τ and the unit normal n of Γ_t at $X(\eta, t)$ are given by

$$\tau(\eta, t) = \frac{X_\eta(\eta, t)}{|X_\eta(\eta, t)|} = \frac{1}{J}\,[(1 + d\kappa^0)\,\tau^0 + d_\eta n^0],$$

$$n(\eta, t) = \frac{1}{J}\,[-d_\eta \tau^0 + (1 + d\kappa^0)\, n^0]$$

where

$$J = J(\eta, d, d_\eta) = |X_\eta(\eta, t)| = \sqrt{(1 + d\kappa^0(\eta))^2 + d_\eta^2}.$$

One can use the formulas

$$V = -X_t(\eta, t) \cdot n(\eta, t), \qquad \kappa = -X_{\eta\eta}(\eta, t) \cdot n(\eta, t)/|X_\eta(\eta, t)|^2$$

to calculate the normal velocity V and the curvature κ of Γ by

$$V = -\frac{1 + d\kappa^0}{J}\, d_t, \tag{2.5}$$

$$\kappa = \frac{1}{J^3}\,[-(1 + d\kappa^0)\, d_{\eta\eta} + (2d_\eta \kappa^0 + d\kappa^0_\eta)\, d_\eta + \kappa^0(1 + d\kappa^0)^2]. \tag{2.6}$$

Since $\partial_s = \dfrac{1}{J}\,\partial_\eta$, equation (2.1) can be written as

$$\begin{aligned} &d_t + \varepsilon J^{-4} d_{\eta\eta\eta\eta} = \mathscr{F}[d] \equiv \mathscr{F}_1[d] + \varepsilon \mathscr{F}_2[d] \quad \text{for } (\eta, t) \in S^1 \times (0, T), \\ &d(\eta, 0) = d^0(\eta) \quad \text{for } \eta \in S^1 \end{aligned} \tag{2.7}$$

where $\mathscr{F}_1[d]$ is the functional defined by

$$\mathscr{F}_1[d](\cdot, t) = -\frac{J}{1 + d\kappa^0}\,[\kappa_n]_{\Gamma_t} \tag{2.8}$$

and $\mathscr{F}_2$ is the function defined by

$$\mathscr{F}_2 = \frac{1}{1+d\kappa^0}\left[-d_{\eta\eta\eta}\frac{\partial}{\partial\eta}\frac{1+d\kappa^0}{J^4} - \frac{\partial}{\partial\eta}\left(\frac{d_{\eta\eta}}{J}\frac{\partial}{\partial\eta}\frac{1+d\kappa^0}{J^3}\right)\right.$$
$$\left. + \frac{\partial}{\partial\eta}\left(\frac{1}{J}\frac{\partial}{\partial\eta}\frac{\kappa^0(1+d\kappa^0)^2 + d_\eta(2d_\eta\kappa^0 + \kappa^0_\eta d)}{J^3}\right)\right]. \tag{2.9}$$

We first study the functional $\mathscr{F}$. It is convenient to introduce a function space $\mathscr{M}_\delta$ defined by

$$\mathscr{M}_\delta = \{h(\eta) \in C^1(S^1) \mid |h|_{C^1(S^1)} < \delta\}$$

where $\delta \in (0, \delta_0)$ is a parameter.

Lemma 2.1. *Let $\mathscr{M}_0 \in C^2$ be given and let h be any function in $\mathscr{M}_{\delta_0}$. Set*

$$\gamma = \{Y(\eta, h(\eta)) \mid \eta \in S^1\}.$$

Then there exists a positive constant C depending only on $\mathscr{M}_0$ and Ω such that if u is a harmonic function in $\Omega\setminus\gamma$ and is continuous across γ with normal derivative vanishing on $\partial\Omega$, then

$$C^{-1}\|\nabla u^\pm\|_{L^2(\gamma)} \leqq \|u_\tau\|_{L^2(\gamma)} \leqq C\|u_n^- - u_n^+\|_{L^2(\gamma)} \tag{2.10}$$

where u^- and u^+ are respectively the restrictions of u on Ω^-, which is the domain enclosed by γ, and on $\Omega^+ \equiv \Omega\setminus(\Omega^- \cup \gamma)$, and u_τ and u_n are respectively the tangential and normal derivatives of u.

Proof. Since γ is a Lipschitz curve and its Lipschitz constant is bounded by a constant independent of γ, the assertion of the lemma thus follows from the classical single-layer potential theory; see, for example, [13, 14, 17]. □

Direct differentiation yields

$$\kappa_\eta = a_1 d_{\eta\eta\eta} + a_2 d^2_{\eta\eta} + a_3 d_{\eta\eta} + a_4,$$

$$\mathscr{F}_2 = b_1 d_{\eta\eta\eta} d_{\eta\eta} + b_2 d_{\eta\eta\eta} + b_3 d^3_{\eta\eta} + b_4 d^2_{\eta\eta} + b_5 d_{\eta\eta} + b_6$$

where a_i and b_j are functions of η, d and d_η only. Thus we obtain the following lemma.

Lemma 2.2. *Assume that $\mathscr{M}_0 \in C^4$ and that $d(\cdot) \in \mathscr{M}_{\delta_0}$. Then there exists a constant C depending only on $\mathscr{M}_0$ and Ω such that $\mathscr{F}_1$ defined in* (2.8) *satisfies*

$$\|\mathscr{F}_1[d]\|^2_{L^2(S^1)} \leqq C[1 + \|d_{\eta\eta\eta}\|^2_{L^3(S^1)} + \|d_{\eta\eta}\|^4_{L^6(S^1)}]$$

and $\mathscr{F}_2$ defined in (2.9) *satisfies*

$$\|\mathscr{F}_2[d]\|^2_{L^2(S^1)} \leqq C[1 + \|d_{\eta\eta\eta}\|^3_{L^3(S^1)} + \|d_{\eta\eta}\|^6_{L^6(S^1)}].$$

Consequently, the functional $\mathscr{F} \equiv \mathscr{F}_1 + \varepsilon \mathscr{F}_2$ *satisfies*

$$\| \mathscr{F}[d] \|^2_{L^2(S^1)} \leqq C_0 \left[\frac{1}{\varepsilon^4} + \varepsilon^2 (\| d_{\eta\eta\eta} \|^3_{L^3(S^1)} + \| d_{\eta\eta} \|^6_{L^6(S^1)}) \right] \tag{2.11}$$

where C_0 *is a constant depending only on* $\mathscr{M}_0$ *and* $\partial\Omega$.

To deal with the high-order growth on the right-hand side of (2.11), we need the following lemma.

Lemma 2.3. *For every* $f \in H^4(S^1)$, *the following inequalities hold*:

$$\| f_{\eta\eta} \|^6_{L^6(S^1)} \leqq 25 \| f_\eta \|^4_{L^\infty(S^1)} \| f_{\eta\eta\eta\eta} \|^2_{L^2(S^1)},$$

$$\| f_{\eta\eta\eta} \|^3_{L^3(S^1)} \leqq 2\sqrt{10} \, \| f_\eta \|_{L^\infty(S^1)} \| f_{\eta\eta\eta\eta} \|^2_{L^2(S^1)}.$$

Proof. Integration by parts yields

$$\begin{aligned}
\int_{S^1} f^6_{\eta\eta} &= - \int_{S^1} f_\eta (f^5_{\eta\eta})_\eta = -\tfrac{5}{2} \int_{S^1} (f^2_\eta)_\eta f^3_{\eta\eta} f_{\eta\eta\eta} \\
&= \tfrac{15}{2} \int_{S^1} f^2_\eta f^2_{\eta\eta} f^2_{\eta\eta\eta} + \tfrac{5}{2} \int_{S^1} f^2_\eta f^3_{\eta\eta} f_{\eta\eta\eta\eta} \\
&\leqq \tfrac{15}{2} \| f_\eta \|^2_{L^\infty(S^1)} \int_{S^1} \tfrac{1}{3} \, (f^3_{\eta\eta})_\eta f_{\eta\eta\eta} + \tfrac{5}{2} \int_{S^1} f^2_\eta f^3_{\eta\eta} f_{\eta\eta\eta\eta} \\
&= -\tfrac{5}{2} \| f_\eta \|^2_{L^\infty(S^1)} \int_{S^1} f^3_{\eta\eta} f_{\eta\eta\eta\eta} + \tfrac{5}{2} \int_{S^1} f^2_\eta f^3_{\eta\eta} f_{\eta\eta\eta\eta} \\
&\leqq 5 \| f_\eta \|^2_{L^\infty(S^1)} \| f_{\eta\eta} \|^3_{L^6(S^1)} \| f_{\eta\eta\eta\eta} \|_{L^2(S^1)}.
\end{aligned}$$

The first assertion of the lemma thus follows. Similarly, one can calculate

$$\begin{aligned}
\| f_{\eta\eta\eta} \|^3_{L^3(S^1)} &= - \int_{S^1} f_{\eta\eta} (| f_{\eta\eta\eta} | \, f_{\eta\eta\eta})_\eta \\
&\leqq 2 \int_{S^1} | f_{\eta\eta} f_{\eta\eta\eta} f_{\eta\eta\eta\eta} | \leqq 2 \| f_{\eta\eta} \|_{L^6(S^1)} \| f_{\eta\eta\eta} \|_{L^3(S^1)} \| f_{\eta\eta\eta\eta} \|_{L^2(S^1)},
\end{aligned}$$

which, together with the first assertion of the lemma, yields the second assertion of the lemma. □

In the sequel, we shall denote by S^1_T the set $S^1 \times (0, T)$ and by $f(t)$ the function $f(\cdot, t)$.

Lemma 2.4. *Assume that* $f \in L^\infty[0, T; H^2(S^1)] \cap H^1[0, T; L^2(S^1)]$. *Then,*

(1) *for every* $0 \leqq \tau < t \leqq T$, f *satisfies*

$$\| f(t) - f(\tau) \|_{C^1(S^1)} \leqq (t - \tau)^{1/8} \| f_t \|^{1/4}_{L^2(S^1_T)} (T^{1/4} \| f_t \|^{1/2}_{L^2(S^1_T)} + 2 \| (f(t) - f(\tau))_{\eta\eta} \|^{1/2}_{L^2(S^1)})^{3/2};$$

(2) *there exists a constant* C *depending only on* T *such that*

$$\| f \|_{C^{1,\frac{3}{8}}(S^1_T)} + \| f_\eta \|_{C^{\frac{1}{2},\frac{1}{8}}(S^1_T)} \leqq C (\| f(0) \|_{L^2(S^1)} + \sup_{0 \leqq t \leqq T} \| f_{\eta\eta}(t) \|_{L^2(S^1)} + \| f_t \|_{L^2(S^1_T)});$$

(3) *the imbedding from* $L^2[0,T;H^4(S^1)] \cap L^\infty[0,T;H^2(S^1)] \cap H^1[0,T;L^2(S^1)]$ *to* $L^3[0, T; W^{3,3}(S^1)] \cap L^6[0, T; W^{2,6}(S^1)]$ *is continuous and compact. Here* $W^{m,p}(S^1)$ *is the Sobolev space consisting of functions whose distributional derivatives, up to order* m, *are in* $L^p(S^1)$.

Proof. Direct calculation yields

$$\|f(t) - f(\tau)\|^2_{L^2(S^1)} = \int_\tau^t \int_{S^1} (f(\eta, t) - f(\eta, \tau))\, f_t(\eta, \tau_1)\, d\eta\, d\tau_1$$
$$\leqq \sqrt{t-\tau}\, \|f(t) - f(\tau)\|_{L^2(S^1)}\, \|f_t\|_{L^2(S^1_T)}.$$

It follows that

$$\|f(t) - f(\tau)\|_{L^2(S^1)} \leqq \sqrt{t-\tau}\, \|f_t\|_{L^2(S^1_T)}.$$

A similar calculation yields

$$\|f_\eta(t) - f_\eta(\tau)\|^2_{L^2(S^1)} = \int_\tau^t \int_{S^1} (f_\eta(\eta, t) - f_\eta(\eta, \tau))\, f_{\eta t}(\eta, \tau_1)\, d\eta\, d\tau_1$$
$$= -\int_\tau^t \int_{S^1} (f_{\eta\eta}(\eta, t) - f_{\eta\eta}(\eta, \tau))\, f_t(\eta, \tau_1)\, d\eta\, d\tau_1$$
$$\leqq \sqrt{t-\tau}\, \|f_{\eta\eta}(t) - f_{\eta\eta}(\tau)\|_{L^2(S^1)}\, \|f_t\|_{L^2(S^1_T)}.$$

It then follows that

$$\|f(t)-f(\tau)\|^2_{L^\infty(S^1)} \leqq \|f(t)-f(\tau)\|^2_{L^2(S^1)} + 2\|f(t)-f(\tau)\|_{L^2(S^1)}\|f_\eta(t)-f_\eta(\tau)\|_{L^2(S^1)}$$
$$\leqq (t-\tau)^{3/4}\|f_t\|^{3/2}_{L^2(S^1_T)}\left(T^{1/4}\|f_t\|^{1/2}_{L^2(S^1_T)} + 2\|f_{\eta\eta}(t)-f_{\eta\eta}(\tau)\|^{1/2}_{L^2(S^1)}\right),$$

and that

$$\|f_\eta(t) - f_\eta(\tau)\|^2_{L^\infty(S^1)}$$
$$\leqq \|f_\eta(t) - f_\eta(\tau)\|^2_{L^2(S^1)} + 2\|f_\eta(t) - f_\eta(\tau)\|_{L^2(S^1)}\, \|f_{\eta\eta}(t) - f_{\eta\eta}(\tau)\|_{L^2(S^1)}$$
$$\leqq (t-\tau)^{1/4}\, \|f_t\|^{1/2}_{L^2(S^1_T)}\, \|(f(t) - f(\tau))_{\eta\eta}\|_{L^2(S^1)}$$
$$\times \left(T^{1/4}\, \|f_t\|^{1/2}_{L^2(S^1_T)} + 2\|(f(t)-f(\tau))_{\eta\eta}\|^{1/2}_{L^2(S^1)}\right).$$

The first assertion of the lemma thus follows.

Using the $L^\infty[0, T; H^2]$ norm to control the η direction and the previous two estimates to control the t direction, one easily obtains the second assertion of the lemma.

To prove the last assertion of the lemma, we need only show that the embedding is compact since Lemma 2.3 and the first assertion of Lemma 2.4 shows that the embedding is bounded, which implies that it is also continuous.

To prove that the imbedding is compact, let $\{f_j\}_{j=1}^\infty$ be a bounded sequence in $L^2[0, T; H^4(S^1)] \cap L^\infty[0, T; H^2(S^1)] \cap H^1[0, T; L^2(S^1)]$. Then the second assertion of the lemma ensures that there exists a subsequence $\{f_{j_k}\}_{k=1}^\infty$ of $\{f_j\}$ such that f_{j_k} converges in $C^{1/16}[0, T; C^1(S^1)]$. Therefore, applying Lemma 2.3 with $f = f_{j_k} - f_{j_{k'}}$, we find that $\{f_{j_k}\}$ is a Cauchy sequence in $L^6[0, T; W^{2,6}(S^1)] \cap L^3[0, T; W^{3,3}(S^1)]$. This shows that the imbedding is compact, thereby completing the proof of the lemma. □

Now we are in a position to prove the following theorem.

Theorem 2.5. *Let $\mathscr{M}_0 \in C^4$ and $\varepsilon \in (0, 1)$ be given and let δ_0 and $Y(\eta, h)$ be defined as in* (2.2) *and* (2.3). *Assume that Γ_0 is given by* (2.4) *with $d^0 \in C^4(S^1) \cap \mathscr{M}_{\frac{1}{2}\delta_0}$. Then*

(1) *there exists a positive constant T (depending on ε) such that the regularized problem* (2.7) (*or* (2.1)) *has a solution $d \in H^1[0, T; L^2(S^1)] \cap L^2[0, T; H^4(S^1)] \cap L^\infty[0, T; H^2(S^1)] \cap C[0, T; \mathscr{M}_{\delta_0}]$;*

(2) *there exists a positive constant δ_1 which depends only on $\mathscr{M}_0$ and Ω such that the solution of* (2.7) (*or* (2.1)) *can be extended as long as*

$$\sup_{0 \leqq t \leqq T} \|d(t) - d^0\|_{C^1(S^1)} < \delta_1; \tag{2.12}$$

more precisely, if $[0, T_\varepsilon)$ is the maximum interval such that (2.7) *has a solution $d \in L^2[0, T; H^4(S^1)] \cap H^1[0, T; L^2(S^1)] \cap L^\infty[0, T; H^2(S^1)]$ and d satisfies* (2.12) *for any $T < T_\varepsilon$, then either $T_\varepsilon = \infty$ or*

$$\sup_{0 \leqq t \leqq T_\varepsilon} \|d(t) - d^0\|_{C^1(S^1)} = \delta_1. \tag{2.13}$$

Proof. Set

$$E_0 = \frac{1}{\varepsilon^6} + \int_{S^1} (|d^0_{\eta\eta\eta\eta}|^2 + |d^0_{\eta\eta\eta}|^3 + |d^0_{\eta\eta}|^6)$$

and

$$\mathscr{K}_{T,M} = \left\{ d \in C[0, T; C^1(S^1)] \;\middle|\; \begin{array}{l} d(t) \in \overline{\mathscr{M}}_{\delta_0} \quad \forall t \in [0, T], \\ \|d_{\eta\eta} - d^0_{\eta\eta}\|^6_{L^6(S^1_T)} \leqq M, \\ \|d_{\eta\eta\eta} - d^0_{\eta\eta\eta}\|^3_{L^3(S^1_T)} \leqq M \end{array} \right\} \qquad \forall T > 0,\ M > 0.$$

Let M be any fixed positive constant, say $M = 1$. Let T be a positive constant to be determined and denote by $\mathscr{X}$ the Banach space

$$C[0, T; C^1(S^1)] \cap L^3[0, T; W^{3,3}(S^1)] \cap L^6[0, T; W^{2,6}(S^1)].$$

Then, the subset defined by

$$\mathscr{K} = \mathscr{K}_{T,M}$$

is closed and convex in $\mathscr{X}$.

For any $\tilde{d} \in \mathscr{K}$, consider the linear evolution problem

$$\begin{aligned} \bar{d}_t + \varepsilon J^{-4}(\eta, \tilde{d}, \tilde{d}_\eta)\, \bar{d}_{\eta\eta\eta\eta} &= \mathscr{F}[\tilde{d}] \qquad \text{in } S^1_T = S^1 \times (0, T), \\ \bar{d}(\eta, 0) &= d^0(\eta) \qquad \text{on } S^1 \times \{0\}. \end{aligned} \tag{2.14}$$

Since $\tilde{d}(t) \in \mathscr{M}_{\delta_0}$ and $\delta_0 \leqq (2\|\kappa^0\|_{L^\infty(S^1)})^{-1}$, it follows that

$$\tfrac{1}{2} \leqq J \leqq 2, \tag{2.15}$$

and by Lemma 2.2,

$$\begin{aligned}\|\mathscr{F}[\tilde{d}]\|^2_{L^2(S^1_T)} &\leqq C_0\{T/\varepsilon^4 + \varepsilon^2\|\tilde{d}_{\eta\eta}\|^6_{L^6(S^1_T)} + \varepsilon^2\|\tilde{d}_{\eta\eta\eta}\|^3_{L^3(S^1_T)}\}\\ &\leqq C_0[T/\varepsilon^4 + \varepsilon^2(2^6E_0T + 2^6M)]\\ &\leqq C\varepsilon^2(E_0T + M). \qquad (2.16)\end{aligned}$$

Hence, from classical semigroup theory [15], the problem (2.14) admits a unique solution $\bar{d} \in L^2[0, T; H^4(S^1)] \cap H^1[0, T; L^2(S^1)]$.

Multiplying (2.14a) by $\varepsilon(\bar{d} - d^0)_{\eta\eta\eta\eta}$ and integrating over $S^1_t \equiv S^1 \times (0, t)$, after routine calculation, we obtain

$$\varepsilon\|\bar{d}_{\eta\eta}(t) - d^0_{\eta\eta}\|^2_{L^2(S^1)} + \varepsilon^2\|(\bar{d} - d^0)_{\eta\eta\eta\eta}\|^2_{L^2(S^1_t)} \leqq C\varepsilon^2[E_0T + M] \quad \forall t \in (0, T).$$

Using equation (2.14a) to estimate $\bar{d}_t$, we then have the estimate

$$\|\bar{d}_t\|^2_{L^2(S^1_T)} + \varepsilon^2\|(\bar{d} - d^0)_{\eta\eta\eta\eta}\|^2_{L^2(S^1_T)} + \varepsilon \sup_{0\leqq t\leqq T}\|\bar{d}_{\eta\eta}(t) - d^0_{\eta\eta}\|^2_{L^2(S^1)} \leqq C_1\varepsilon^2[E_0T + M] \qquad (2.17)$$

for some constant C_1 depending only on C_0.

Lemma 2.3 then implies that

$$\|\bar{d}_{\eta\eta} - d^0_{\eta\eta}\|^6_{L^6(S^1_T)} \leqq C_2 \sup_{0\leqq t\leqq T}\|\bar{d}(t) - d^0\|^4_{C^1(S^1)}[E_0T + M], \qquad (2.18)$$

$$\|\bar{d}_{\eta\eta\eta} - d^0_{\eta\eta\eta}\|^3_{L^3(S^1_T)} \leqq C_3 \sup_{0\leqq t\leqq T}\|\bar{d}(t) - d^0\|_{C^1(S^1)}[E_0T + M], \qquad (2.19)$$

whereas Lemma 2.4(1) implies that

$$\sup_{0\leqq t\leqq T}\|\bar{d}(t) - d^0\|_{C^1(S^1)} \leqq C_4T^{1/8}(\varepsilon^2[E_0T + M])^{1/2}(T^{1/4} + \varepsilon^{-1/2})^{3/2}. \qquad (2.20)$$

Hence, if we take T small enough (depending on ε and M), then $\bar{d} \in \mathscr{K}$; that is, the mapping $\mathscr{T}: \tilde{d} \to \bar{d}$ maps $\mathscr{K}$ into itself. Notice that $\mathscr{T}(\mathscr{K})$ is a bounded set in $L^2[0, T; H^4] \cap L^\infty[0, T; H^2(S^1)] \cap H^1[0, T; L^2(S^1)]$; it follows from Lemma 2.4(3) that $\mathscr{T}$ is also compact. Since the mapping $\tilde{d} \to J(\eta, \tilde{d}, \tilde{d}_\eta)$ is continuous from $\mathscr{K}$ to $C(S^1_T)$ and the mapping $\tilde{d} \to \mathscr{F}[\tilde{d}]$ is continuous from $\mathscr{K} \subset \mathscr{X}$ to $L^2(S^1_T)$, one can easily verify that $\mathscr{T}$ is continuous. Therefore, the Schauder fixed-point theorem implies that $\mathscr{T}$ has a fixed point, which clearly is a solution to (2.7). This establishes the first assertion of the theorem.

Let C_2 and C_3 be the constants in (2.18) and (2.19), and set

$$\delta_1 = \min\{\tfrac{1}{2}\delta_0, \ (4C_3)^{-1}, \ (4C_2)^{-1/4}\}.$$

We now prove that the second assertion of the theorem holds for this δ_1.

Let $T \in (0, \infty)$ be a constant such that (2.7) has a solution d in $[0, T]$ and d satisfies

$$\sup_{0\leqq t\leqq T}\|d(t) - d^0\|_{C^1(S^1)} \leqq \delta_1 - \mu \quad \text{for some } \mu > 0. \qquad (2.21)$$

We show that the solution can be extended to $T + \mu_1$ for some positive constant μ_1 depending on μ.

Define

$$N(t) = \|d_{\eta\eta\eta} - d^0_{\eta\eta\eta}\|^3_{L^3(S^1_T)} + \|d_{\eta\eta} - d^0_{\eta\eta}\|^6_{L^6(S^1_t)}, \quad t \in [0, T].$$

Then, we can proceed as before to show that for all $t \in [0, T]$, $N(t)$ satisfies

$$N(t) \leqq \sup_{0 \leqq \tau \leqq t} [C_2\|d(\tau) - d^0\|^4_{C^1(S^1)} + C_3\|d(\tau) - d^0\|_{C^1(S^1)}](E_0 t + N(t)).$$

It then follows from (2.21) and the definition of δ that

$$N(t) \leqq E_0 t \quad \forall t \in [0, T].$$

As in the proof of (2.17), we can show that for all $t \in [0, T]$,

$$\|d_t\|^2_{L^2(S^1_t)} + \varepsilon^2\|(d - d^0)_{\eta\eta\eta\eta}\|^2_{L^2(S^1_t)} + \varepsilon \sup_{0 \leqq t \leqq T}\|(d(t) - d^0)_{\eta\eta}\|^2_{L^2(S^1)} \leqq 2C_1 E_0 t. \tag{2.22}$$

Now let μ_1 be a small positive constant to be determined. Define

$$\mathscr{K} = \{\tilde{d} \in \mathscr{K}_{T+\mu_1, E_0 T+1} \mid \tilde{d} = d \quad \forall t \in [0, T]\}$$

and proceed as before to define $\bar{d}$ for every $\tilde{d} \in \mathscr{K}$. As in the proof of (2.20) (noting that $\|\bar{d}_{\eta\eta}(t) - \bar{d}_{\eta\eta}(\tau)\| \leqq \|\bar{d}_{\eta\eta}(t) - d^0_{\eta\eta}\| + \|\bar{d}_{\eta\eta}(\tau) - d^0_{\eta\eta}\|$) we can show that for all $0 \leqq \tau < t \leqq T + \mu_1$, d satisfies

$$\|\bar{d}(t) - \bar{d}(\tau)\|_{C^1(S^1)} \leqq 2C_4(t - \tau)^{1/8}(\varepsilon^2[(T + \mu_1)E_0 + E_0 T + 1])^{1/2} \times ((T + \mu_1)^{1/4} + \varepsilon^{-1/2})^{3/2}.$$

It then follows that

$$\begin{aligned}\|\bar{d}(t) - d^0\|_{C^1(S^1)} &\leqq \|\bar{d}(t) - \bar{d}(T)\|_{C^1(S^1)} + \|\bar{d}(T) - d^0\|_{C^1(S^1)} \\ &\leqq C(T, \varepsilon)\mu_1^{1/8} + \delta_1 - \mu \quad \forall t \in [T, T + \mu_1],\end{aligned} \tag{2.23}$$

where we have used (2.21) and the fact that $\bar{d} = \tilde{d} = d$ for $t \in [0, T]$. And therefore, we can prove that $\mathscr{T}$ maps $\mathscr{K}$ into itself if μ_1 is small enough. Applying the same argument as before leads to the existence of a solution of (2.7) in the interval $[0, T + \mu_1]$. From (2.23), we can take μ_1 so small that the solution still satisfies (2.12). Also, if T is bounded from above and μ is bounded away from zero, then μ_1 is bounded away from zero. Therefore, either $T_\varepsilon = \infty$ or $T_\varepsilon < \infty$ and (2.13) holds. This completes the proof of the theorem. □

In the sequel, we shall always assume that T_ε is the maximum interval defined in Theorem 2.5. Notice that if the initial data are smooth (the norm may depend on ε), we can use a boot-strap argument to show that the solution is smooth. Therefore, we shall also assume that the solution is as smooth as we wish.

3. Energy Identities

In this section, we derive several energy identities for the solution of (2.1); i.e., we prove the following theorem.

Theorem 3.1. *Assume that* $\Gamma_0 \in C^4$ *and* $\partial\Omega \in C^2$. *Let* $\Gamma^\varepsilon = \bigcup_{0\leqq t<T_\varepsilon} \Gamma_t^\varepsilon \times \{t\}$ *be the solution of* (2.1) *and let* V^ε, κ^ε, *and* u^ε *be the corresponding normal velocity, curvature, and harmonic function. Denote by* $S^\varepsilon(t)$ *the arc length of* Γ^ε. *Then, for all* $t \in (0, T_\varepsilon)$, *the following identities hold*:

$$\frac{d}{dt} S^\varepsilon(t) + \varepsilon \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2}\, ds + \iint_\Omega |\nabla u^\varepsilon(t)|^2\, dx = 0, \tag{3.1}$$

$$\frac{d}{dt} \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} + 2\varepsilon \int_{\Gamma_t^\varepsilon} \kappa_{ss}^{\varepsilon 2} + \iint_{\Omega\setminus\Gamma_t^\varepsilon} |D_x^2 u^\varepsilon(t)|^2$$
$$= \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 3} V^\varepsilon + \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon [(u_n^{\varepsilon+})^2 - (u_n^{\varepsilon-})^2] - \int_{\partial\Omega} \kappa_\partial (u_s^{\varepsilon+})^2, \tag{3.2}$$

$$\frac{d}{dt} \iint_\Omega |\nabla u^\varepsilon(t)|^2 + \varepsilon \frac{d}{dt} \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2} + 2 \int_{\Gamma_t^\varepsilon} V_s^{\varepsilon 2}$$
$$= 2 \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} V^{\varepsilon 2} - \int_{\Gamma_t^\varepsilon} V^\varepsilon [(u_n^{\varepsilon+})^2 - (u_n^{\varepsilon-})^2] + \varepsilon \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon V^\varepsilon \kappa_s^{\varepsilon 2} \tag{3.3}$$

where D_x^2 *refers to all the second derivatives with respect to the space variables,* κ_∂ *is the curvature of* $\partial\Omega$, *and* $u^{\varepsilon-}(\cdot, t)$ *and* $u^{\varepsilon+}(\cdot, t)$ *are respectively the restrictions of* $u^\varepsilon(\cdot, t)$ *on* $\Omega_t^{\varepsilon-}$, *the domain enclosed by* Γ_t^ε, *and on* $\Omega_t^{\varepsilon+} \equiv \Omega\setminus(\Omega_t^{\varepsilon-} \cup \Gamma_t^\varepsilon)$.

Proof. Multiplying (2.1a) by κ^ε and integrating over Γ_t^ε yield

$$\int_{\Gamma_t^\varepsilon} V^\varepsilon \kappa^\varepsilon - \varepsilon \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2} = \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon [\kappa_n^\varepsilon]_{\Gamma_t^\varepsilon} = \int_{\Gamma_t^\varepsilon} (u^{\varepsilon-} u_n^{\varepsilon-} - u^{\varepsilon+} u_n^{\varepsilon+}) = \iint_\Omega |\nabla u^\varepsilon(t)|^2\, dx.$$

To calculate the first term on the left-hand side, we parametrize Γ^ε in terms of the arc length parameter s, i.e., we write Γ^ε as

$$\Gamma^\varepsilon = \{X(s, t) \mid s \in \mathscr{R}^1,\ t \in [0, T_\varepsilon)\}$$

where for all $t \in [0, T_\varepsilon)$ and $s \in \mathscr{R}^1$, $X(s, t)$ satisfies

$$X(s, t) = X(s + S^\varepsilon(t), t), \tag{3.4}$$

$$X_s(s, t) = \tau(s, t), \tag{3.5}$$

$$X_{ss}(s, t) = -\kappa^\varepsilon(s, t)\, n(s, t), \tag{3.6}$$

$$V^\varepsilon(s, t) = -X_t(s, t) \cdot n(s, t), \tag{3.7}$$

$$\kappa^\varepsilon(s, t) = -X_{ss}(s, t) \cdot n(s, t), \tag{3.8}$$

$$n_s(s, t) = \kappa^\varepsilon(s, t)\, \tau(s, t). \tag{3.9}$$

It then follows that

$$\int_{\Gamma_t^\varepsilon} V^\varepsilon \kappa^\varepsilon = \int_0^{S^\varepsilon(t)} X_{ss} \cdot X_t\, ds = X_s \cdot X_t \Big|_0^{S^\varepsilon(t)} - \int_0^{S^\varepsilon(t)} X_s \cdot X_{st} = -\frac{d}{dt} S^\varepsilon(t)$$

since $X_s \cdot X_{st} = \frac{1}{2}(|X_s|^2)_t = 0$ and

$$X_s \cdot X_t|_0^{S^\varepsilon(t)} = -\frac{d}{dt} S^\varepsilon(t), \tag{3.10}$$

which follows by differentiating (3.4) with respect to s and t and multiplying the resulting equations. The identity (3.1) thus follows.

We proceed to show (3.2). Using the conventional summation notation over double indices, we can compute

$$\iint_{\Omega\setminus\Gamma_t^\varepsilon} |D_x^2 u^\varepsilon(t)|^2 = \iint_{\Omega\setminus\Gamma_t^\varepsilon} \nabla(u_{x_i}^\varepsilon \nabla u_{x_i}^\varepsilon)$$

$$= \int_{\Gamma_t^\varepsilon} (u_{x_i}^{\varepsilon-} u_{x_i x_j}^{\varepsilon-} n_j - u_{x_i}^{\varepsilon+} u_{x_i x_j}^{\varepsilon+} n_j) + \int_{\partial\Omega} u_{x_i}^{\varepsilon+} u_{x_i x_j}^{\varepsilon+} n_j$$

where n_j is the jth component of n. Since $\partial_{x_i} = \tau_i \partial_s + n_i \partial_n$, it follows that for either $u = u^{\varepsilon-}$ or $u = u^{\varepsilon+}$,

$$\int_{\Gamma_t^\varepsilon} u_{x_i} u_{x_i x_j} n_j = \int_{\Gamma_t^\varepsilon} u_s \tau_i u_{x_i x_j} n_j + u_n n_i u_{x_i x_j} n_j$$

$$= \int_{\Gamma_t^\varepsilon} \kappa_s^\varepsilon[(u_{x_j} n_j)_s - u_{x_j}(n_j)_s] + u_n[\Delta u - ((u_{x_j}\tau_j)_s - u_{x_j}(\tau_j)_s)]$$

$$= \int_{\Gamma_t^\varepsilon} [\kappa_s^\varepsilon (u_n)_s - \kappa_s^\varepsilon u_{x_j}(\kappa^\varepsilon \tau_j)] + u_n[0 - \kappa_{ss}^\varepsilon + u_{x_j}(-\kappa^\varepsilon n_j)]$$

$$= -2\int_{\Gamma_t^\varepsilon} \kappa_{ss}^\varepsilon u_n - \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon \kappa_s^{\varepsilon 2} - \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon u_n^2$$

where we have used the identities $\partial_s = \tau_i \partial_{x_i}$ and $\Delta u = \tau_i u_{x_i x_j} \tau_j + n_i u_{x_i x_j} n_j$ in the second equality, the identities $n_s = \kappa^\varepsilon \tau$ and $\tau_s = -\kappa^\varepsilon n$ in the third equality, and integration by parts in the last equality. Similarly, using $u_n^\varepsilon = 0$ on $\partial\Omega$, we can derive

$$\int_{\partial\Omega} u_{x_i}^{\varepsilon+} u_{x_i x_j}^{\varepsilon+} n_j = -\int_{\partial\Omega} \kappa_\partial (u_s^{\varepsilon+})^2.$$

Hence,

$$\iint_{\Omega\setminus\Gamma_t^\varepsilon} |D_x^2 u^\varepsilon(t)|^2 = -2\int_{\Gamma_t^\varepsilon} \kappa_{ss}^\varepsilon [u_n^\varepsilon]_{\Gamma_t^\varepsilon} + \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon[(u_n^{\varepsilon+})^2 - (u_n^{\varepsilon-})^2] - \int_{\partial\Omega} \kappa_\partial (u_s^{\varepsilon+})^2.$$

Substituting (2.1a) into the first term on the right-hand side yields

$$\iint_{\Omega\setminus\Gamma_t^\varepsilon} |D_x^2 u^\varepsilon(t)|^2 + 2\varepsilon \int_{\Gamma_t^\varepsilon} \kappa_{ss}^{\varepsilon 2}$$

$$= -2\int_{\Gamma_t^\varepsilon} \kappa_{ss}^\varepsilon V^\varepsilon + \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon[(u_n^{\varepsilon+})^2 - (u_n^{\varepsilon-})^2] - \int_{\partial\Omega} \kappa_\partial (u_s^{\varepsilon+})^2. \tag{3.11}$$

Using the identities (3.5)–(3.10), we can calculate the first integral on the right-hand side by

$$\int_{\Gamma_t^\varepsilon} \kappa_{ss}^\varepsilon V^\varepsilon = \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon V_{ss}^\varepsilon = -\int_{\Gamma_t^\varepsilon} \kappa^\varepsilon (X_t \cdot n)_{ss}$$

$$= -\int_{\Gamma_t^\varepsilon} \kappa^\varepsilon [X_{sst} \cdot n + 2X_{st} \cdot n_s + X_t \cdot n_{ss}] =$$

$$= - \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon[-(\kappa^\varepsilon n)_t \cdot n + 2X_{st} \cdot (\kappa^\varepsilon \tau) + X_t \cdot (\kappa^\varepsilon \tau)_s]$$

$$= - \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon[-\kappa_t^\varepsilon - \kappa^\varepsilon n_t \cdot n + 2\tau_t \cdot \kappa^\varepsilon \tau + \kappa_s^\varepsilon X_t \cdot \tau + \kappa^\varepsilon X_t \cdot \tau_s]$$

$$= - \int_{\Gamma_t^\varepsilon} [-\tfrac{1}{2}(\kappa^{\varepsilon 2})_t + 0 + 0 + \tfrac{1}{2}(\kappa^{\varepsilon 2} X_t \cdot \tau)_s - \tfrac{1}{2}\kappa^{\varepsilon 2}(X_t \cdot \tau)_s - \kappa^{\varepsilon 3} X_t \cdot n]$$

$$= \tfrac{1}{2}\frac{d}{dt}\int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} - \tfrac{1}{2}\kappa^{\varepsilon 2}(S^\varepsilon(t), t)\frac{d}{dt}S^\varepsilon(t) - \tfrac{1}{2}\kappa^{\varepsilon 2} X_t \cdot \tau \Big|_0^{S^\varepsilon(t)} - \tfrac{1}{2}\int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 3} V^\varepsilon$$

$$= \tfrac{1}{2}\frac{d}{dt}\int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} - \tfrac{1}{2}\int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 3} V^\varepsilon.$$

Substituting the last equality into (3.11), we obtain the identity (3.2).

Finally we prove (3.3). Direct differentiation yields

$$\frac{d}{dt}\iint_\Omega |\nabla u^\varepsilon(t)|^2 = 2\iint_{\Omega \setminus \Gamma_t^\varepsilon} \nabla u^\varepsilon \nabla u_t^\varepsilon + \int_{\Gamma_t^\varepsilon} V^\varepsilon[|\nabla u^{\varepsilon+}|^2 - |\nabla u^{\varepsilon-}|^2]$$

$$= 2\int_{\Gamma_t^\varepsilon} (u_n^{\varepsilon-} u_t^{\varepsilon-} - u_n^{\varepsilon+} u_t^{\varepsilon+}) + \int_{\Gamma_t^\varepsilon} V^\varepsilon[(u_n^{\varepsilon+})^2 - (u_n^{\varepsilon-})^2]. \tag{3.12}$$

Differentiating the identity $\kappa^\varepsilon(s, t) = u^{\varepsilon\pm}(X(s, t), t)$ with respect to t yields

$$\kappa_t^\varepsilon = \nabla u^{\varepsilon\pm} \cdot X_t + u_t^{\varepsilon\pm} = u_t^{\varepsilon\pm} + \kappa_s^\varepsilon X_t \cdot \tau - u_n^{\varepsilon\pm} V^\varepsilon.$$

It follows that

$$\int_{\Gamma_t^\varepsilon} (u_n^{\varepsilon-} u_t^{\varepsilon-} - u_n^{\varepsilon+} u_t^{\varepsilon+})$$

$$= \int_{\Gamma_t^\varepsilon} [u_n^\varepsilon]_{\Gamma_t^\varepsilon}(\kappa_t^\varepsilon - \kappa_s^\varepsilon X_t \cdot \tau) - V^\varepsilon[(u_n^{\varepsilon+})^2 - (u_n^{\varepsilon-})^2]$$

$$= \int_{\Gamma_t^\varepsilon} (V^\varepsilon + \varepsilon\kappa_{ss}^\varepsilon)(\kappa_t^\varepsilon - \kappa_s^\varepsilon X_t \cdot \tau) - V^\varepsilon[(u_n^{\varepsilon+})^2 - (u_n^{\varepsilon-})^2]. \tag{3.13}$$

We calculate

$$\int_{\Gamma_t^\varepsilon} V^\varepsilon(\kappa_t^\varepsilon - \kappa_s^\varepsilon X_t \cdot \tau)$$

$$= \int_{\Gamma_t^\varepsilon} V^\varepsilon[-X_{sst} \cdot n - X_{ss} \cdot n_t - \kappa_s^\varepsilon X_t \cdot \tau]$$

$$= \int_{\Gamma_t^\varepsilon} V^\varepsilon[(-(X_t \cdot n)_{ss} + 2X_{st} \cdot n_s + X_t \cdot n_{ss}) + \kappa^\varepsilon n \cdot n_t - \kappa_s^\varepsilon X_t \cdot \tau]$$

$$= \int_{\Gamma_t^\varepsilon} V^\varepsilon[(V_{ss}^\varepsilon + 2\tau_t \cdot (\kappa^\varepsilon \tau) + X_t \cdot (\kappa^\varepsilon \tau)_s) + 0 - \kappa_s^\varepsilon X_t \cdot \tau]$$

$$= -\int_{\Gamma_t^\varepsilon} V_s^{\varepsilon 2} + \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} V^{\varepsilon 2} \tag{3.14}$$

and

$$\begin{aligned}\int_{\Gamma_t^\varepsilon} \kappa_{ss}^\varepsilon(\kappa_t^\varepsilon - \kappa_s^\varepsilon X_t \cdot \tau) &= \kappa_s^\varepsilon \kappa_t^\varepsilon \Big|_0^{S^\varepsilon(t)} - \int_{\Gamma_t^\varepsilon} [\kappa_s^\varepsilon \kappa_{st}^\varepsilon + \kappa_{ss}^\varepsilon \kappa_s^\varepsilon X_t \cdot \tau] \\ &= \kappa_s^\varepsilon \kappa_t^\varepsilon \Big|_0^{S^\varepsilon(t)} - \tfrac{1}{2} \frac{d}{dt} \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2} + \tfrac{1}{2} \kappa_s^{\varepsilon 2}(S^\varepsilon(t), t) \frac{d}{dt} S^\varepsilon(t) \\ &\quad - \tfrac{1}{2} \kappa_s^{\varepsilon 2} X_t \cdot \tau \Big|_0^{S^\varepsilon(t)} + \tfrac{1}{2} \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2}(X_t \cdot \tau)_s \\ &= -\tfrac{1}{2} \frac{d}{dt} \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2} + \tfrac{1}{2} \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2} \kappa^\varepsilon V^\varepsilon \end{aligned} \tag{3.15}$$

where in the last equality, we have used equation (3.10) and the relation $\kappa_t^\varepsilon \kappa_s^\varepsilon \big|_0^{S^\varepsilon(t)} = -\kappa_s^{\varepsilon 2} \dfrac{d}{dt} S^\varepsilon(t)$, which follows by differentiating the identity $\kappa^\varepsilon(0, t) = \kappa^\varepsilon(S^\varepsilon(t), t)$ with respect to s and t and multiplying the resulting equations. Substituting (3.14) and (3.15) into (3.13) and (3.12), we obtain the identity (3.3). This completes the proof of the theorem. □

4. A Priori Estimates

We now use Theorem 3.1 to estimate the solution of (2.1) (or (2.7)). For any $t \in [0, T_\varepsilon)$, define

$$A(t) = S^\varepsilon(t) + \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} + \iint_\Omega |\nabla u^\varepsilon(t)|^2 + \varepsilon \int_{\Gamma_t^\varepsilon} \kappa_s^\varepsilon, \tag{4.1}$$

$$B(t) = \iint_\Omega |\nabla u^\varepsilon(t)|^2 + \iint_{\Omega \setminus \Gamma_t^\varepsilon} |D_x^2 u^\varepsilon(t)|^2 + 2 \int_{\Gamma_t^\varepsilon} V_s^{\varepsilon 2} + \varepsilon \int_{\Gamma_t^\varepsilon} \kappa_s^{\varepsilon 2} + 2\varepsilon \int_{\Gamma_t^\varepsilon} \kappa_{ss}^{\varepsilon 2}. \tag{4.2}$$

Adding up the three identities in Theorem 3.1 yields

$$\begin{aligned}\frac{d}{dt} A(t) + B(t) &= \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 3} V^\varepsilon + 2 \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} V^{\varepsilon 2} + \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon [(u_n^{\varepsilon +})^2 - (u_n^{\varepsilon -})^2] \\ &\quad - \int_{\Gamma_t^\varepsilon} V^\varepsilon [(u_n^{\varepsilon +})^2 - (u_n^{\varepsilon -})^2] - \int_{\partial\Omega} \kappa_\partial (u_s^{\varepsilon +})^2 + \varepsilon \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon V^\varepsilon \kappa_s^{\varepsilon 2}. \end{aligned} \tag{4.3}$$

In order to estimate the right-hand side, we need the following lemma.

Lemma 4.1. (a) *The function* $S^\varepsilon(t)$ *is monotonically decreasing in* $[0, T_\varepsilon)$ *and satisfies*

$$4\pi \geqq S^\varepsilon(0) \geqq S^\varepsilon(t) \geqq \pi \quad \forall t \in [0, T_\varepsilon). \tag{4.4}$$

(b) *There exists a constant C_5 depending only on $\mathscr{M}_0$ and $\partial\Omega$ such that for all $t \in [0, T_\varepsilon)$ the following inequalities hold*:

$$\| f \|_{L^2(\Gamma_t^\varepsilon)} \leqq C_5 \| f \|^{1/2}_{L^2(\Omega_t^{\varepsilon\pm})} (\| \nabla f \|^2_{L^2(\Omega_t^{\varepsilon\pm})} + \| f \|^2_{L^2(\Omega_t^{\varepsilon\pm})})^{1/4} \quad \forall f \in H^1(\Omega_t^\pm), \tag{4.5}$$

$$\| f \|_{L^2(\partial\Omega)} \leqq C_5 \| f \|^{1/2}_{L^2(\Omega_t^{\varepsilon+})} (\| \nabla f \|^2_{L^2(\Omega_t^{\varepsilon+})} + \| f \|^2_{L^2(\Omega_t^{\varepsilon+})})^{1/4} \quad \forall f \in H^1(\Omega_t^{\varepsilon+}), \tag{4.6}$$

$$\| f \|_{L^3(\Gamma_t^\varepsilon)} \leqq C_5 (\| \nabla f \|^2_{L^2(\Omega)} + \| f \|^2_{L^2(\Gamma_t^\varepsilon)})^{1/2} \quad \forall f \in H^1(\Omega). \tag{4.7}$$

Proof. The monotonicity of $S^\varepsilon(t)$ is an immediate consequence of (3.1) whereas (4.4) follows from (2.15) and the formula $S^\varepsilon(t) = \int_0^{2\pi} J(\eta, d, d_\eta)\, d\eta$.

Since $d(\cdot, t) \in \mathscr{M}_{\delta_0}$, the curve Γ_t^ε is Lipschitz continuous and its Lipschitz character depends only on $\mathscr{M}_0$. Hence, the second assertion of the lemma follows from the Sobolev imbedding theorem. This completes the proof of the lemma. □

Applying (4.5) with $f = \nabla u^{\varepsilon\pm}$ yields

$$\| \nabla u^{\varepsilon\pm} \|_{L^2(\Gamma_t^\varepsilon)} \leqq C_5 \| \nabla u^\varepsilon \|^{1/2}_{L^2(\Omega)} (\| D_x^2 u^\varepsilon \|^2_{L^2(\Omega \setminus \Gamma_t^\varepsilon)} + \| \nabla u^\varepsilon \|^2_{L^2(\Omega)})^{1/2} \leqq C_5 A^{1/4} B^{1/4}.$$

We now estimate the right-hand side of (4.3).

Since

$$\int_{\Gamma_t^\varepsilon} V^\varepsilon = \int_{\Gamma_t^\varepsilon} [u_n^\varepsilon] - \varepsilon \int_{\Gamma_t^\varepsilon} \kappa_{ss}^\varepsilon = - \int_{\partial\Omega} u_n^{\varepsilon+} = 0, \tag{4.8}$$

it follows that

$$\| V^\varepsilon \|^2_{L^\infty(\Gamma_t^\varepsilon)} \leqq \| (V^{\varepsilon 2})_s \|_{L^1(\Gamma_t^\varepsilon)} \leqq 2 \| V^\varepsilon \|_{L^2(\Gamma_t^\varepsilon)} \; \| V_s^\varepsilon \|_{L^2(\Gamma_t^\varepsilon)} \leqq \sqrt{2} \, \| V^\varepsilon \|_{L^2(\Gamma_t^\varepsilon)} B^{1/2}. \tag{4.9}$$

Using equation (2.1a), we have

$$\begin{aligned} \| V^\varepsilon \|_{L^2(\Gamma_t^\varepsilon)} &= \| \varepsilon \kappa_{ss}^\varepsilon - [u_n^\varepsilon]_{\Gamma_t^\varepsilon} \|_{L^2(\Gamma_t^\varepsilon)} \\ &\leqq \varepsilon \| \kappa_{ss}^\varepsilon \|_{L^2(\Gamma_t^\varepsilon)} + \| \nabla u^{\varepsilon+} \|_{L^2(\Gamma_t^\varepsilon)} + \| \nabla u^{\varepsilon-} \|_{L^2(\Gamma_t^\varepsilon)} \\ &\leqq \sqrt{\varepsilon/2}\, B^{1/2} + 2 C_5 A^{1/4} B^{1/4}. \end{aligned}$$

Substituting the last inequality into (4.9) yields

$$\| V^\varepsilon \|_{L^\infty(\Gamma_t^\varepsilon)} \leqq \varepsilon^{1/4} B^{1/2} + 2\sqrt{C_5}\, A^{1/8} B^{3/8}. \tag{4.10}$$

It follows that

$$\begin{aligned} \int_{\Gamma_t^\varepsilon} | \kappa^{\varepsilon 3} V^\varepsilon | &\leqq \| \kappa^\varepsilon \|^3_{L^3(\Gamma_t^\varepsilon)} \; \| V^\varepsilon \|_{L^\infty(\Gamma_t^\varepsilon)} \\ &\leqq C_5^3 (\| \nabla u^\varepsilon \|^2_{L^2(\Omega)} + \| u^\varepsilon \|^2_{L^2(\Gamma_t^\varepsilon)})^{3/2} \; \| V^\varepsilon \|_{L^\infty(\Gamma_t^\varepsilon)} \\ &\leqq C_5^3 A^{3/2} (\varepsilon^{1/4} B^{1/2} + 2\sqrt{C_5}\, A^{1/8} B^{3/8}) \\ &\leqq (\varepsilon^{1/2} + \mu) AB + C_\mu A^2 \quad \forall \mu > 0, \end{aligned}$$

where we have used (4.7) in the second inequality and (4.10) in the third inequality.

Similarly, the other terms on the right-hand side of (4.3) can be estimated as follows:

$$\begin{aligned}\int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2} V^{\varepsilon 2} &\leqq \|\kappa^\varepsilon\|^2_{L^2(\Gamma_t^\varepsilon)} \|V^\varepsilon\|^2_{L^\infty(\Gamma_t^\varepsilon)} \\ &\leqq A\,(\varepsilon^{1/4} B^{1/2} + 2\sqrt{C_5}\, A^{1/8} B^{3/8})^2 \\ &\leqq (\varepsilon^{1/2} + \mu)\, AB + C_\mu A^2 \qquad \forall \mu > 0,\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_t^\varepsilon} |\kappa^\varepsilon (u_n^{\varepsilon\pm})^2| &\leqq \|\kappa^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \|\nabla u^{\varepsilon\pm}\|^2_{L^2(\Gamma_t^\varepsilon)} \\ &\leqq (\|\kappa^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} + \|\kappa_s^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)}) \|\nabla u^{\varepsilon\pm}\|^2_{L^2(\Gamma_t^\varepsilon)} \\ &\leqq (A^{1/2} + C_5 A^{1/4} B^{1/4})\, C_5^2 A^{1/2} B^{1/2} \\ &\leqq \mu AB + C_\mu (1 + A),\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_t^\varepsilon} |V^\varepsilon (u_n^{\varepsilon\pm})^2| &\leqq \|V^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \|\nabla u^{\varepsilon\pm}\|^2_{L^2(\Gamma_t^\varepsilon)} \\ &\leqq (C_5^2 \varepsilon^{1/4} + \mu)\, A^{1/2} B + C_\mu A^{3/2} \qquad \forall \mu > 0,\end{aligned}$$

$$\begin{aligned}\int_{\partial\Omega} |\kappa_\partial (u_s^{\varepsilon +})^2| &\leqq \|\kappa_\partial\|_{L^\infty(\partial\Omega)} C_5^2 \|\nabla u^\varepsilon\|_{L^2(\Omega_t^{\varepsilon+})} (\|\nabla u\|^2_{L^2(\Omega_t^{\varepsilon+})} + \|D^2 u^\varepsilon\|^2_{L^2(\Omega_t^{\varepsilon+})})^{1/2} \\ &\leqq C A^{1/2} B^{1/2} \leqq \mu AB + C_\mu \qquad \forall \mu > 0,\end{aligned}$$

$$\begin{aligned}\varepsilon \int_{\Gamma_t^\varepsilon} |\kappa^\varepsilon V^\varepsilon \kappa_s^{\varepsilon 2}| &\leqq \varepsilon \|\kappa_s^\varepsilon\|^2_{L^2(\Gamma_t^\varepsilon)} \|\kappa^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \|V^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \\ &\leqq \varepsilon^{1/2} (\varepsilon^{1/2} \|\kappa_s^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)}) \|\kappa_s^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} (\|\kappa^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} + \|\kappa_s^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)}) \|V^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \\ &\leqq \varepsilon^{1/2} A^{1/2} C_5 A^{1/4} B^{1/4} (A^{1/2} + C_5 A^{1/4} B^{1/4})\, (\varepsilon^{1/4} B^{1/2} + 2\sqrt{C_5}\, A^{1/8} B^{3/8}) \\ &\leqq C \varepsilon^{1/2} (AB + CA^2).\end{aligned}$$

Substituting these estimates into (4.3), we have the following lemma.

Lemma 4.2. *For every* $\mu > 0$, *there exist positive constants* $C_6 = C_6(\mathscr{M}_0, \Omega)$ *and* $C_\mu = C(\mu, \mathscr{M}_0, \Omega)$ *such that the function* $A(t)$ *and* $B(t)$ *defined in* (4.1) *and* (4.2) *satisfy*

$$\frac{d}{dt} A + B \leqq (C_6 \varepsilon^{1/2} + \mu)(A + A^{1/2})\, B + C_\mu (1 + A^2) \qquad \forall t \in (0, T_\varepsilon). \tag{4.11}$$

We shall use the following lemma to replace the classical Gronwall inequality.

Lemma 4.3. *Assume that* $A(t)$ *and* $B(t)$ *are non-negative smooth functions in* $[0, T)$ *and that for some positive constant* M *they satisfy*

$$\frac{d}{dt} A(t) + \left(1 - \frac{A(t) + A^{1/2}(t)}{4(A(0) + 1)}\right) B(t) \leqq M(1 + A^2(t)) \qquad \forall t \in (0, T). \tag{4.12}$$

Then the following inequalities hold:

$$A(t) \leqq A(0) + 1 \quad \forall t \in [0, T_1), \tag{4.13}$$

$$\int_0^{T_1} B(\tau)\, d\tau \leqq 2A(0) + 2M(1 + (A(0) + 1)^2)\, T_1 \tag{4.14}$$

where

$$T_1 = \min\left\{T, \frac{1}{M} \arctan \frac{1}{1 + A(0) + A(0)^2}\right\}. \tag{4.15}$$

Proof. Set

$$\bar{T}_1 \equiv \sup\{t \in [0, T) \mid A(\tau) \leqq 1 + A(0) \quad \forall \tau \in [0, t]\}.$$

Then (4.12) implies that

$$\frac{d}{dt} A(t) \leqq M(1 + A^2(t)) \quad \forall t \in (0, \bar{T}_1)$$

which yields

$$A(t) \leqq \tan(Mt + \arctan A(0)) = \frac{\tan(Mt) + A(0)}{1 - A(0) \tan(Mt)} \quad \forall t \in [0, \bar{T}_1).$$

Since the right-hand side is less than $A(0) + 1$ when $t \in [0, T_1]$ where T_1 is as in (4.15), it follows from the definition of $\bar{T}_1$ that $\bar{T}_1 \geqq T_1$, and therefore inequality (4.13) holds. Integrating (4.12) over $(0, T_1)$ and using the estimate (4.13), we obtain inequality (4.14). This completes the proof of the lemma. □

Introduce

$$A_0^\varepsilon = \int_{\Gamma_0} (\kappa(0))^2 + \iint_{\Omega} |\nabla u(0)|^2 + \varepsilon \int_{\Gamma_0} (\kappa_s(0))^2 \tag{4.16}$$

where $u(\cdot, 0)$ is the harmonic function with the boundary value $\kappa(\cdot, 0)$ (the curvature of Γ_0) on Γ_0.

Lemma 4.4. *Assume that the conditions of Theorem 2.5 hold and let A_0^ε be the constant defined by* (4.16). *Assume that* $\sup_{0 \leqq \varepsilon \leqq 1} A_0^\varepsilon \leqq A_0$ *for some positive constant A_0. Then there exist positive constants ε_0, M_0, and T_0 which depend only on $\mathscr{M}_0$, $\partial\Omega$, and on A_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, the following inequality holds*:

$$\sup_{0 \leqq t < \min\{T_0, T_\varepsilon\}} A(t) + \int_0^{\min\{T_0, T_\varepsilon\}} B(\tau)\, d\tau \leqq M_0$$

where $A(t)$ and $B(t)$ are defined in (4.1) *and* (4.2).

Proof. With ε_0 and μ taken so small that

$$C_6 \varepsilon_0^{1/2} + \mu \leqq \frac{1}{4(A_0 + 1)},$$

the assertion of the lemma follows immediately from Lemma 4.2 and Lemma 4.3. □

Lemma 4.5. *Assume that the conditions of Lemma 4.4 are satisfied and let T_ε be the constant defined in Theorem 2.5. Then there exists a positive constant T^* depending only on $\mathscr{M}_0$, Ω, and A_0 such that*

$$T_\varepsilon > T^* \quad \forall \varepsilon \in (0, \varepsilon_3)$$

where ε_0 is the constant in Lemma 4.4.

Proof. Parametrize Γ^ε as in §2 and denote by $d^\varepsilon(\eta, t)$ the solution of (2.7). By the second assertion of Theorem 2.5, we need only show that for some $T^* > 0$,

$$\sup_{0 \leqq t \leqq \min\{T_\varepsilon, T^*\}} \| d^\varepsilon(t) - d^0 \|_{C^1(S^1)} \leqq \tfrac{1}{2} \delta_1 \quad \forall \varepsilon \in (0, \varepsilon_0)$$

where δ_1 is as in (2.13).

Lemma 4.4 and inequality (4.10) imply that

$$\int_0^{\min\{T_0, T_\varepsilon\}} \| V^\varepsilon(\cdot, t) \|^2_{L^\infty(\Gamma_t^\varepsilon)} \, dt \leqq C M_0 \quad \forall \varepsilon \in (0, \varepsilon_0).$$

It follows from (2.5) that

$$\int_0^{\min\{T_0, T_\varepsilon\}} \sup_{S^1} d_t^{\varepsilon 2}(\cdot, t) \, dt \leqq \sup_{S^1 \times (0, T_\varepsilon)} \left(\frac{J}{(1 + d^\varepsilon \kappa^0)^2} \right) \int_0^{\min\{T_0, T_\varepsilon\}} \| V^\varepsilon(\cdot, t) \|^2_{L^\infty(\Gamma_t^\varepsilon)} \, dt$$
$$\leqq C_7 M_0 \quad \forall \varepsilon \in (0, \varepsilon_0)$$

since $d^\varepsilon(t) \in \mathscr{M}_{\delta_0}$. Consequently, for every $\eta \in S^1$, $t \in [0, \min\{T_0, T_\varepsilon\})$ and $\varepsilon \in (0, \varepsilon_0)$, one has

$$|d^\varepsilon(t) - d^0| \leqq \int_0^t |d_t^\varepsilon(\eta, \tau)| \, d\tau \leqq \sqrt{t} \left(\int_0^t |d_t^\varepsilon(\eta, \tau)|^2 \, d\tau \right)^{1/2} \leqq (C_7 M_0)^{1/2} \sqrt{t}. \tag{4.17}$$

Similarly, from (2.6) and Lemma 4.4 it follows that for all $t \in [0, \min\{T_0, T_\varepsilon\})$ and $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{S^1} |d_{\eta\eta}^\varepsilon(\eta, t)|^2 \, d\eta \leqq C \left(1 + \int_{\Gamma_t^\varepsilon} \kappa^{\varepsilon 2}(\cdot, t) \right) \leqq C(M_0 + 1).$$

Inequality (4.17) and the interpolation inequality

$$\| f \|_{C^1(S^1)} \leqq C \| f \|^{1/3}_{L^\infty(S^1)} (\| f_{\eta\eta} \|_{L^2(S^1)} + \| f \|_{L^2(S^1)})^{2/3} \quad \forall f \in H^2(S^1)$$

then imply that

$$\| d^\varepsilon(t) - d^0 \|_{C^1(S^1)} \leqq C_8 \sqrt{M_0 + 1} \; t^{1/6} \quad \forall t \in [0, \min\{T_0, T_\varepsilon\}), \ \varepsilon \in (0, \varepsilon_0). \tag{4.18}$$

Set

$$T^* \equiv \min \left\{ T_0, \ \frac{\delta_1^2}{16 C_7 M_0}, \ \frac{\delta_1^6}{4^6 C_8^6 (M_0 + 1)^3} \right\}.$$

Then it follows from (4.17) and (4.18) that

$$\| d^\varepsilon(t) - d^0 \|_{C^1(S^1)} \leqq \tfrac{1}{2} \delta_1 \quad \forall t \in [0, \min\{T^*, T_\varepsilon\}).$$

It follows from the definition of T_ε in Theorem 2.5 that

$$T_\varepsilon > \min\{T^*, T_\varepsilon\} \quad \forall \varepsilon \in (0, \varepsilon_0).$$

The assertion of the lemma thus follows. $\square$

We conclude this section with the following theorem which is a consequence of Lemmas 4.4 and 4.5.

Theorem 4.6. *Assume that the conditions of Theorem 2.5 hold and that* $\sup_{0<\varepsilon\leqq 1} A_0^\varepsilon \leqq A_0 < \infty$ *for some* $A_0 > 0$. *Then there exist positive constants* ε_0, T, *and* M *which depend only on* $\mathscr{M}_0$, Ω, *and* A_0 *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *the problem* (2.1) (*or* (2.7)) *has a solution in* $[0, T]$ *satisfying*

$$\sup_{0\leqq t\leqq T}\left(\left|\frac{d}{dt} S^\varepsilon(t)\right| + \|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega)} + \|\kappa^\varepsilon\|^2_{L^2(\Gamma_t^\varepsilon)} + \varepsilon\|\kappa_s^\varepsilon\|^2_{L^2(\Gamma_t^\varepsilon)}\right) \leqq M, \qquad (4.19)$$

$$\int_0^T [\|D_x^2 u^\varepsilon(t)\|^2_{L^2(\Omega\setminus\Gamma_t^\varepsilon)} + \|V_s^\varepsilon\|^2_{H^1(\Gamma_t^\varepsilon)} + \varepsilon\|\kappa_{ss}\|^2_{L^2(\Gamma_t^\varepsilon)}]\, dt \leqq M, \qquad (4.20)$$

$$\int_0^T \|d_t^\varepsilon(t)\|^2_{H^1(S^1)}\, dt + \sup_{0\leqq t\leqq T}\|d^\varepsilon_{\eta\eta}(\cdot, t)\|^2_{L^2(S^1)} \leqq M \qquad (4.21)$$

where $d^\varepsilon(\eta, t)$ *is the solution of* (2.7).

5. Local Existence of a Solution to the Hele-Shaw Problem

We first define a solution of the Hele-Shaw problem in a Sobolev space. Let $G(x, y)$ be the Green function for Δ in the region Ω with the homogeneous Neumann boundary condition, i.e., for each $y \in \Omega$, $G(\cdot, y)$ satisfies

$$-\Delta_x G = \delta(x-y) \quad \text{in } \Omega,$$

$$\frac{\partial G}{\partial n_x} = -\frac{1}{l(\partial\Omega)} \quad \text{on } \partial\Omega,$$

$$\int_{\partial\Omega} G = 0$$

where $l(\partial\Omega)$ is the arc length of $\partial\Omega$. It follows from the Green formula that for any Lipschitz curve $\gamma \subset \Omega$ and any function $f \in L^2(\gamma)$, there exists a solution to

$$-\Delta u = 0 \quad \text{in } \Omega,$$

$$[u]_\gamma = 0 \quad \text{on } \gamma,$$

$$[u_n]_\gamma = f \quad \text{on } \gamma,$$

$$u_n = 0 \quad \text{on } \partial\Omega$$

if and only if $\int_\gamma f = 0$; if $\int_\gamma f = 0$, then the solution is unique up to an additive constant and is given by

$$u = \int_\gamma G(x, y) f(y)\, dS_y + c \tag{5.1}$$

where c is an arbitrary constant.

Notice that if the restriction of u on γ is equal to the curvature of γ, then u satisfies

$$\int_\gamma u(x)\, dS_x = 2\pi,$$

and therefore the constant c in (5.1) is uniquely given by

$$c = \frac{1}{S}\left[2\pi - \int_\gamma\int_\gamma G(x, y) f(y)\, dS_y\, dS_x\right]$$

where S is the arc length of γ. This leads to the following definition for the solution of the Hele-Shaw problem.

Definition 5.1. A family of curves $\bigcup_{0 \leqq t \leqq T}(\Gamma_t \times \{t\}) =: \Gamma$ is called a *solution to the Hele-Shaw problem* (1.1) if there exists a homeomorphism $X: S^1 \times [0, T] \to \Gamma$ such that $X(\eta, t) = (x(\eta, t), t)$, $x(\eta, t) = (x_1(\eta, t), x_2(\eta, t))$,

$$\Gamma_t = \{x(\eta, t) \mid \eta \in S^1\} \quad \forall t \in [0, T],$$

$$x_\eta(\eta, t),\ |x_\eta(\eta, t)|^{-1} \in C(S^1 \times [0, T]),$$

$$x(\eta, t) \in H^{2,1}(S^1_T) \equiv \{x \in L^2(S^1_T) \mid x_{\eta\eta},\ x_t \in L^2(S^1_T)\}.$$

If n denotes the (outward) normal $(x_{2,\eta}, -x_{1,\eta})/|x_\eta|$, if V denotes the (inward) normal velocity $-x_t \cdot n$, and if κ denotes the curvature $-x_{\eta\eta} \cdot n/|x_\eta|^2$, then

$$\int_{S^1} V(\xi, t)\, |x_\xi(\xi, t)|\, d\xi = 0 \quad \text{in } L^2((0, T)), \tag{5.2}$$

$$\kappa(\eta, t) = \int_{S^1} G(x(\eta, t), x(\xi, t))\, V(\xi, t)\, |x_\xi(\xi, t)|\, d\xi + c(t) \quad \text{in } L^2(S^1 \times (0, T)) \tag{5.3}$$

where

$$c(t) = \frac{1}{S(t)}\left[2\pi - \int_{S^1}\int_{S^1} G(x(\eta, t), x(\xi, t))\, V(\xi, t)\, |x_\eta(\eta, t)|\, |x_\xi(\xi, t)|\, d\xi\, d\eta\right],$$

$$S(t) = \int_{S^1} |x_\xi(\xi, t)|\, d\xi.$$

In the sequel, we shall denote by κ_γ the curvature of γ and by u_γ the function harmonic in $\Omega \backslash \gamma$ taking value κ_γ on γ and having a zero normal derivative on $\partial\Omega$.

Our main result is the following:

Theorem 5.1. *Let $\mathscr{M}_0$ be a C^4 one-dimensional manifold embedded in Ω and define δ_0 and Y as in* (2.2) *and* (2.3). *Assume that Γ_0 is a simply connected curve given by* (2.4) *with $d^0 \in \mathscr{M}_{\delta_0/2}$ and that for some positive constant A_0,*

$$\|\kappa_{\Gamma_0}\|^2_{L^2(\Gamma_0)} + \|\nabla u_{\Gamma_0}\|^2_{L^2(\Omega)} \leqq A_0.$$

Then the Hele-Shaw problem starting from Γ_0 has at least one solution in $[0, T]$ *for some positive constant* T *depending only on* $\mathscr{M}_0$, Ω, *and* A_0.

Proof. Notice that $\Gamma_0 \in \mathscr{M}_{\delta_0/2}$ and that $\kappa_{\Gamma_0} \in L^2(\Gamma_0)$ implies that $d^0 \in C^{3/2}(S^1)$. Therefore there exists a sequence $\{d_0^j(\cdot)\}_{j=1}^{\infty}$ in $C^\infty(S^1) \cap \mathscr{M}_{\delta_0/2}$ such that

$$d_0^j \to d_0 \text{ in } C^{3/2}(S^1), \quad \kappa_{\Gamma_0^j}(\eta) \to \kappa_{\Gamma_0}(\eta) \text{ in } L^2(S^1), \quad u_{\Gamma_0^j} \to u_{\Gamma_0} \text{ in } H^1(\Omega)$$

where $\Gamma_0^j = \{Y(\eta, d_0^j(\eta)) \mid \eta \in S^1\}$.

Set

$$\varepsilon_j = (j + \|(\kappa_{\Gamma_0^j})_s\|^2_{L^2(\Gamma_0^j)})^{-1},$$

$$A^j(0) = \|k_{\Gamma_0^j}\|^2_{L^2(\Gamma_0^j)} + \|\nabla u_{\Gamma_0^j}\|^2_{L^2(\Omega)} + \varepsilon_j \|(k_{\Gamma_0^j})_s\|^2_{L^2(\Gamma_0^j)}.$$

Clearly, $A^j(0) \leqq 2A_0 + 1$ if j is large enough. It then follows from Theorem 4.6 that there exist positive constants T and M which depend only on $\mathscr{M}_0$, Ω, and A_0 such that if j is large enough, then the problem (2.7) with $\varepsilon = \varepsilon_j$ and initial value $d_0^j(\cdot)$ has a solution $d^j(\eta, t)$ in the time interval $[0, T]$, and the solution satisfies

$$\sup_{0 \leqq t \leqq T} \|d^j_{\eta\eta}(t)\|^2_{L^2(S^1)} + \int_0^t \|d_t^j\|^2_{L^2(S^1)} \leqq M.$$

Consequently, Lemma 2.4(2) implies that

$$\|d^j\|_{C^{1,3/8}(S^1 \times [0,T])} + \|d^j_\eta\|_{C^{1/2,1/8}(S^1 \times [0,T])} \leqq CM$$

for all j large enough. Therefore, there exists a subsequence of $\{\varepsilon_j\}_{j=1}^\infty$, which we still denote by $\{\varepsilon_j\}$, and a function $d \in H^{2,1}(S^1_T) \cap L^\infty[0, T; H^2(S^1)] \cap C^{1/8}[0, T; \overline{\mathscr{M}_{\delta_0}}]$ such that as $j \to \infty$,

$$d^j(\eta, t) \rightharpoonup d(\eta, t) \qquad \text{weakly in } H^{2,1}(S^1_T), \tag{5.4}$$

$$d^j(\eta, t) \to d(\eta, t) \qquad \text{in } C^{1-\mu, 3/8-\mu}(S^1 \times [0, T]), \tag{5.5}$$

$$d^j_\eta(\eta, t) \to d_\eta(\eta, t) \qquad \text{in } C^{1/2-\mu, 1/8-\mu}(S^1 \times [0, T]) \tag{5.6}$$

for any $\mu > 0$.

Define

$$x^j(\eta, t) = Y(\eta, d^j(\eta, t)), \quad x(\eta, t) = Y(\eta, d(\eta, t)).$$

It follows from (5.4)–(5.6) that, as $j \to \infty$,

$$x^j(\eta, t) \rightharpoonup x(\eta, t) \qquad \text{weakly in } H^{2,1}(S^1_T), \tag{5.7}$$

$$x^j(\eta, t) \to x(\eta, t) \qquad \text{in } C^{1-\mu, 3/8-\mu}(S^1 \times [0, T]), \tag{5.8}$$

$$x^j_\eta(\eta, t) \to x_\eta(\eta, t) \qquad \text{in } C^{1/2-\mu, 1/8-\mu}(S^1 \times [0, T]) \tag{5.9}$$

for any $\mu > 0$. In addition, $d(\cdot, t) \in \overline{\mathscr{M}_{\delta_0}}$ implies that

$$\tfrac{1}{2} \leqq |x_\eta(\eta, t)| \leqq 2 \quad \forall \eta \in S^1, \ t \in [0, T].$$

We now show that

$$\Gamma \equiv \{(x(\eta, t), t) \mid \eta \in S^1, \ t \in [0, T]\}$$

is a solution to the Hele-Shaw problem. Clearly, it suffices to show (5.2) and (5.3).

Since x_η^j converges to x_η in $C^{1/8-\mu}(S^1 \times [0, T])$, it follows that the arc length $S^j(t)$ of Γ_t^j converges to the arc length $S(t)$ of Γ_t in $C^{1/8-\mu}([0, T])$ and the normal $n^j(\eta, t)$ of Γ_t^j converges to the normal $n(\eta, t)$ of Γ_t in $C^{1/8-\mu}(S^1 \times [0, T])$. Also, the curvature $\kappa^j(\eta, t)$ of Γ_t^j converges to the curvature $\kappa(\eta, t)$ of Γ_t weakly in $L^2(S_T^1)$, and the normal velocity $V^j(\eta, t)$ of Γ^j converges to the normal velocity $V(\eta, t)$ of Γ weakly in $L^2(S_T^1)$. Since $\int_{\Gamma_t^j} V^j = \int_{S^1} V^j(\eta, t) |x_\eta^j(\eta, t)| \, d\eta = 0$, it follows that $\int_{S^1} V(\eta, t) |x_\eta(\eta, t)| \, d\eta = 0$ in $L^2(0, T)$; i.e., (5.2) holds.

It remains to show (5.3). Since d^j satisfies (2.7), we have for all $\xi \in S^1$ and $t \in [0, T]$ that

$$\kappa^j(\xi, t) = \int_{S^1} G(x^j(\xi, t), x^j(\eta, t)) \left(V^j(\eta, t) + \varepsilon_j \kappa_{ss}^j(\eta, t)\right) |x_\eta^j(\eta, t)| \, d\eta + c^j(t)$$

where

$$c^j(t) = \frac{1}{S^j(t)} \left[2\pi - \int_{S^1} \int_{S^1} G(x^j(\xi, t), x^j(\eta, t)) \left(V^j(\eta, t) + \varepsilon_j \kappa_{ss}^j(\eta, t)\right) \times |x_\xi^j(\xi, t)| \, |x_\eta^j(\eta, t)| \, d\xi \, d\eta \right].$$

Recall that $S^j(t) \to S(t)$ strongly and $\kappa^j \to \kappa$, $V^j \to V$ weakly; it follows that to show (5.3), we need only show that as $\varepsilon_j \to 0$,

$$G(x^j(\xi, t), x^j(\eta, t)) \ \to \ G(x(\xi, t), x(\eta, t)) \quad \text{in } L^2(0, T; L^2(S^1) \times L^2(S^1)), \tag{5.10}$$

$$\varepsilon_j \int_{S^1} G(x^j(\xi, t), x^j(\eta, t)) \, \kappa_{ss}^j(\eta, t) \, |x_\eta^j(\eta, t)| \, d\eta \ \to \ 0 \quad \text{in } L^2(S^1 \times (0, T)). \tag{5.11}$$

Write G as

$$G(x, y) = -\frac{1}{2\pi} \ln |x - y| + h(x, y)$$

where $h(x, y)$ is a smooth function in $\Omega \times \Omega$. It follows that

$$\begin{aligned} |G(x^j(\eta, t), x^j(\xi, t))| &\leqq \frac{1}{2\pi} \left| \ln |x^j(\eta, t) - x^j(\xi, t)| \right| + |h(x^j(\eta, t), x^j(\xi, t))| \\ &\leqq C(\left| \ln |\eta - \xi| \right| + 1), \end{aligned}$$

where in the second inequality, we have used the inequality $\frac{1}{2} \leqq |x_\eta^j| \leqq 2$ and the fact that x^j is uniformly bounded away from $\partial\Omega$ (so that h is uniformly bounded). Therefore, (5.10) follows from (5.8) and the dominated convergence theorem.

To prove (5.11), recall that Theorem 4.6 implies that

$$\varepsilon_j \int_0^T \int_{\Gamma_t^\varepsilon} \kappa_{ss}^{j2} \leqq M$$

for all j large enough. It follows that

$$\left\| \varepsilon_j \int_{S^1} G(x^j(\xi, t), x^j(\eta, t))\, \kappa_{ss}^j(\eta, t)\, |x_\eta^j(\eta, t)|\, d\eta \right\|_{L^2(S^1 \times (0,T))}$$

$$\leqq \sqrt{\varepsilon_j} \sup_{x \in \Omega, t \in [0,T]} \left(2\pi \int_{\Gamma_t^j} G^2(x, y)\, dS_y \right)^{1/2} \left(\varepsilon_j \int_0^T \int_{\Gamma_t^j} \kappa_{ss}^{j2} \right)^{1/2} \leqq CM \sqrt{\varepsilon_j}\,,$$

which implies (5.11). This completes the proof of Theorem 5.1. □

Theorem 5.2. *Under the same conditions as in Theorem 5.1, the solution of the Hele-Shaw problem given by Theorem 5.1 has the following properties*:

(1) *The arc-length function $S(t)$ is Lipschitz continuous and monotonically decreasing, the function $u(x, t) := u_{\Gamma_t}$ is in $L^\infty[0, T; H^1(\Omega)]$, and these functions satisfy the relation*

$$\frac{d}{dt} S(t) + \|\nabla u(t)\|_{L^2(\Omega)}^2 = 0 \quad \textit{for almost all } t \in (0, T). \tag{5.12}$$

(2) *The function $H(t) = \|\kappa(t)\|_{L^2(\Gamma_t)}^2$ is absolutely continuous on $[0, T]$, $D_x^2 u$ is in $L^2(\Omega_T \backslash \Gamma)$, $\kappa^3 V$ is in $L^1(\Gamma)$ and*

$$\frac{d}{dt} \|\kappa\|_{L^2(\Gamma_t)}^2 + \|D^2 u(t)\|_{L^2(\Omega \backslash \Gamma_t)} = \int_{\Gamma_t} \{\kappa^3 V + \kappa[(u_n^+)^2 - (u_n^-)^2]\} - \int_{\partial\Omega} \kappa_\partial (u_s^+)^2. \tag{5.13}$$

(3) *The function $E(t) = \|\nabla u(t)\|_{L^2(\Omega)}^2$ is absolutely continuous on $[0, T]$, $V(\eta, t)$ is in $L^2[0, T; H^1(S^1)]$, κV is in $L^2(S_T^1)$, and*

$$\frac{d}{dt} \|\nabla u(t)\|_{L^2(\Omega)}^2 + 2\|V_s\|_{L^2(\Gamma_t)}^2 = 2\|\kappa V\|_{L^2(\Gamma_t)}^2 - \int_{\Gamma_t} V[(u_n^+)^2 - (u_n^-)^2]. \tag{5.14}$$

Proof. Since $\{u^j\}$ is uniformly bounded in $L^\infty[0, T; H^1(\Omega)]$, it follows that $u \in L^\infty[0, T; H^1(\Omega)]$. Multiplying both sides of (5.3) by V, integrating over Γ_t, and using the geometric identity $\frac{d}{dt} S = -\int \kappa V$ which is proved in §3, we obtain (5.12). This proves the first assertion of the lemma. The second and third assertions of the lemma can be similarly proved by the method in §3. □

Remark 5.1. With slight modifications, the method exploited in the current and previous sections can be applied to non-simply connected domains, to non-homogeneous Neumann boundary conditions, as well as to the Dirichlet boundary conditions. Also, it can be applied to the two-dimensional Stefan problem with the Gibbs-Thomson relation for the melting temperature. It is possible that the method can be applied to higher dimensions.

Remark 5.2. The regularity and uniqueness of the solution is still an open problem.

6. Global Existence of a Solution to the Hele-Shaw Problem

In this section, we establish the global existence of a solution to the Hele-Shaw problem when the initial curve is close to a circle. For simplicity, we assume that $\Omega = \mathscr{R}^2$. All the previous results still hold if we replace the boundary condition $\partial_n u = 0$ on $\partial\Omega$ by

$$\nabla u = O(|x|^{-2}) \quad \text{when } |x| \to \infty. \tag{6.1}$$

In the sequel, we denote by u_γ the function harmonic in $\mathscr{R}^2 \backslash \gamma$, equal to the curvature of γ on γ, and satisfying (6.1).

We shall prove the following theorem:

Theorem 6.1. *There exists a positive constant δ_2 such that if Γ_0 given by*

$$\Gamma_0 = \{(R_0(\theta)\cos\theta,\ R_0(\theta)\sin\theta) \mid \theta \in S^1\}$$

satisfies

$$\|R_0(\cdot) - 1\|_{C^1(S^1)} \leqq \delta_2, \quad \|\nabla u_{\Gamma_0}\|_{L^2(\mathscr{R}^2)} \leqq \delta_2,$$

then the Hele-Shaw problem starting from Γ_0 has a solution for all $t \in [0, \infty)$.

To prove this theorem, we need the following lemma, which is purely geometric.

Lemma 6.2. (1) *Assume that γ is a curve enclosing a region with area π. Then, its arc length (if finite) is no less than 2π.*

(2) *Assume that γ is a simply-connected curve with arc length S and that the curvature κ of γ is in $L^2(\gamma)$ and satisfies*

$$\int_\gamma |\kappa - \bar{\kappa}| \leqq \tfrac{1}{5}$$

where $\bar{\kappa} = \dfrac{1}{S}\int_\Gamma \kappa = \dfrac{2\pi}{S}$ is the average of κ over γ. Then there exists a point $(x_0, y_0) \in \mathscr{R}^2$ and a $C^{3/2}(S^1)$ function $R(\theta)$ such that

$$\gamma = \{(x_0, y_0) + (R(\theta)\cos\theta,\ R(\theta)\sin\theta) \mid \theta \in S^1\}; \tag{6.2}$$

in addition, the function $R(\cdot)$ satisfies

$$\|R(\cdot) - \bar{R}\|_{C^0(S^1)} \leqq \tfrac{8}{5}\bar{R}\int_\gamma |\kappa - \bar{\kappa}|, \quad \|R_\theta(\cdot)\|_{C^0(S^1)} \leqq \tfrac{15}{8}\bar{R}^2\int_\gamma |\kappa - \bar{\kappa}|$$

where $\bar{R} = \bar{\kappa}^{-1} = \dfrac{S}{2\pi}$.

Proof. The first assertion of the lemma follows from the well-known geometric fact that a circle of radius 1 has the minimum arc length among all the curves enclosing regions of area π.

To prove the second assertion of the lemma, let $x = x(s)$, $y = y(s)$ be a parametrization of γ where s is the arc-length parameter. It follows that $x_s^2 + y_s^2 = 1$, so that there exists a function $\varphi(s)$ satisfying

$$x_s = -\sin\varphi(s), \quad y_s = \cos\varphi(s) \quad \forall s \in [0, S].$$

In addition, the curvature κ of γ is equal to φ_s. Therefore $\varphi_s(s) = \kappa(s) \in L^2([0, S])$. The Sobolev imbedding theorem then implies that $\varphi \in C^{1/2}([0, S])$.

Integrating the equation $x_s = -\sin\varphi$ and using the relation $\varphi_s = \kappa$ yields

$$x(s) = x(0) - \int_0^s \sin\varphi(s) = [x(0) - \bar{R}\cos\varphi(0)] + \bar{R}\cos\varphi(s) + \bar{R}\int_0^s [\kappa - \bar{\kappa}]\sin\varphi.$$

Denoting by x_0 the constant $x(0) - \bar{R}\cos\varphi(0)$, and by Δ_1 the function $\bar{R}\int_0^s [\kappa - \bar{\kappa}]\sin\varphi$, we then obtain

$$x(s) = x_0 + \bar{R}\cos\varphi + \Delta_1.$$

Similarly, integrating $y_s = \cos\varphi$ yields

$$y(s) = y_0 + \bar{R}\sin\varphi + \Delta_2$$

where $\Delta_2(s) = -\bar{R}\int_0^s(\kappa - \bar{\kappa})\cos\varphi$.

Set

$$\tilde{R}(s) = \sqrt{(x(s) - x_0)^2 + (y(s) - y_0)^2}.$$

It then follows from the expression of $x(s)$ and $y(s)$ that

$$\tilde{R}^2 - \bar{R}^2 = \Delta_3 \equiv 2\bar{R}\Delta_1\cos\varphi + 2\bar{R}\Delta_2\sin\varphi + \Delta_1^2 + \Delta_2^2.$$

One can estimate Δ_3 by

$$|\Delta_3| \leqq 2\bar{R}^2\left(1 + \int_\gamma |\kappa - \bar{\kappa}|\right)\int_\gamma |\kappa - \bar{\kappa}| \leqq 2\bar{R}^2(1 + \tfrac{1}{5})\tfrac{1}{5} \leqq \tfrac{1}{2}\bar{R}^2.$$

It follows that $\frac{1}{2}\sqrt{2}\,\bar{R} \leqq \tilde{R} \leqq \frac{1}{2}\sqrt{6}\,\bar{R}$ and

$$|\tilde{R} - \bar{R}| \leqq \frac{1}{\tilde{R} + \bar{R}}|\Delta_3| \leqq \tfrac{8}{5}\bar{R}\int_\gamma |\kappa - \bar{\kappa}|. \tag{6.3}$$

Define

$$\theta(s) = \operatorname{Arctan}\frac{y(s) - y_0}{x(s) - x_0}, \quad s \in [0, S].$$

Then we can compute

$$\begin{aligned}
\theta_s(s) &= \frac{1}{\tilde{R}^2}[(x - x_0)y_s - (y - y_0)x_s] \\
&= \frac{1}{\tilde{R}^2}[(\bar{R}\cos\varphi + \Delta_1)\cos\varphi - (\bar{R}\sin\varphi + \Delta_2)(-\sin\varphi)] \\
&= \frac{1}{\tilde{R}^2}(\bar{R} + \Delta_1\cos\varphi + \Delta_2\sin\varphi).
\end{aligned}$$

Since $|\Delta_1 \cos\varphi + \Delta_2 \sin\varphi| \leqq \bar{R}\int_\gamma |\kappa - \bar{\kappa}| \leqq \frac{1}{5}\bar{R}$, it follows that $\frac{8}{15}\bar{R} \leqq \theta_s \leqq \frac{12}{5}\bar{R}$. Therefore, $\theta(s)$ is monotonic and has an inverse $s = s(\theta)$ satisfying $\frac{5}{12}\bar{R} \leqq s_\theta \leqq \frac{15}{8}\bar{R}$. Since γ is a simply connected curve, we can easily verify that $\theta(S) = \theta(0) + 2\pi$. Define $R(\theta) = \tilde{R}(s(\theta))$; then $R(\theta)$ is a periodic function of period 2π and the curve γ has the representation (6.2).

Notice that if $\varphi(s) \in C^{1/2}$, then $\theta_s(s) \in C^{1/2}$, so that $s(\theta) \in C^{3/2}$. Direct calculation yields

$$R_\theta = \frac{R_s}{\theta_s} = \left.\frac{R^2(\Delta_2 \cos\varphi - \Delta_1 \sin\varphi)}{\bar{R} + \Delta_1 \cos\varphi + \Delta_2 \sin\varphi}\right|_{\varphi=\varphi(s(\theta))},$$

which implies that $R(\theta) \in C^{3/2}(S^1)$. Since $|\Delta_2 \cos\varphi - \Delta_1 \sin\varphi| \leqq \bar{R}\int_\gamma |\kappa - \bar{\kappa}|$, it follows that $|R_\theta| \leqq \frac{15}{8}\bar{R}^2 \int_\gamma |\kappa - \bar{\kappa}|$. Combining this with (6.3), we obtain the second assertion of the lemma. □

We also need the following lemma to prove Theorem 6.1.

Lemma 6.3. *The Hele-Shaw flow preserves the area; that is, if* $\bigcup_{0\leqq t<T}(\Gamma_t \times \{t\})$ *is a solution to the Hele-Shaw problem, then the region enclosed by* Γ_t, $0 < t < T$, *has the same area as the region enclosed by* Γ_0.

This lemma follows immediately from the fact that $\int_{\gamma_t} V = 0$ and the geometric identity

$$\frac{d}{dt}\,\mathrm{Area}(t) = \int_{\Gamma_t} V$$

where $\mathrm{Area}(t)$ is the area of the region enclosed by Γ_t.

Proof of Theorem 6.1. By the local existence Theorem 5.1 and a continuation argument, we need only show that for each time t in the existence interval, the following two conditions are satisfied:

(1) There exists a point $(x_0(t), y_0(t)) \in \mathscr{R}^2$ and a function $R(\cdot, t) \in C^1(S^1)$ such that

$$\Gamma_t = \{(x_0(t), y_0(t)) + (R(\theta, t)\cos\theta, R(\theta, t)\sin\theta) \mid \theta \in S^1\} \tag{6.4}$$

with

$$\|R(\cdot, t) - 1\|_{C^1(S^1)} \leqq \tfrac{1}{5}. \tag{6.5}$$

(2)

$$\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} \leqq 1 \quad \forall t \in [0, T). \tag{6.6}$$

Notice that (1) implies that at each time $t \geqq 0$, we can always take the reference manifold $\mathscr{M}_0$ as the unit circle, whereas (2) implies that Γ_t satisfies the conditions required for the initial curve Γ_0 in Theorem 5.1. Therefore, the solution can always be extended by a fixed amount of time; that is, a global solution can be obtained.

Similar to the harmless assumption that $d \in \mathscr{M}_{\delta_0}$, which we used to prove Theorem 5.1, is the following assumption, which we use to prove the above two conditions: For each time t in the existence interval of the Hele-Shaw problem, there exists a point $(x_0(t), y_0(t)) \in \mathscr{R}^2$ and a function $R(\cdot, t) \in C^1(S^1)$ such that Γ_t is given by (6.4) and $R(\cdot, t)$ satisfies

$$\|R(\cdot, t) - 1\|_{C^1(S^1)} \leqq \tfrac{1}{2}. \tag{6.7}$$

Under the assumption (6.7), the assertion of Lemma 2.1 and the following Sobolev inequality

$$\|\kappa - \bar{\kappa}\|_{L^p(\Gamma_t)} \leqq C_p \|\nabla u(t)\|_{L^2(\mathscr{R}^2)} \qquad \forall p \in [1, \infty)$$

hold. It then follows from Lemma 6.2 that to prove (6.5) and (6.6), it suffices to show that $\|\nabla u(t)\|_{L^2(\mathscr{R}^2)}$ is small enough. The following lemma completes the proof of Theorem 6.1.

Lemma 6.4. *There exists a positive constant δ_3 such that if Γ_0 satisfies the assumption of Theorem 6.1 with $\delta \in (0, \delta_3)$, then in the interval where the solution satisfies* (6.7), *the solution also satisfies*

$$\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} \leqq \|\nabla u_{\Gamma_0}\|_{L^2(\mathscr{R}^2)} + C(S(0) - S_\infty)$$

where $S_\infty = 2\pi\sqrt{A(0)/\pi}$ is the length of the circle of area $A(0)$, $A(0) = \frac{1}{2}\int_0^{2\pi} R_0^2(\theta)\, d\theta$ is the area of the region enclosed by Γ_0, and C is a universal constant.

Proof. We use the energy identities (5.12) and (5.14) to prove this lemma. For simplicity, we assume that the region enclosed by Γ_0 has area π. It follows from Lemma 6.3 that the region enclosed by Γ_t also has area π. Therefore, by Lemma 6.2 we have

$$S(0) \geqq S(t) \geqq S_\infty = 2\pi \quad \text{and} \quad \bar{R}(0) \geqq \bar{R}(t) \geqq 1$$

where $\bar{R} \equiv 1/\bar{\kappa}(t) = S(t)/(2\pi)$.

Although we have estimated the two terms on the right-hand side of (5.14) in § 4, the estimate there cannot lead to the global estimate. Since here we have the exact equation $V = [\kappa_n]$, a much better estimate can be obtained.

To estimate the last term in (5.14), let w be the (unique) function harmonic in $\mathscr{R}^2 \backslash \Gamma_t$, equal to V^2 on Γ_t and satisfying the decay condition (6.1). Then

$$\begin{aligned}
-\int_{\Gamma_t} V[(u_n^+)^2 - (u_n^-)^2] &= \int_{\Gamma_t} V^2[u_n^+ + u_n^-] = \int_{\Gamma_t} (\kappa - \bar{\kappa})[w_n^+ + w_n^-] \\
&\leqq \|\kappa - \bar{\kappa}\|_{L^2(\Gamma_t)} (\|\nabla w^+\|_{L^2(\Gamma_t)} + \|\nabla w^-\|_{L^2(\Gamma_t)}) \\
&\leqq C\|\nabla u\|_{L^2(\mathscr{R}^2)} \|w_\tau\|_{L^2(\Gamma_t)} \leqq C'\|\nabla u\|_{L^2(\mathscr{R}^2)} \|V_s\|^2_{L^2(\Gamma_t)}
\end{aligned} \tag{6.8}$$

where we have used Green's formula in the second equality and Lemma 2.1 in the last inequality.

To estimate the term $\int_{\Gamma_t} \kappa^2 V^2$, write it as

$$\int_{\Gamma_t} \kappa^2 V^2 = \int_{\Gamma_t} (\kappa - \bar{\kappa})^2 V^2 + 2\bar{\kappa} \int_{\Gamma_t} (\kappa - \bar{\kappa}) V^2 + \bar{\kappa}^2 \int_{\Gamma_t} V^2. \tag{6.9}$$

The first two terms can be estimated by

$$\int_{\Gamma_t} (\kappa - \bar{\kappa})^2 V^2 \leqq \| V\|^2_{L^\infty(\Gamma_t)} \|\kappa - \bar{\kappa}\|^2_{L^2(\Gamma_t)} \leqq C\| V_s\|^2_{L^2(\Gamma_t)} \|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)}, \tag{6.10}$$

$$\bar{\kappa}\int (\kappa - \bar{\kappa})\, V^2 \leqq C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} \| V_s\|^2_{L^2(\Gamma_t)}. \tag{6.11}$$

Substituting the estimates (6.8)–(6.11) into the right-hand side of (5.14) yields

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} + \left(2 - C\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} - C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)}\right) \| V_s\|^2_{L^2(\Gamma_t)}$$
$$\leqq 2\bar{\kappa}^2 \| V\|^2_{L^2(\Gamma_t)}. \tag{6.12}$$

To estimate $\bar{\kappa}^2\| V\|^2_{L^2(\Gamma_t)}$, let v be the harmonic extension of V in $\mathscr{R}^2$ such that v satisfies (6.1). Then

$$\bar{\kappa}^2 \| V\|^2_{L^2(\Gamma_t)} = \bar{\kappa}^2 \int_{\Gamma_t} v[u_n^- - u_n^+] = \bar{\kappa}^2 \int_{\Gamma} (\kappa - \bar{\kappa})\,[v_n^- - v_n^+]$$
$$\leqq C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} \| V_s\|_{L^2(\Gamma_t)} \leqq \tfrac{1}{2}\| V_s\|^2_{L^2(\Gamma_t)} + C\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)}.$$

Substituting this estimate in the right-hand side of (6.12) and integrating from 0 to t yields

$$\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} + \int_0^t \left[\tfrac{1}{2} - C\|\nabla u(\tau)\|_{L^2(\mathscr{R}^2)} - C\|\nabla u(\tau)\|^2_{L^2(\mathscr{R}^2)}\right] \| V_s\|^2_{L^2(\Gamma_\tau)}\, d\tau$$
$$\leqq \|\nabla u(0)\|^2_{L^2(\mathscr{R}^2)} + C\int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathscr{R}^2)}$$
$$= \|\nabla u(0)\|^2_{L^2(\mathscr{R}^2)} + C(S(0) - S(t)) \tag{6.13}$$
$$\leqq \Delta_0 \equiv \|\nabla u_{\Gamma_0}\|^2_{L^2(\mathscr{R}^2)} + C(S(0) - S_\infty) \tag{6.14}$$

where we have used (5.12) in the first equality. Therefore, if $C(\Delta_0 + \sqrt{\Delta_0}) \leqq \frac{1}{4}$, we can use an argument similar to that used in proving Lemma 4.3 to prove

$$\|\nabla u(t)\|^2_{L^2(R^2)} \leqq \Delta_0, \qquad \int_0^t \| V_s\|^2_{L^2(\Gamma_\tau)}\, d\tau \leqq 4\Delta_0$$

for all t in the interval specified in the assumption of Lemma 6.4. The assertion of the lemma thus follows. Since $\Delta_0 \leqq C\delta$ for some positive constant C, this also completes the proof of Theorem 6.1.

Remark 6.1. If we simply use the Sobolev imbedding $\| V\|_{L^2(\Gamma_t)} \leqq M\| V_s\|_{L^2(\Gamma_t)}$ to control the term $\bar{\kappa}\| V\|_{L^2(\Gamma_t)}$, then the best constant for M is $\bar{R}(t) = 1/\bar{\kappa}(t)$, which is not applicable in proving Lemma 6.4. However, in the next section, we shall use the fact that the Hele-Shaw problem has a three-dimensional manifold of equilibria to show that the constant M can be reduced to $1/2\bar{\kappa}$, and therefore, an exponential decay estimate can be obtained.

7. Asymptotic Long-Time Behavior of the Solution

If one applies linear analysis to the Hele-Shaw problem near its equilibrium to find solutions of the form $\Gamma_t = \{(R(\theta, t)\cos\theta, R(\theta, t)\sin\theta) \mid \theta \in S^1\}$ with $R(\theta, t) = 1 + \varepsilon R(\theta)\, e^{\sigma t} + O(\varepsilon^2)$, then one may find that $\sigma = 0$ for three linearly independent modes and σ is negative for all the other modes. Observe that every equilibrium of the Hele-Shaw problem is a circle and all the equilibria consist of a three-dimensional manifold parametrized by the radius, and the x and y coordinates of the center of the circles. Clearly, this three-dimensional manifold contributes to the zero-growth modes of the linearized Hele-Shaw problem near any equilibrium. According to the general theory of dynamical systems, the manifold of the equilibria should be exponentially stable. In fact, we shall prove the following theorem.

Theorem 7.1. *Assume that Γ_0 is a simply connected curve satisfying the conditions in Theorem 6.1, and let $\bigcup_{0 \leqq t \leqq \infty} (\Gamma_t \times \{t\})$ be the global solution given by Theorem 6.1. Denote by A_0 the area of the region enclosed by Γ_0 and by $\kappa_\infty = \sqrt{\pi/A_0}$ the curvature of a circle with area A_0. Then there exists a positive constant C such that*

$$\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \leqq Ce^{-24\kappa^3_\infty t} \qquad \forall t \geqq 0, \tag{7.1}$$

$$\int_t^\infty [\|V_s\|^2_{L^2(\Gamma_\tau)} + \|V\|^2_{L^2(\Gamma_\tau)}]\, d\tau \leqq Ce^{-24\kappa^3_\infty t} \qquad \forall t > 0, \tag{7.2}$$

$$\int_t^\infty \|V\|_{L^\infty(\Gamma_\tau)}\, d\tau \leqq Ce^{-12\kappa^3_\infty t} \qquad \forall t > 0. \tag{7.3}$$

Furthermore, there exists a circle $\mathscr{C}$ with area A_0 such that

$$\operatorname{dist}(\Gamma_t, \mathscr{C}) \leqq Ce^{-12\kappa^3_\infty t} \qquad \forall t \in [0, \infty)$$

where $\operatorname{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|$ *is the distance between the two sets A and B.*

Remark 7.1. The exponential rate of decay in Theorem 7.1 is sharp in the sense that the linearized Hele-Shaw problem has a mode which decays with the rate $e^{-12\kappa^3_\infty t}$.

The key to the proof of Theorem 7.1 is the following lemma:

Lemma 7.2. *Assume that γ is a simply-connected curve with length S. Denote by κ the curvature of γ and by $\bar\kappa = 2\pi/S$ the average of κ over γ. Assume that $\kappa \in L^2(\gamma)$ and that*

$$\int_\gamma |\kappa - \bar\kappa| \leqq \tfrac{1}{5}.$$

Let u be the harmonic extension of k in $\mathscr{R}^2$ such that u satisfies (6.1). *Define $V = u_n^- - u_n^+$. Then there exists a universal constant C such that*

$$\|V_s\|^2_{L^2(\gamma)} \geqq 4\bar{\kappa}^2[1 - C\|\kappa - \bar{\kappa}\|_{L^1(\gamma)}]\,\|V\|^2_{L^2(\gamma)}, \tag{7.4}$$

$$\|\nabla u\|^2_{L^2(\mathscr{R}^2)} \leqq \frac{1}{4\bar{\kappa}}\,[1 + C\|\kappa - \bar{\kappa}\|_{L^1(\gamma)}]\,\|V\|^2_{L^2(\gamma)}. \tag{7.5}$$

The proof will be given at the end of this section.

Proof of Theorem 7.1. Since $\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} \leqq \|\nabla u(\tau)\|_{L^2(\mathscr{R}^2)} + C(S(\tau) - S_\infty)$ and $\int_0^\infty \|\nabla u(\tau)\|^2_{L^2(\mathscr{R}^2)}\,d\tau < \infty$, we can assume that $\|\nabla u(t)\|_{L^2(\mathscr{R}^2)}$ is as small as we wish if we assume that t is sufficiently large.

Using (7.4) and (7.5), we obtain from (6.12) that

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \leqq -[2 - C\|\nabla u(t)\|_{L^2(\Gamma_t)}]\,\|V_s\|_{L^2(\Gamma_t)} + 2\bar{\kappa}^2\|V\|^2_{L^2(\Gamma_t)} \tag{7.6}$$

$$\leqq -[6 - C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} - C\|\kappa - \bar{\kappa}\|_{L^1(\Gamma_t)}]\,\bar{\kappa}^2\|V\|^2_{L^2(\Gamma_t)} \tag{7.7}$$

$$\leqq -[24 - C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} - C\|\kappa - \bar{\kappa}\|_{L^1(\Gamma_t)}]\,\bar{\kappa}^3\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)}$$

$$\leqq -24\bar{\kappa}^3[1 - C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)}]\,\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \tag{7.8}$$

where we have used the inequality $\|\kappa - \bar{\kappa}\|_{L^2(\Gamma_t)} \leqq C\|\nabla u(t)\|_{L^2(R^2)}$ in the last inequality.

Since $\bar{\kappa} = 2\pi/S(t) \nearrow \kappa_\infty$ and $\lim_{t\to\infty}\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} = 0$, it follows that for any $\alpha \in (0, 24\kappa^3_\infty)$, there exists a time t_α, such that when $t \geqq t_\alpha$, one has inequality $24\bar{\kappa}(t)^3[1 - C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)}] \geqq \alpha$. Therefore, (7.8) implies that

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \leqq -\alpha\|\nabla u\|^2_{L^2(\mathscr{R}^2)} \quad \forall t \geqq t_\alpha.$$

It follow that

$$\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \leqq e^{\alpha(t_\alpha - t)}\|\nabla u(t_\alpha)\|^2_{L^2(R^2)} \quad \forall t \geqq t_\alpha.$$

Consequently,

$$|\bar{\kappa}(t) - \kappa_\infty| = \left|\frac{2\pi}{S(t)} - \frac{2\pi}{S_\infty}\right| \leqq C(S(t) - S_\infty) = C\int_t^\infty \|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \leqq Ce^{-\alpha t}.$$

Substituting this back into (7.8) yields

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} + 24\kappa^3_\infty\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \leqq Ce^{-\alpha t}\|\nabla u(t)\|^2_{L^2(\mathscr{R}^2)} \leqq Ce^{-40\kappa^3_\infty t}$$

if we take $\alpha = 20\kappa^3_\infty$. Inequality (7.1) thus follows from Gronwalls's inequality, whereas inequality (7.2) follows by integrating (7.7) and (7.6) from t to ∞.

Using (7.2) one can estimate

$$\begin{aligned}\int_t^\infty \|V\|_{L^\infty(\Gamma_\tau)}\,d\tau &\leqq C\int_t^\infty \|V_s\|_{L^2(\Gamma_\tau)}\,d\tau \\ &\leqq C\left(\int_t^\infty e^{-12\kappa_\infty^3\tau}\right)^{1/2}\left(\int_t^\infty \|V_s\|^2_{L^2(\Gamma_\tau)}\,e^{12\kappa_\infty^3\tau}\right)^{1/2} \\ &\leqq Ce^{-6\kappa_\infty^3 t}\left[e^{12\kappa_\infty^3 t}\int_t^\infty \|V_s\|^2_{L^2(\Gamma_\tau)} + \int_t^\infty e^{12\kappa_\infty^3\tau}\left(\int_t^\infty \|V_s\|^2_{L^2(\Gamma_{\tau_1})}\,d\tau_1\right)d\tau\right]^{1/2} \\ &\leqq Ce^{-12\kappa_\infty^3 t}.\end{aligned}$$

Inequality (7.3) thus follows.

To prove the last assertion of the theorem, let $\mathscr{B}_t$ be the smallest rectangle bounded by four lines $x = x_1(t)$, $x = x_2(t)$, $y = y_1(t)$, and $y = y_2(t)$ such that Γ_t is contained in $\mathscr{B}_t$. Then

$$\left|\frac{d}{dt}X_i(t)\right|,\ \left|\frac{d}{dt}Y_i(t)\right| \leqq \|V\|_{L^\infty(\Gamma_t)}, \qquad i = 1, 2.$$

It follows from (7.17) that

$$\int_t^\infty \sum_{i=1}^2 \left(\left|\frac{d}{dt}X_i(t)\right| + \left|\frac{d}{dt}Y_i(t)\right|\right) < Ce^{-12\kappa_\infty^3 t}.$$

Therefore, there exist $X_i(\infty)$ and $Y_i(\infty)$ such that

$$|X_i(t) - X_i(\infty)| + |Y_i(t) - Y_i(\infty)| \leqq Ce^{-12\kappa_\infty^3 t} \quad \forall t > 0.$$

The assertion of the theorem then follows from Lemma 6.2 and the fact that

$$\|\kappa - \bar{\kappa}\|_{L^2(\Gamma_t)} \leqq C\|\nabla u(t)\|_{L^2(\mathscr{R}^2)} \leqq Ce^{-12\kappa_\infty^3 t}.$$

To finish the proof of Theorem 7.1, it remains to prove Lemma 7.2.

Proof of Lemma 7.2. By the scaling $u(x) \to \frac{1}{\bar{\kappa}}\,u\left(\frac{1}{\bar{\kappa}}\,x\right)$, we can assume that $\bar{\kappa} = \bar{R} = 1$, so that $s = 2\pi$. Let $X(s) = \big(x(s), y(s)\big)$ be a parametrization of γ where s is an arc-length parameter, and let $\varphi(s)$, $\theta(s)$ and $R(\theta)$ be the functions introduced in the proof of Lemma 6.2.

Since $\int_\gamma V = \int_\gamma [u_n^- - u_n^+] = 0$, we can write

$$V(X(s)) = \frac{1}{\sqrt{\pi}}\sum_{i=1}^\infty \big(a_i\cos(is) + b_i\sin(is)\big).$$

It follows that

$$\|V_s\|^2_{L^2(\gamma)} = \sum_{i=1}^\infty i^2(a_i^2 + b_i^2) \geqq 4\|V\|^2_{L^2(\gamma)} - 3(a_1^2 + b_1^2). \tag{7.9}$$

Hence, to show the first assertion of the lemma, we need only to estimate a_1 and b_1.

First we want to show that if we express $\kappa(s) = u(X(s))$ in terms of its Fourier expansion, then the coefficients of $\cos s$ and $\sin s$ are "small". For this purpose, define

$$u_1(x) = \frac{1}{4\pi} \int_\gamma \ln|x-y|\, V(y)\, dS_y = \frac{1}{4\pi} \int_{S^1} \ln|x - X(s)|\, V(X(s))\, ds.$$

Then u_1 is harmonic in $\mathscr{R}^2 \backslash \gamma$ and continuous in $\mathscr{R}^2$ with a jump V in the normal derivative across γ. Since $\int_\gamma V = 0$, one can easily verify that $u_1(x) = O(1/|x|)$ as $|x| \to \infty$. Therefore, there exists a constant m such that

$$u(x) = u_1(x) + m \qquad \forall x \in \mathscr{R}^2.$$

Notice that $u = \kappa = \varphi_s$ on γ; it follows that

$$\int_{S^1} u(X(s)) \cos(\varphi(s) - \alpha)\, ds = \int_{S^1} \varphi_s(s) \cos(\varphi(s) - \alpha)\, ds = 0 \qquad \forall \alpha \in S^1.$$

In addition, we have the identities $\int_{S^1} \cos\varphi(s)\, ds = \int_\gamma n_x = \int_{\Omega^-} \Delta(x)\, dx\, dy = 0$, $\int_0^{2\pi} \sin\varphi(s)\, ds = \int_\gamma n_y = 0$, where Ω^- is the region enclosed by γ and n_x and n_y are the normal components of the normal of γ. Hence, for any $\alpha \in S^1$,

$$\begin{aligned} &\left| \int_{S^1} u_1(X(s)) \cos(s-\alpha) \right| \\ &\qquad = \left| \int_{S^1} (u(X(s)) - m)\, [\cos(s-\alpha) - \cos(\varphi(s) - \varphi(0) - \alpha)] \right| \\ &\qquad \leqq \|\varphi(s) - \varphi(0) - s\|_{L^\infty(S^1)} \|u_1\|_{L^1(\gamma)} \\ &\qquad \leqq C\|\kappa - \bar\kappa\|_{L^1(\gamma)} \|u_1\|_{L^1(\gamma)} \leqq C'\|\kappa - \bar\kappa\|_{L^1(\gamma)} \|V\|_{L^1(\gamma)}. \end{aligned}$$

Now we want to express a_1 and b_1 in terms of a function close to u_1. Define $\tilde X(s) = (\cos s, \sin s)$ and introduce a function $\tilde u_1$ defined by

$$\tilde u_1(x) = \frac{1}{4\pi} \int_{S^1} \ln|x - \tilde X(s)|\, V(X(s))\, ds.$$

One then recognizes that $\tilde u_1$ is harmonic in $\mathscr{R}^2 \backslash \mathscr{C}$ and is continuous in $\mathscr{R}^2$ with jump V in the normal derivative across $\mathscr{C}$, where $\mathscr{C}$ is the unit circle centered at the origin. Also, $\tilde u_1(x) \to 0$ as $|x| \to \infty$. Therefore, one can easily verify that $\tilde u_1$ is given by

$$\tilde u_1(r\tilde X(s)) = \begin{cases} \dfrac{1}{2\sqrt{\pi}} \displaystyle\sum_{i=1}^\infty \frac{1}{ir^i} (a_i \cos(is) + b_i \sin(is)) & \text{if } r \geqq 1, \\[2ex] \dfrac{1}{2\sqrt{\pi}} \displaystyle\sum_{i=1}^\infty \frac{r^i}{i} (a_i \cos(is) + b_i \sin(is)) & \text{if } r < 1. \end{cases}$$

We can estimate the difference between $u_1(X(s))$ and $\tilde{u}_1(\tilde{X}(s))$ by

$$|u_1(X(s)) - \tilde{u}_1(\tilde{X}(s))| \leqq \frac{1}{4\pi} \sup_{s,s_1 \in S^1} \left| \ln \frac{|X(s) - X(s_1)|}{|\tilde{X}(s) - \tilde{X}(s_1)|} \right| \|V\|_{L^1(\gamma)}$$

$$\leqq C[\|R(\theta(\cdot)) - 1\|_{C^1(S^1)} + \|\theta_s(\cdot) - 1\|_{C^0(S^1)}] \|V\|_{L^1(\gamma)}$$

$$\leqq C\|\kappa - \bar{\kappa}\|_{L^1(\gamma)} \|V\|_{L^1(\gamma)}.$$

It then follows that

$$|a_1| = \frac{2}{\sqrt{\pi}} \left| \int_{S^1} \tilde{u}(\tilde{X}(s)) \cos s \right|$$

$$\leqq C\|u_1(X(s)) - \tilde{u}(\tilde{X}(s))\|_{L^1(S^1)} + \frac{2}{\sqrt{\pi}} \left| \int_{S^1} u_1(X(s)) \cos s \right|$$

$$\leqq C\|\kappa - \bar{\kappa}\|_{L^1(\gamma)} \|V\|_{L^2(\gamma)}.$$

Similarly we can estimate b_1. Hence, upon substituting these estimates into (7.9), inequality (7.4) follows.

To prove (7.5), we can use the expression of $\tilde{u}_1$ to compute

$$\|\tilde{u}_1(\tilde{X})\|^2_{L^2(S^1)} = \tfrac{1}{4} \sum_{i=1}^{\infty} \frac{a_i^2 + b_i^2}{i^2} \leqq \tfrac{1}{16} \|V\|^2_{L^2(\gamma)} + \tfrac{3}{16} (a_1^2 + b_1^2)$$

$$\leqq [\tfrac{1}{16} + C\|\kappa - \bar{\kappa}\|^2_{L^1(\gamma)}] \|V\|^2_{L^2(\gamma)}.$$

Therefore

$$\|\kappa - \bar{\kappa}\|_{L^2(\gamma)} = \inf_{c \in R^1} \|u - c\|_{L^2(\gamma)} \leqq \|u_1\|_{L^2(\gamma)}$$

$$\leqq \|u_1(X) - \tilde{u}_1(\tilde{X})\|_{L^2(S^1)} + \|\tilde{u}_1(\tilde{X})\|_{L^2(S^1)}$$

$$\leqq \tfrac{1}{4}[1 + C\|\kappa - \bar{\kappa}\|_{L^1(\gamma)}] \|V\|_{L^2(\gamma)}.$$

Hence, we have the estimate

$$\|\nabla u\|^2_{L^2(\mathscr{R}^2)} = \int_{\gamma} (\kappa - \bar{\kappa}) [u_n^- - u_n^+] \leqq \|\kappa - \bar{\kappa}\|_{L^2(\gamma)} \|V\|_{L^2(\gamma)}$$

$$\leqq [\tfrac{1}{4} + C\|\kappa - \bar{\kappa}\|_{L^1(\gamma)}] \|V\|^2_{L^2(\gamma)}.$$

This completes the proof of the lemma and also the proof of Theorem 7.1. □

Remark 7.2. The curvature $\kappa(s)$ does not have the term $a_1 \cos s + b_1 \sin s$ in its Fourier expansion because the curve given by $r = R(\theta)$ with $R(\theta) = 1 + \varepsilon(a_1 \cos\theta + b_1 \sin\theta)$ is close to a (center-shifted) circle up to an order of ε^2.

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Department of Mathematics
University of Pittsburgh
Pittsburgh, Pennsylvania 15260

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