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Spectral Density Analysis of Noisy Repetitive Pulses. Models for Continuously Operating Mode-Locked Lasers

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Abstract A theoretical description and the interpretation of the power spectra of high-repetition-rate laser pulses showing fluctuations in time, intensity and shape are presented.

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Motivation for this paper results from the approximated theoretical descriptions in the existing literature, commonly used to interpret experimental data concerning power spectra measurements of high-repetition-rate laser pulses showing fluctuations in both intensity and time. It is intented that this work gives exact results which can be related to physical and experimental parameters characterizing continuously operating mode-locked lasers [1–3]. Practical insight for this study and mathematical background are found in a very helpful manner in fundamental textbooks [4, 5]. The paper is organized as follows. At first it presents the well-known result of the spectral density of a periodic and deterministic pulse train; in particular the influence of the pulse durations and/or of the finite response of the detector on the spectrum is examined. The effect of amplitude fluctuations on the power spectrum in a sequence of equidistant pulses is analyzed by considering its stochastic properties. Generally speaking, the effect of intensity noise is to produce new discrete components: the more correlation in the noise, the less the spectral distribution is affected. An example drawn from laser studies is given; here the spectral density is derived from pulse shape fluctuations involving either constant maximum amplitude pulses or correlated random intensity variations. Finally, the power spectrum is evaluated in the presence of interpulse jitter and amplitude dispersion. The main concern is with both forms of fluctuations depending on the stochastic characteristics of the noise and leading to the appearance of a selective attenuation of the spectral lines as well as a continuous spectral density component.

1. Periodic and Deterministic Signal

First let us consider $f(t)$ a sequence of equidistant and identical pulses $f_0(t)$,

$$f(t)=\sum_{k=-\infty}^{+\infty} f_0(t+kt_0)=f_0(t)\otimes\sum_{k=-\infty}^{+\infty}\delta(t+kt_0)$$
$$=f_0(t)\otimes s(t,t_0)\,, \qquad (1)$$

where t_0 is the period and where the symbol $\otimes$ represents the convolution. The autocorrelation $R_s(\tau)$ of the train of impulses $s(t,t_0)$ is also a sequence of pulses [6, 7]

$$R_s(\tau)=t_0^{-1}s(\tau,t_0) \qquad (2)$$

and its Fourier transform is

$$S_s(\omega)=\omega_0 t_0^{-1}\sum_{k=-\infty}^{+\infty}\delta(\omega+k\omega_0)\,, \qquad (3)$$

where $\omega_0=2\pi/t_0$. The power spectrum $S_f(\omega)$ corresponding to $f(t)$ is the Fourier transform of the autocorrelation function $R_f(\tau)$ (Fig. 1) [6, 7]:

$$R_f(\tau)=f_0(\tau)\otimes f_0(-\tau)\otimes R_s(\tau)$$
$$=R_{f_0}(\tau)\otimes R_s(\tau) \qquad (4)$$

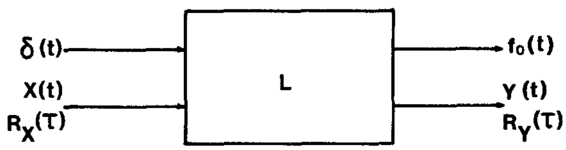


Fig. 1. If $f_0(t)$ is the response function of the linear system L and $X(t)$ [resp. $Y(t)$] is a random function of time at the input [resp. output] of L with autocorrelation $R_X(\tau)$ [resp. $R_Y(\tau)$], then [8, 9] $Y(t)=X(t)\otimes f_0(t)$, $R_Y(\tau)=R_x(\tau)\otimes f_0(\tau)\otimes f_0(-\tau)$

with

$$R_{f_0}(\tau)=\int_{-\infty}^{+\infty} f_0(t)\, f_0(t+\tau)\, dt$$

$$S_f(\omega)=\sum_k \omega_0 t_0^{-1}|\hat{f}_0(\omega)|^2\,\delta(\omega+k\omega_0)$$
$$=\sum_k \omega_0 t_0^{-1}|\hat{f}_0(k\omega_0)|^2\,\delta(\omega+k\omega_0), \tag{5}$$

where $\sum_k$ stands for $\sum_{k=-\infty}^{+\infty}$ and $\hat{f}_0(\omega)$ is the Fourier transform of $f_0(t)$.

2. Periodic Sequence of Pulses with a Randomly Modulated Amplitude

Let us define the random impulse train

$$F(t)=\sum_k [F_0(kt_0)+m_{F_0}]\,\delta(t+kt_0)=F_1(t)+F_2(t) \tag{6}$$

with

$$F_1(t)=\sum_k F_0(kt_0)\,\delta(t+kt_0),$$
$$F_2(t)=m_{F_0}s(t,t_0), \tag{7}$$

where $F_0(t)$ is a stationary stochastic process with zero mean. The expectation and the autocorrelation of $F(t)$ are given by

$$E[F(t)]=m_{F_0}s(t,t_0), \tag{8}$$

$$R_F(\tau)=E[F(t+\tau)\,F(t)]. \tag{9}$$

Since $F_1(t)$ and $F_2(t)$ are orthogonal (i.e., $E[F_1F_2]=0$)

$$R_F(\tau)=R_{F_1}(\tau)+R_{F_2}(\tau)$$
$$=R_{F_1}(\tau)+m_{F_0}^2R_s(\tau). \tag{10}$$

Using the fundamental result [6] that for a periodic sequence of impulses

$$S(t,t_0)=\sum_n a_n\delta(t+nt_0) \tag{11}$$

the autocorrelation $R_S(\tau)$ has the form of

$$R_S(\tau)=\sum_m b_m\delta(\tau+mt_0) \tag{12}$$

with

$$b_m=t_0^{-1}\lim_{l\to\infty}(2l)^{-1}\sum_{n=-l}^{n=l} a_n a_{m+n}. \tag{13}$$

For the random part $F_1(t)$ of $F(t)$, b_m takes the form [8, 9]

$$b_m=t_0^{-1}E\left[\lim_{l\to\infty}(2l)^{-1}\sum_{n=-l}^{+l} F_0(nt_0)\,F_0(nt_0+mt_0)\right]$$
$$=t_0^{-1}R_{F_0}(mt_0) \tag{14}$$

so that

$$R_{F_1}(\tau)=\sum_k t_0^{-1}R_{F_0}(kt_0)\,\delta(\tau+kt_0). \tag{15}$$

For the deterministic part F_2 of $F(t)$, we evidently have

$$R_{F_2}(\tau)=m_{F_0}^2R_s(\tau)=t_0^{-1}m_{F_0}^2s(\tau,t_0) \tag{16}$$

and thus

$$R_F(\tau)=t_0^{-1}\sum_k [R_{F_0}(kt_0)+m_{F_0}^2]\,\delta(\tau+kt_0). \tag{17}$$

It is interesting to remark that the power spectrum $S_{F_0}(\omega)$ of $F_0(t)$ is related to $R_{F_0}(kt_0)$ by

$$R_{F_0}(kt_0)=(2\pi)^{-1}\int_{-\infty}^{+\infty} d\omega S_{F_0}(\omega)\exp(\mathrm{j}kt_0\omega) \tag{18}$$

and also that

$$R_{F_0}(kt_0)=E\{F_0(lt_0)\,F_0[(l+k)t_0]\}. \tag{19}$$

Hence the spectral density of $F(t)$ is given by

$$S_F(\omega)=t_0^{-2}\sum_k S_{F_0}(\omega+k\omega_0)+m_{F_0}^2\omega_0t_0^{-1}\sum_k\delta(\omega+k\omega_0). \tag{20}$$

Indeed, the Fourier transform of $R_{F_1}(\tau)$ is

$$S_{F_1}(\omega)=\sum_k t_0^{-1}R_{F_0}(kt_0)\exp(-\mathrm{j}kt_0\omega) \tag{21}$$

and applying the Poisson's sum formula (Appendix), this expression becomes the first term of the right-hand side member of (20). The second term $S_{F_2}(\omega)$ is readily obtained from (3).

At this stage it is of interest to note that the autocorrelation $R_G(\tau)$ and the spectral density $S_G(\omega)$ of a sequence of equidistant pulses $G(t)$ with a fluctuating amplitude and defined by

$$G(t)=f_0(t)\otimes F(t) \tag{22}$$

can be easily deduced by considering Fig. 1. Thus

$$R_G(\tau)=f_0(\tau)\otimes f_0(-\tau)\otimes R_F(\tau) \tag{23}$$

and

$$S_G(\omega)=|\hat{f}_0(\omega)|^2\,S_F(\omega). \tag{24}$$

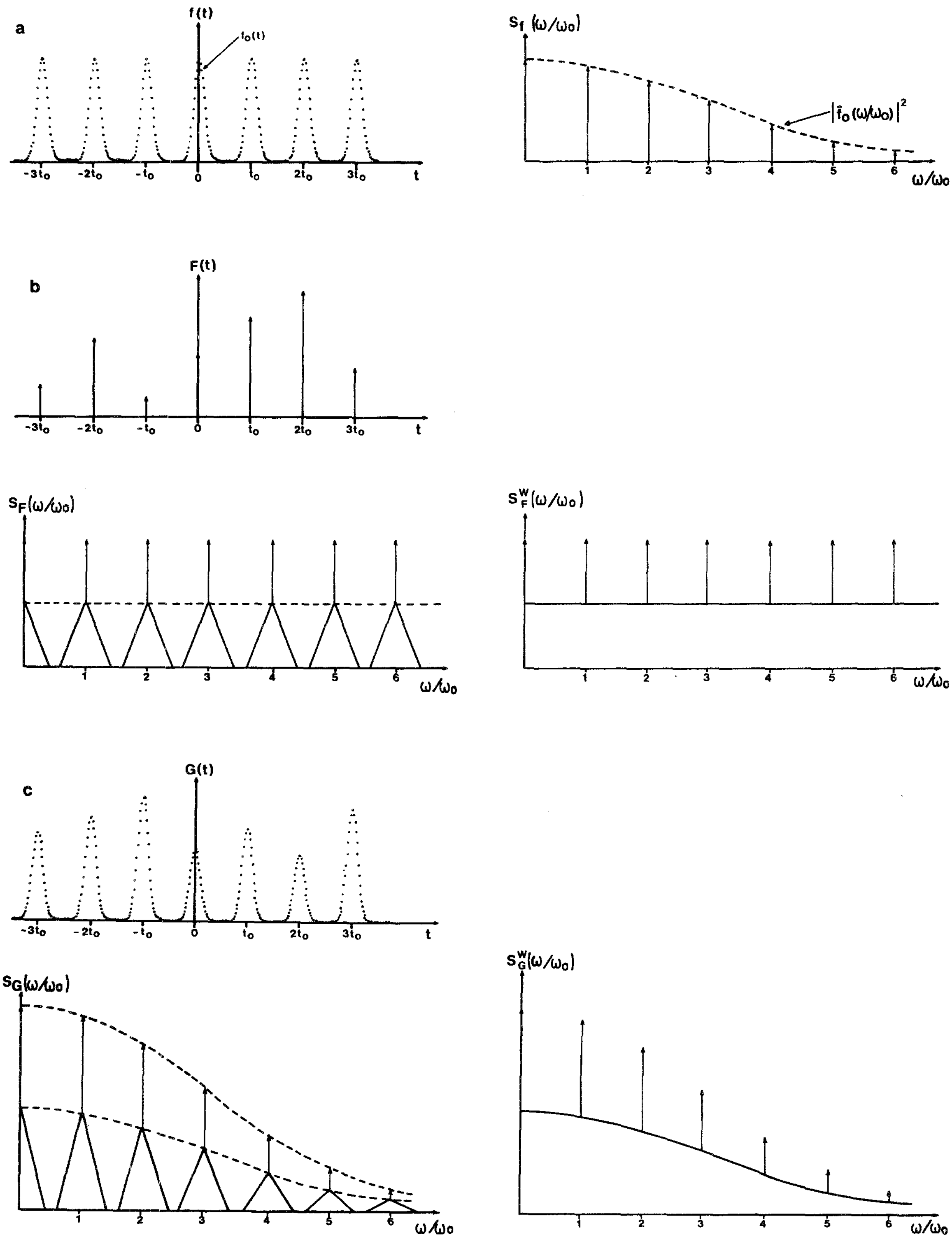


Fig. 2a–c. Reduced power spectra $S(\omega/\omega_0)$ of periodic signals with deterministic and randomly modulated amplitudes: **a** pulse train with fixed amplitude: $f(t)$, (1) and $S_f(\omega)$, (5); **b** impulse train with fluctuating amplitude: $F(t)$, (6), $S_F(\omega)$ [(20) with $S_{F_0}(\omega)$ of triangular shape], and $S_F^W(\omega)$ [(26), white noise case]; **c** pulse train with fluctuating amplitude: $G(t)$ (22), $S_G(\omega)$ (24), and $S_G^W(\omega)$ (27). $\hat{f}_0(\omega)$ is the same as in Fig. 2a

The white noise situation deserves particular interest. Indeed $F_0(lt_0)$ and $F_0[(l+k)t_0]$ are independent; by changing R_F by $R_{F_0}^W$

$$R_{F_0}^W(\tau)=\sigma_{F_0}^2\,\delta(\tau) \tag{25}$$

the power spectra of $F^W(t)$ and $G^W(t)$ are

$$S_F^W(\omega)=t_0^{-1}\sigma_{F_0}^2+m_{F_0}^2\omega_0 t_0^{-1}\sum_k \delta(\omega+k\omega_0), \tag{26}$$

$$S_G^W(\omega)=|\hat{f}_0(\omega)|^2\,S_F^W(\omega), \tag{27}$$

as we easily see from (19 and 20).

The spectrum $S_F^W(\omega)$ (26) consists of (i) a continuous component whose amplitude is given by the variance of the white noise; (ii) a discrete part with elements at multiples of the repetition rate $1/t_0$. In the case of finite duration pulses $f_0(t)$, the continuous spectrum is shaped by that of the pulse, i.e. $|\hat{f}_0(\omega)|^2$.

It is noteworthy to mention that in the white noise case, $G(t)$ can be equivalently expressed by

$$G^W(t)=\sum_k A_k f_0(t)\otimes\delta(t+kt_0), \tag{28}$$

where the random variables A_k are mutually independent, identically distributed and where the expectation and variance are supposed to exist.

In Fig. 2 are illustrated the plots of the spectral densities of the different signals $f(t)$, $F(t)$, $G(t)$, $F^W(t)$, and $G^W(t)$.

3. Sequence of Pulses Exhibiting Width Fluctuations

Let $h(t,\Theta_k)$ be a function of the random variable Θ_k, which represents for $\Theta_k=\theta$ the temporal variation of a pulse characterized by the width θ, and let $L(t)$ be the random function defined by

$$\begin{aligned}L(t)&=\sum_k h(t+kt_0;\Theta_k)\\ &=\sum_k h(t;\Theta_k)\otimes\delta(t+kt_0).\end{aligned} \tag{29}$$

It is assumed that $\{\Theta_k\}$ are mutually independent and equally distributed with a probability density $f_\Theta(\theta)$. The expectation of $L(t)$ is given by

$$\begin{aligned}E[L(t)]&=E[h(t;\Theta_k)]\otimes\sum_k\delta(t+kt_0)\\ &=\int_{-\infty}^{+\infty} h(t;\theta)\,f_\Theta(\theta)\,d\theta\otimes\sum_k\delta(t+kt_0)\end{aligned} \tag{30}$$

which represents a periodic function in contrast to the general case of a particular outcome, (29). The autocorrelation is calculated directly from the general definition [6, 8]

$$\begin{aligned}R_L(\tau)&=E\int L(t)\,L(t+\tau)\,dt\\ &=EL(\tau)\otimes L(-\tau)\\ &=E\sum_m h_m(\tau;\{\Theta_k\})\otimes\delta(\tau+mt_0)\end{aligned} \tag{31}$$

with

$$h_m(\tau;\{\Theta_k\})=t_0^{-1}\lim_{l\to\infty}(2l)^{-1}\sum_{n=-l}^{n=+l}h(\tau;\Theta_n)\otimes h(-\tau;\Theta_{m+n}). \tag{32}$$

Before evaluating the spectral density of $L(t)$, it is of interest to remark that for $m\neq 0$

$$E[h_m(\tau;\{\Theta_k\})]=E[h(\tau;\Theta)]\otimes E[h(-\tau;\Theta)] \tag{33}$$

and for $m=0$

$$E[h_0(\tau;\{\Theta_k\})]=E[h(\tau;\Theta)\otimes h(-\tau;\Theta)], \tag{34}$$

where Θ is a random variable with the same probability density as Θ_k.

The spectral density is evidently given by

$$\begin{aligned}S_L(\omega)&=\int_{-\infty}^{+\infty} e^{j\omega\tau}R_L(\tau)\,d\tau\\ &=t_0^{-1}\{E[|\hat{h}(\omega;\Theta)|^2]-E^2[|\hat{h}(\omega;\Theta|]\\ &\quad+\omega_0[E^2|\hat{h}(\omega;\Theta)|]\sum_k\delta(\omega+k\omega_0)\},\end{aligned} \tag{35}$$

where $\hat{h}(\omega;\Theta)$ is the Fourier transform of $h(t;\Theta)$.

The random pulse duration modulation gives rise to a continuous power spectrum and to a decrease of the power of the discrete components. To make the calculation look less abstract, let us consider the following examples.

– For Gaussian-shaped pulses with constant amplitude maximum n_{F_0} and with a standard deviation θ

$$h(t,\theta)=n_{F_0}\exp(-t^2/2\theta^2) \tag{36}$$

it is easy to show that

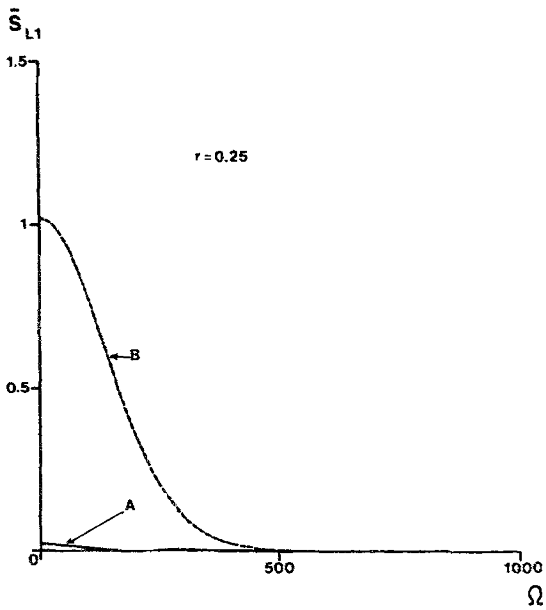
$$\hat{h}(\omega,\theta)=n_{F_0}\theta\sqrt{2\pi}\exp(-\theta^2\omega^2/2). \tag{37}$$

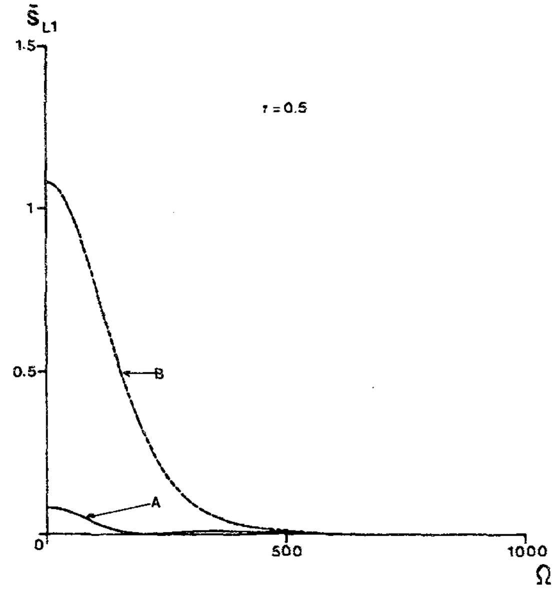
To go further, it will be assumed that the random variable Θ is uniformly distributed between $\theta_m(1-r)$ and $\theta_m(1+r)$ with $0<r<1$. By defining the reduced parameters: $\Omega=\omega/\omega_0$, $\Omega_T=1/\theta_m\omega_0$, one gets the reduced expressions of the expectations:

$$\begin{aligned}E[|\hat{h}(\Omega,\Theta)|]&=n_{F_0}\sqrt{2\pi}\,\theta_m\\ &\quad\times\frac{1}{r}\frac{\Omega_T^2}{\Omega^2}\exp[-\Omega^2(1+r^2)/2\Omega_T^2]\sinh(r\Omega^2/\Omega_T^2)\end{aligned} \tag{38}$$

and

$$\begin{aligned}E[|\hat{h}(\Omega,\Theta|^2]&=n_{F_0}^2 2\pi\theta_m^2\frac{1}{4r}\frac{\Omega_T^2}{\Omega^2}\\ &\quad\times\Big\{(1-r)\exp[-\Omega^2(1-r)^2/\Omega_T^2]\\ &\quad-(1+r)\exp[-\Omega^2(1+r)^2/\Omega_T^2]\\ &\quad+\frac{\sqrt{\pi}}{2}\frac{\Omega_T}{\Omega}(\mathrm{erf}[\Omega(1+r)/\Omega_T]-\mathrm{erf}[\Omega(1-r)/\Omega_T])\Big\}\end{aligned} \tag{39}$$





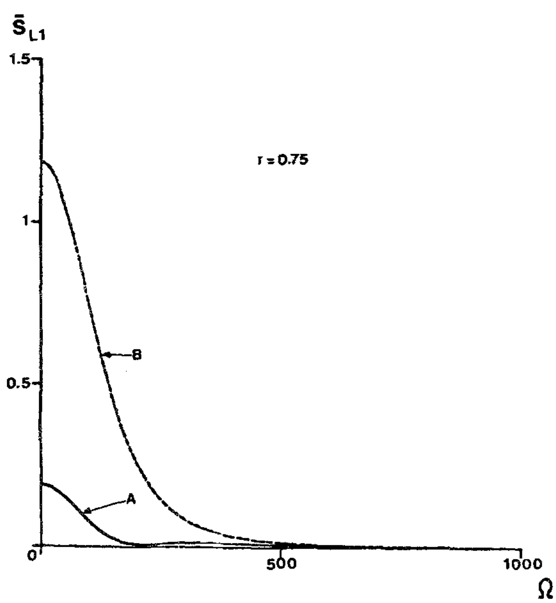


Fig. 3. Reduced power spectra $\bar{S}_{L1}(\Omega)=t_0(2\pi n_{F_0}^2\theta_m^2)^{-1}S_{L1}(\Omega)$ of periodic signals (Gaussian-shaped pulses) with fixed amplitude (n_{F_0}) and widths uniformly distributed between $\theta_m(1-r)$ and $\theta_m(1+r)$, calculated, (35, 38, and 39), with $\Omega_T=(\theta_m\omega_0)^{-1}=200$ which corresponds for example to a repetition rate of 79.6 MHz and a mean width θ_m of 10 ps. The curve A represents the continuous part of the spectrum. The discrete lines $\sum_k \delta(\Omega+k)$ are superimposed on curve A and limited by the envelope B

– The Gaussian-pulses which exhibit both correlated amplitude and duration fluctuations can be modeled by

$$h(t,\theta)=(\theta\sqrt{2\pi})^{-1}m_{F_0}\exp(-t^2/2\theta^2). \tag{40}$$

It should be noted that in contrast to the previous example, the integral $\int_{-\infty}^{+\infty} dt\, h(t,\theta)=m_{F_0}$ is a constant. The Fourier transform gives

$$\hat{h}(\omega,\theta)=m_{F_0}\exp(-\theta^2\omega^2/2). \tag{41}$$

With the same uniform distribution as above, and with the same definitions of the reduced parameters, one can write

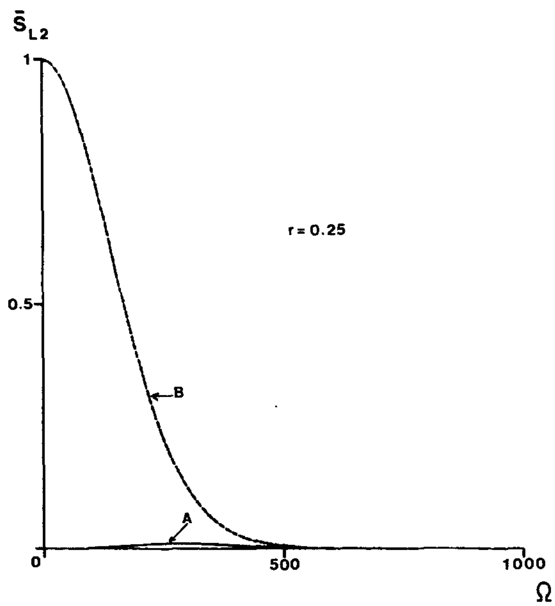
$$E[|\hat{h}(\Omega,\Theta)|]=m_{F_0}\frac{\sqrt{2\pi}}{4}\frac{\Omega_T}{r\Omega}\times\{\mathrm{erf}[\Omega(1+r)/\sqrt{2}\Omega_T]-\mathrm{erf}[\Omega(1-r)/\sqrt{2}\Omega_T]\}, \tag{42}$$

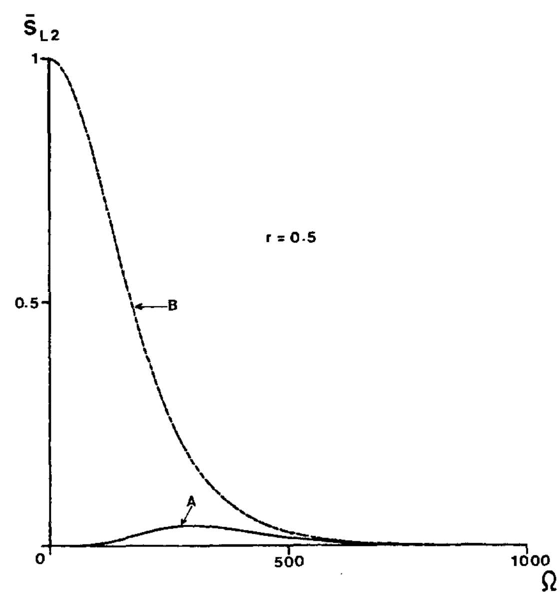
$$E[|h(\Omega,\Theta)|^2]=m_{F_0}^2\frac{\sqrt{\pi}}{4}\frac{\Omega_T}{r\Omega}\times\{\mathrm{erf}[\Omega(1+r)/\Omega_T]-\mathrm{erf}[\Omega(1-r)/\Omega_T]\}. \tag{43}$$

In Figs. 3 and 4 are represented, for different values of r, the reduced power spectra, see (35), $\bar{S}_L(\Omega)=t_0S_L(\Omega)/\alpha$ for fixed amplitude ($\alpha=n_{F_0}^2 2\pi\theta_m^2$) and constant energy ($\alpha=m_{F_0}^2$) pulses.

4. General Case

We now turn to the problem of evaluating the spectral density of a sequence of pulses for which both the amplitudes and the occurrence times are random. It is assumed that the two processes are independent and stationary. In the following, $H(t)$ denotes the random function characterizing this pulse train. $H(t)$ can be





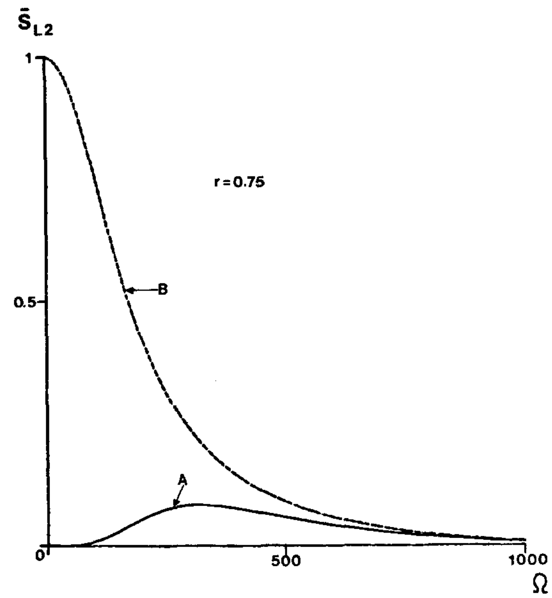


Fig. 4. Reduced power spectra $\bar{S}_{L2}(\Omega)=t_0 S_{L2}(\Omega)/m_{F_0}^2$ of periodic signals with correlated amplitude and width fluctuations, constant energy; (35, 42, and 43). See legend of Fig. 3 for the parameters description

expressed in the form of

$$H(t)=f_0[t-T(t)]\otimes F(t) \tag{44}$$

where $f_0(t)$ is the impulse response function of a linear system undergoing a stochastic delay and $F(t)$ a random function (Fig. 5). The autocorrelation $R_H(\tau)$ is given by

$$\begin{aligned} R_H(\tau) &= E[H^*(t)\,H(t+\tau)] \\ &= E\iint_{-\infty}^{+\infty} du\,dv\,f_0[t-u-T(u)] \\ &\quad \times F(u)\,f_0[t+\tau-v-T(v)]\,F(v)\,. \end{aligned} \tag{45}$$

Recalling that $T(t)$ and $F(t)$ are independent processes one has

$$\begin{aligned} R_H(\tau) &= E\iint_{-\infty}^{+\infty} du\,dv\,f_0[t-u-T(u)] \\ &\quad \times f_0[t+\tau-v-T(v)]\,R_F(v-u)\,, \end{aligned} \tag{46}$$

where $F(t)$ and $R_F(\tau)$ are, respectively, defined by (6 and 17).

As previously $\hat{f}_0(\omega)$ is the Fourier transform of $f_0(t)$; the above expression can be written as

$$\begin{aligned} R_H(\tau) &= \frac{1}{4\pi^2} E\iint_{-\infty}^{+\infty} du\,dv \iint_{-\infty}^{+\infty} d\omega_1\,d\omega_2\,R_F(v-u) \\ &\quad \times \hat{f}_0^*(\omega_1)\exp\{-j\omega_1[t-u-T(u)]\} \\ &\quad \times \hat{f}_0(\omega_2)\exp\{+j\omega_2[t+\tau-v-T(v)]\}\,. \end{aligned} \tag{47}$$

If $\phi_T(\omega_1,\omega_2;v-u)$ is the second order characteristic function of the stationary random function $T(t)$ given by

$$\phi_T(\omega_1,\omega_2;v-u)=E\{\exp[j\omega_1 T(u)+j\omega_2 T(v)]\} \tag{48}$$

the last relationship can be written as

$$\begin{aligned} R_H(\tau) &= \frac{1}{4\pi^2}\iiiint_{-\infty}^{+\infty} du\,dv\,d\omega_1\,d\omega_2\,\hat{f}_0^*(\omega_1) \\ &\quad \times \hat{f}_0(\omega_2)\,\phi_T(\omega_1,-\omega_2;v-u) \\ &\quad \times R_F(v-u)\exp[-j\omega_1(t-u)+j\omega_2(t+\tau-v)]\,. \end{aligned} \tag{49}$$

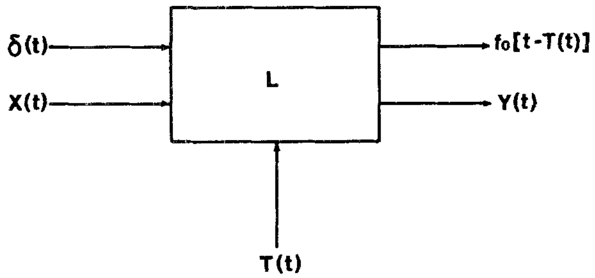


Fig. 5. Schematic of a stochastic delay with $X(t)$ and $Y(t) = f_0[(t-T(t)]\otimes X(t)$ being random functions of time

Remarking that $(2\pi)^{-1}\int dv \exp[-j(\omega_1-\omega_2)v] = \delta(\omega_1-\omega_2)$ and performing the change of variables $v-u=w$ and $v=v$ and the neglect of the subscript of ω_1

$$R_H(\tau)=\frac{1}{2\pi}\iint_{-\infty}^{+\infty} d\omega\, dw |\hat{f}_0(\omega)|^2\ \phi_T(\omega,-\omega;w)\, R_F(w) \times \exp[j\omega(\tau-w)]\,. \tag{50}$$

Substituting $R_F(w)$ by (17) and using the Poisson's sum formula (Appendix) for $\sum \delta(\tau+kt_0)$, $R_H(\tau)$ is simply equal to

$$R_H(\tau)=\frac{t_0^{-2}}{2\pi}\iint_{-\infty}^{+\infty} d\omega\, dw |\hat{f}_0(\omega)|^2\ \Phi_T(\omega,-\omega;w)\exp(j\omega\tau) \times [R_{F_0}(w)+m_{F_0}^2]\sum_k \exp[-jw(\omega+k\omega_0)]\,. \tag{51}$$

Defining the Fourier transforms $\varphi(\omega,\Omega)$ and $\psi(\omega,\Omega)$, respectively, of $R_{F_0}(w)\,\phi_T(\omega,-\omega;w)$ and $\phi_T(\omega,-\omega;w)$, it is not difficult to show, by using the frequency shifting theorem, that the general expression of the power spectrum $S_H(\omega)$ has the following form

$$S_H(\omega)=t_0^{-2}|\hat{f}_0(\omega)|^2\sum_k[\varphi(\omega,\omega+k\omega_0) + m_{F_0}^2\psi(\omega,\omega+k\omega_0)]\,. \tag{52}$$

In order to calculate the spectral density $S_H(\omega)$ of an impulse train with time and amplitude fluctuations, let us first define the random function $I(t)$ by

$$H(t)=I(t)\otimes f_0(t) \tag{53}$$

so that

$$S_H(\omega)=|\hat{f}_0(\omega)|^2\, S_I(\omega)\,. \tag{54}$$

We return to (50) and (17) from which we see that

$$S_I(\omega)=t_0^{-1}\int_{-\infty}^{+\infty} dw \exp(-j\omega w)\sum_k \phi_T(\omega,-\omega;kt_0) \times [R_{F_0}(kt_0)+m_{F_0}^2]\,\delta(w+kt_0)\,. \tag{55}$$

In the white noise case for the arrival times $\{T_k=T(kt_0)\}$ is taken as a sequence of mutually independent and identically distributed random variables, so that the characteristic function $\Psi_T(\omega)$ verifies

$$|\Psi_T(\omega)|^2=\phi_T(\omega,-\omega;w) \quad \text{for} \quad w\neq 0\,. \tag{56}$$

In this case the spectral density of the impulse train becomes

$$S_I^W(\omega)=t_0^{-1}\int_{-\infty}^{+\infty} dw \exp(-j\omega w) \times \Big\{[1-|\Psi_T(\omega)|^2]\,[R_{F_0}(0)+m_{F_0}^2]\,\delta(w) + \sum_k |\Psi_T(\omega)|^2\,[R_{F_0}(kt_0)+m_{F_0}^2]\,\delta(w+kt_0)\Big\}\,. \tag{57}$$

By substituting for $R_{F_0}(0)$ from (18) and by using (20) for the Fourier transform of $R_F(w)$, it is easy to obtain the final result

$$S_I^W(\omega)=(2\pi t_0)^{-1}[1-|\Psi_T(\omega)|^2] \times \left[\int_{-\infty}^{+\infty} S_{F_0}(\omega)\,d\omega+2\pi m_{F_0}^2\right] + |\Psi_T(\omega)|^2\sum_k[t_0^{-2}S_{F_0}(\omega+k\omega_0) + m_{F_0}^2\omega_0 t_0^{-1}\delta(\omega+k\omega_0)]\,. \tag{58}$$

$S_I^W(\omega)$ may be considered as the power spectrum of a sequence of impulses whose amplitudes and interpulse durations are randomly varying. The first term of the right-hand side of (58) clearly reveals the coupling between the amplitude fluctuations and time jitter. It leads to a high-band frequency power spectrum whose intensity is related to the average power of the amplitude noise $\sigma_{F_0}^2+m_{F_0}^2$ with $\sigma_{F_0}^2=(2\pi)^{-1}\int S_{F_0}(\omega)\,d\omega = R_{F_0}(0)$. The cut-off frequency ω^c of the equivalent high pass filter is given by $|\Psi_T(\omega^c)|^2-2^{-1}=0$. Let us recall the general statements $\Psi_T(0)=1$ and $\Psi_T(\infty)=0$. The second term gives rise to a series of lines at multiples of ω_0 attenuated at high frequencies ($\omega>\omega^c$).

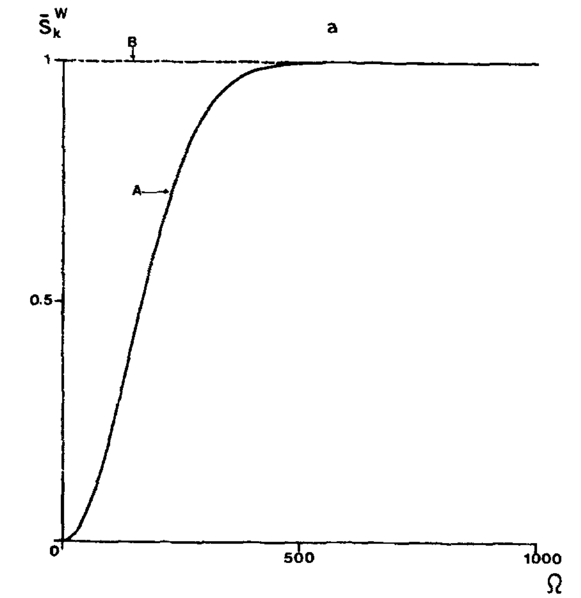
The situation where the pulses are fixed in amplitude but undergo white noise fluctuations in position can be examined by means of (58) in which we set $\sigma_F^2=0$. A straightforward calculation shows that the spectral density $S_k^W(\omega)$ of the random function describing the jittered pulse train given by

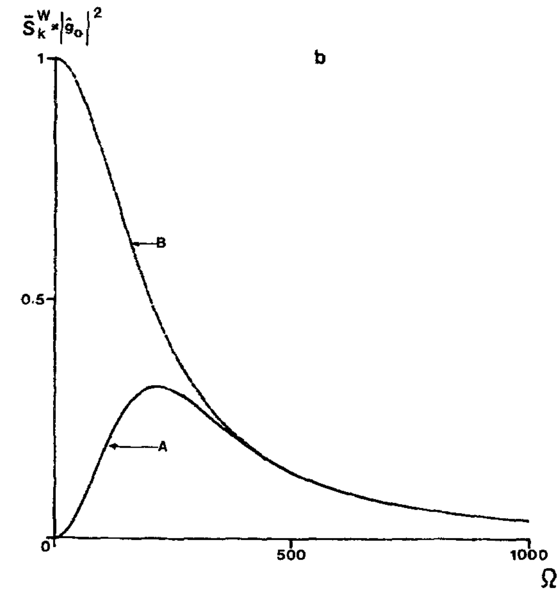
$$K(t)=\sum_k m_{F_0}\delta(t+kt_0+T_k) \tag{59}$$

is of the form of

$$S_k^W(\omega)=m_{F_0}^2\Big\{t_0^{-1}[1-|\Psi_T(\omega)|^2] + \omega_0 t_0^{-1}|\Psi_T(\omega)|^2\sum_k\delta(\omega+k\omega_0)\Big\}\,. \tag{60}$$

As previously mentioned, in the situation of interpulse duration fluctuations, the effect of the temporal jitter is to diminish the power available in the high frequency part of the discrete components since





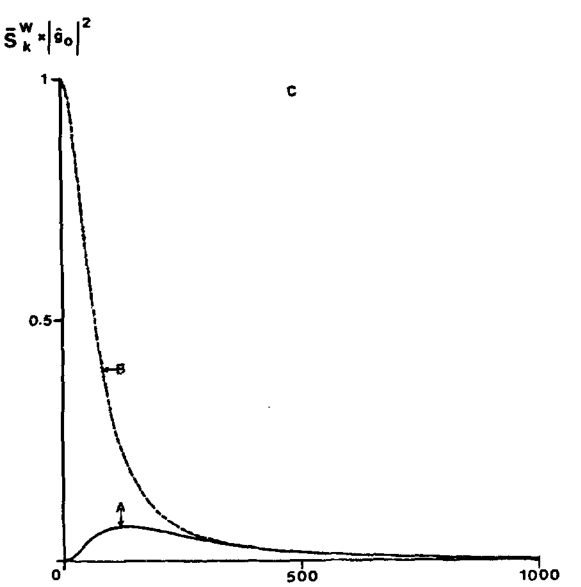


Fig. 6a–c. Reduced power spectra $\bar{S}_K^W(\Omega)$ of an impulse train with fixed amplitude ($\omega_0 = 5\times10^8\ \mathrm{s}^{-1}$) showing white noise fluctuations in position ($\sigma_T = 10$ ps); **a** without any perturbation by the detector, (64); **b, c** by taking into account the response of the detector [(66) with $q=1$ and $q=3$]. The curve A represents the continuous part due to the interpulse jitter and the envelope B is obtained by adding the discrete lines $\sum_k \delta(\Omega+k)$

$|\Psi_T(\omega)|^2 < 1$ and to generate a continuous high frequency spectrum $m_{F_0}^2 t_0^{-1}[1-|\Psi_T(\omega)|^2]$.

In the general case, a practical and quantitative evaluation of $S_H(\omega)$ depends on the kind of assumption made regarding the temporal jitter and the amplitude fluctuations as well as the type of detector used to measure the power spectrum. The bandwidth limitation of the detection system whose impulse response is $g_0(t)$ gives rise to a limited power spectrum

$$S_J(\omega) = |\hat{g}_0(\omega)|^2\, S_H(\omega)\,. \tag{61}$$

Assuming that the high cut-off frequencies ω_g^c and ω_f^c of $|\hat{g}_0(\omega)|$ and $|\hat{f}_0(\omega)|$ verify $\omega_g^c \lll \omega_f^c$, the spectral density can be readily approximated by

$$S_J(\omega) = |\hat{g}_0(\omega)|^2\, S_I(\omega)\,. \tag{62}$$

To go further, the impulse response of the detector is taken as that of the simplest integrating device, i.e. $g_0(t) = (1/\tau_D)\exp(-t/\tau_D)$ and thus $|\hat{g}_0(\omega)|^2 = 1/(1+\omega^2\tau_D^2)$. The time jitter will be supposed normal with zero mean and variance σ_T^2; then

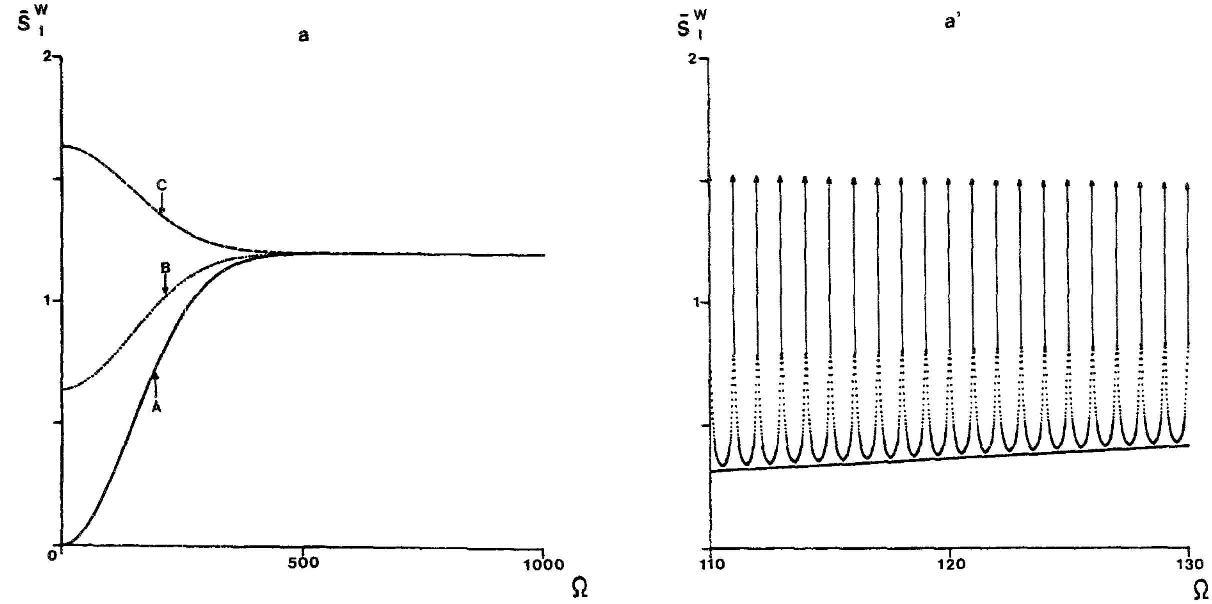
$$\Psi_T(\omega) = \exp(-\sigma_T^2\omega^2/2)\,. \tag{63}$$

In Fig. 6a is plotted the reduced power spectrum

$$\begin{aligned}\bar{S}_K^W(\Omega) &= t_0 S_K^W(\Omega)/m_{F_0}^2\\ &= 1-\exp[-(\Omega/\Omega_T)^2]\\ &\quad+\exp[-(\Omega/\Omega_T)^2]\sum_k \delta(\Omega+k)\end{aligned} \tag{64}$$

deduced from (60 and 63) with $\Omega = \omega/\omega_0$ and $\Omega_T = (\sigma_T\omega_0)^{-1}$ and for the values $\sigma_T = 10$ ps and $\omega_0 = 5\cdot10^8\ \mathrm{s}^{-1}$ ($t_0 = 12.56$ ns) which would correspond to an impulse train with stable amplitude ($\sigma_{F_0}^2 = 0$) and showing white noise fluctuations in position like those observed in a synchronously pumped picosecond dye laser beam. If such a sequence of impulses is also subjected to an amplitude noise characterized by the autocorrelation function $R_F(\tau) = \sigma_{F_0}^2\exp(-|\tau|/\tau_{F_0})$ such that

$$S_{F_0}(\omega) = 2\sigma_{F_0}^2\tau_{F_0}/(1+\omega^2\tau_{F_0}^2) \tag{65}$$



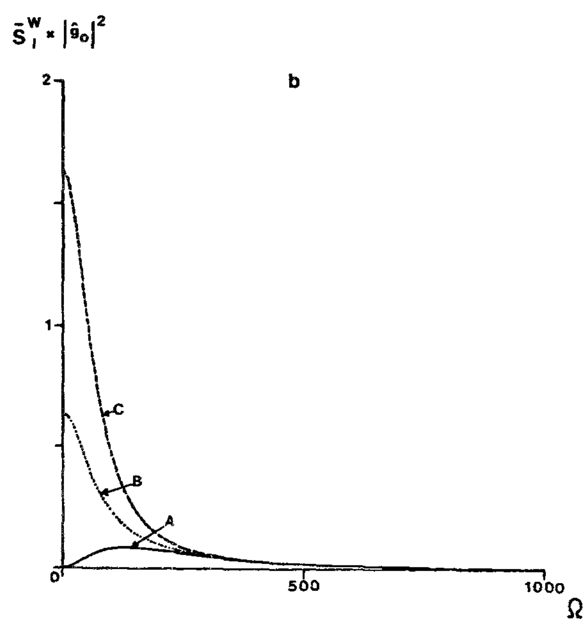


Fig. 7a, b. Spectra $\bar{S}_I^W(\Omega)$, (58), obtained by adding the contribution of an amplitude noise characterized by (65) with $\sigma_{F_0}^2 = 0.2m_{F_0}^2$ and $\omega_0\tau_{F_0}=10$ to those of Fig. 6. (**a'**): detail of (**a**); (**b**): $q=3$. The amplitude noise spectra $\sum_k S_{F_0}(\Omega+k)$ [resp. the discrete lines $\sum_k \delta(\Omega+k)$] are superimposed on curves A (resp. B) and limited by the envelopes B (resp. C)

the power spectrum $S_I^W(\omega)$ given by (58) takes the form illustrated in Fig. 7a.

The influence of the detector is made evident in Figs. 6b, c, and 7b representing the reduced power spectra

$$\bar{S}_{K,I}^W(\Omega) \times |\hat{g}_0(\Omega)|^2 = \bar{S}_{K,I}^W(\Omega) \times (1+q^2\Omega^2/\Omega_T^2)^{-1}, \qquad (66)$$

where $q=\tau_D/\sigma_T$.

Appendix

Poisson's sum formula [6]

$$\sum_k f(t+kt_0) = t_0^{-1} \sum_k \hat{f}(k\omega_0)\exp(jk\omega_0 t).$$

With the particular form obtained with $t=0$

$$\sum_k f(kt_0) = t_0^{-1} \sum_k \hat{f}(k\omega_0).$$

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