THE GRAVITATIONAL POTENTIAL OF A HOMOGENEOUS POLYHEDRON OR DON'T CUT CORNERS

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Abstract. A polyhedron can model irregularly shaped objects such as asteroids, comet nuclei, and small planetary satellites. With minor effort, such a model can incorporate important surface features such as large craters. Here we develop closed-form expressions for the exterior gravitational potential and acceleration components due to a constant-density polyhedron. An equipotential surface of Phobos is illustrated.

Key words: Irregularly shaped body, polyhedron, gravitational potential, Phobos.

1. Introduction

Plans are being made to orbit the solar system's smaller bodies such as planetary satellites, comets, and asteroids. Images show many of these bodies to have elongated, non-spherical shapes, some with large craters (Figure 1). Conventional spherical harmonic representations of the gravitational potential of such bodies require high degree and order expansions which are difficult to obtain.

The surface of such a body can be observed by remote imaging or radar. If complete coverage of the surface can be obtained, a polyhedral model of the body can be constructed (Figure 2). The polyhedron's faces can be large or small, and follow important surface features such as large craters and ridges.

Here we develop closed-form expressions for the exterior gravitational potential and acceleration components experienced by a unit mass due to a homogeneous (constant-density) polyhedron. Using the Gauss Divergence Theorem and Green's Theorem, the potential and acceleration components can be expressed in terms of the polyhedron's edges and vertex angles. Accuracy of our results is determined by how closely the polyhedron models the actual body and by density variations within.

Section 2 of this paper shows how to represent the gravitational potential of a homogeneous body in terms of its surface. In Section 3 we derive the contribution of a single planar triangular face to the gravitational potential of a polyhedron, and in Section 4 the corresponding acceleration expressions. In Section 5 we compare two equipotential surfaces of the inner Martian satellite Phobos, one generated by this paper's technique and the other by conventional spherical harmonics. Section 6 cites related work by other researchers.

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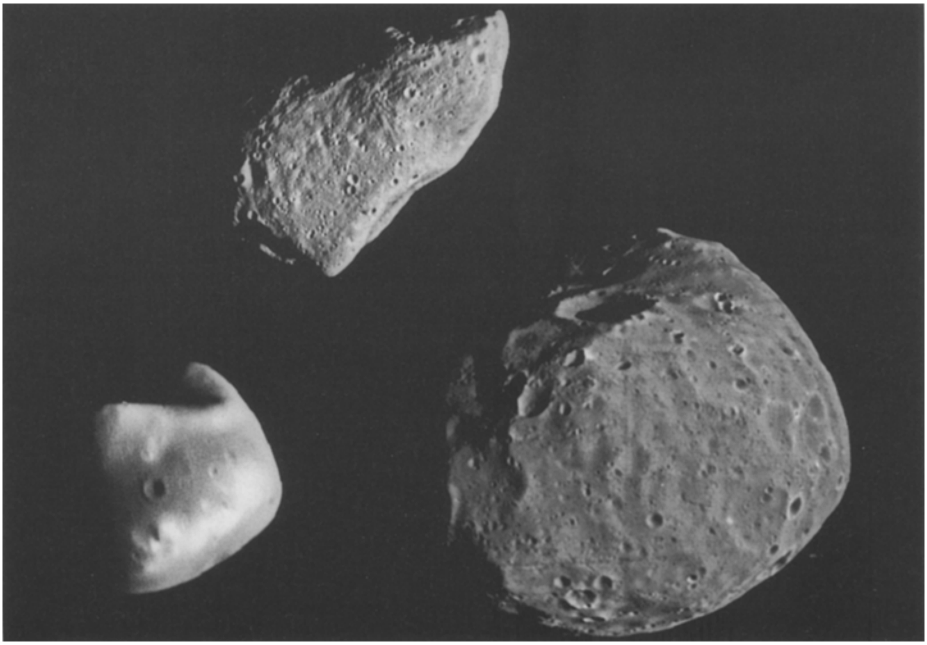

Fig. 1. This montage shows asteroid 951 Gaspra (top) and the Martian satellites Deimos (lower left) and Phobos (lower right). The three bodies appear at the same scale and nearly the same lighting conditions. Gaspra is about 17 km long. (Photograph courtesy of NASA/JPL).

2. Gravitational Potential in Terms of a Body's Surface

The gravitational force vector f experienced by a unit mass located at (x, y, z) due to another point mass m at (ξ, η, ζ) can be expressed as the gradient of a potential U:

$$\mathrm{f} = \nabla U = \nabla\left(G\,\frac{m}{r}\right)$$

where r is the distance between the unit mass and m, and G is the gravitational constant. ∇ represents the three-dimensional del operator whose Cartesian form is $\hat{\imath}\partial/\partial x + \hat{\jmath}\partial/\partial y + \hat{\mathrm{k}}\partial/\partial z$. When the other mass m is part of an extended body, the potential expression becomes an integral over the volume of the body:

$$U = G \iiint\limits_{\text{body}} r^{-1}\,\mathrm{d}m$$

The usual approach is to expand r^{-1} in terms of Legendre polynomials and U in terms of spherical harmonics (Kaula, 1966; Heiskanen *et al.*, 1967). But there is another way to proceed.

If we can find some vector-valued function $\mathrm{w} = \mathrm{w}(\xi, \eta, \zeta)$ such that its divergence $\nabla \bullet \mathrm{w} = r^{-1}$, then we can use the Gauss Divergence Theorem to manipulate

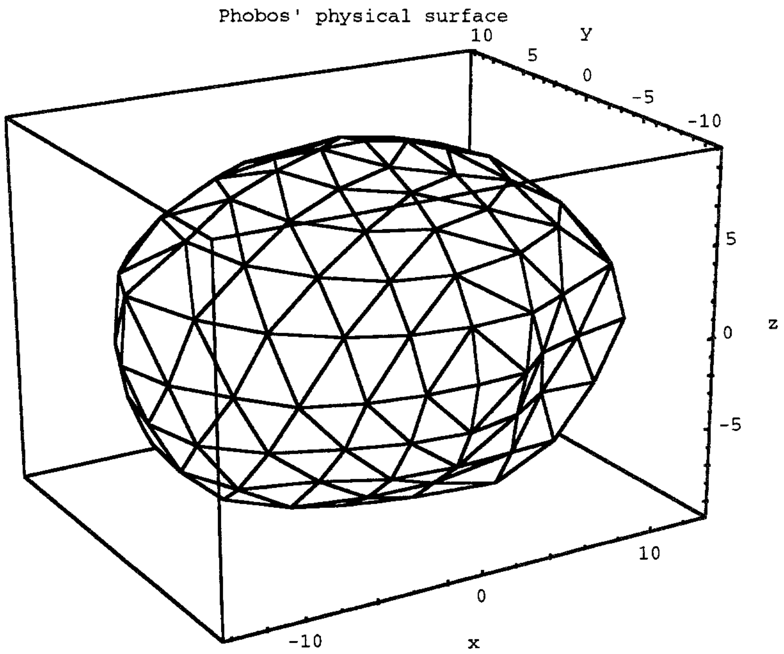

Fig. 2. A polyhedral model of the Martian satellite Phobos (Turner, 1978). North is up and Mars is to the right in the $+x$ direction. The large depression near the right limb represents the crater Stickney. The semiaxes dimensions of a fitted ellipsoid are $a = 13.5$ km, $b = 10.7$ km, and $c = 9.4$ km.

the potential expression (here, the del operator acts on ξ, η, ζ). This theorem states the divergence of a vector field w integrated throughout a volume V is equal to the inner product of the surface normal $\hat{\mathrm{n}}$ and the vector field, integrated over the surface S of the volume:

$$\iiint_V \nabla \bullet \mathrm{w} \, \mathrm{d}V = \iint_S \hat{\mathrm{n}} \bullet \mathrm{w} \, \mathrm{d}S$$

The volume V must be bounded and connected, its surface S must be piecewise smooth and orientable, and the vector w and its first derivative must exist and be continuous throughout V and on S (Greenberg, 1978).

The volume integral for the potential contains $\mathrm{d}m$ while the volume integral for the Gauss Divergence Theorem contains $\mathrm{d}V$. We eliminate this problem by assuming the body has constant density. For brevity, we omit the density factor $\sigma = \mathrm{d}m/\mathrm{d}V$ and G, the gravitational constant, from all subsequent expressions.

The desired vector is $\mathrm{w} = \hat{\mathrm{r}}/2$, where $\hat{\mathrm{r}}$ is a unit vector from the unit mass directed toward the differential volume element. (In this paper, unit vectors such as $\hat{\mathrm{r}}$ and $\hat{\mathrm{n}}$ wear a hat.) Under the assumption of constant density, the gravitational potential can be expressed as

$$U = \iiint\limits_{V} r^{-1}\,\mathrm{d}V = \tfrac{1}{2} \iint\limits_{S} \hat{\mathbf{n}} \bullet \hat{\mathbf{r}}\,\mathrm{d}S \tag{2.1}$$

We emphasize that this expresses the gravitational potential of a volume, not a surface.

If the unit mass is on or within the body, the direction of $\hat{\mathbf{r}}$ is undefined for the volume element coincident with the unit mass, and a precondition for the Gauss Divergence Theorem is violated. Thus, we claim our expressions are valid only when the unit mass is exterior to the body.

Equation (2.1) holds true for any constant-density body which satisfies the preconditions of the Divergence Theorem. If the surface integral is tractable, we can develop a closed-form expression for the potential. For example, it is not difficult to show that the exterior gravitational potential due to a homogeneous sphere is equivalent to the potential generated by the sphere's mass concentrated at its center. In this paper we proceed by evaluating Equation (2.1) in the form of a polyhedron.

3. Potential Contribution due to a Triangular Face

We might model a body's surface by a polyhedron. This is attractive because $\hat{\mathbf{n}}$ is constant on each face, say $\hat{\mathbf{n}}_f$. We compute a polyhedron's potential by first integrating $\hat{\mathbf{n}}_f \bullet \hat{\mathbf{r}}$ over the area of each face, then summing over all faces:

$$2U = \iint\limits_{S} \hat{\mathbf{n}} \bullet \hat{\mathbf{r}}\,\mathrm{d}S = \sum_{\text{faces}} \left[\iint\limits_{\text{face}} \hat{\mathbf{n}}_f \bullet \hat{\mathbf{r}}\,\mathrm{d}S \right].$$

In this section, we integrate $\hat{\mathbf{n}}_f \bullet \hat{\mathbf{r}}$ over a planar region, then specialize to a convex polygon, and finally to a triangle.

3.1. Planar Face

If the body is a polyhedron, the faces must be polygons – planar regions with straight edges. Initially however, we can derive slightly more general results by assuming only that the faces are planar. The edges of these planar faces need not be straight.

For each planar face, establish a right-handed Cartesian 'face' coordinate system with basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ parallel to the face plane and $\hat{\mathbf{k}}$ aligned with the face's outward-pointing normal $\hat{\mathbf{n}}_f$. The specific direction of $\hat{\mathbf{i}}$ or $\hat{\mathbf{j}}$ is unimportant. In the face coordinate system, all points in the face share the same ζ coordinate which we label ζ_f. A typical point on the face has coordinates (ξ, η, ζ_f).

In the face coordinate system, the unit mass has coordinates (x, y, z). For much of our derivation it is convenient to use coordinates relative to the unit mass, so we define $\Delta x = \xi - x$, $\Delta y = \eta - y$, and $\Delta z = \zeta_f - z$. Then we have $\hat{\mathbf{r}} = \hat{\mathbf{i}}\,\frac{\Delta x}{r} + \hat{\mathbf{j}}\,\frac{\Delta y}{r} + \hat{\mathbf{k}}\,\frac{\Delta z}{r}$, where Δz is the constant, signed coordinate of the face

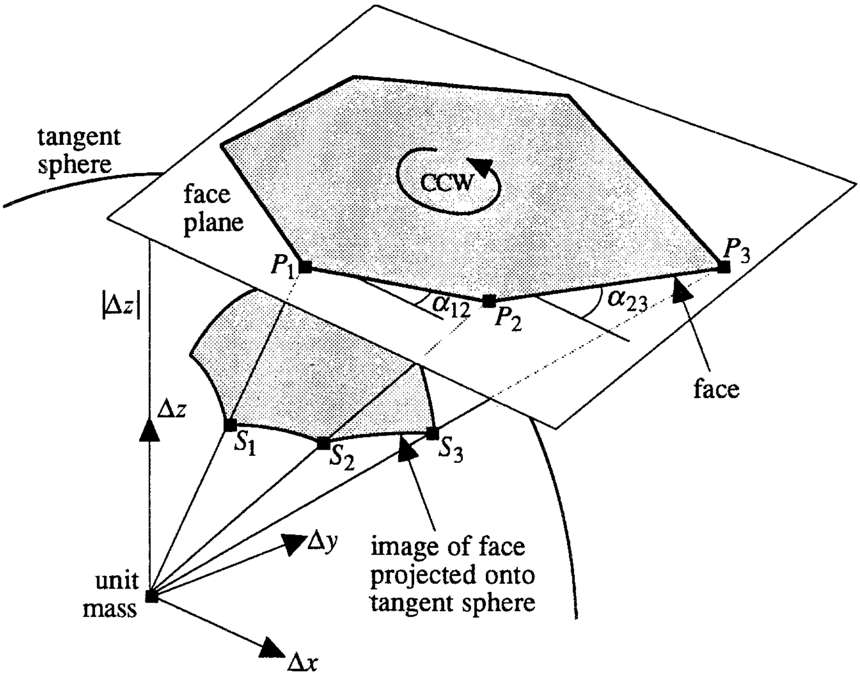

Fig. 3. An imaginary sphere is centered on the unit mass and just touches the face plane. The face is projected onto this tangent sphere and the area of its image computed in terms of the spherical vertex angles. The value of a spherical angle (e.g. S_2) can be calculated from the coordinates of three consecutive vertices of the corresponding planar polygon (P_1, P_2, and P_3).

plane relative to the unit mass, and $r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. The differential surface element is $\mathrm{d}S = \mathrm{d}\xi\,\mathrm{d}\eta = \mathrm{d}(\Delta x)\,\mathrm{d}(\Delta y)$.

It is convenient to illustrate a lemma before proceeding. Imagine a sphere of radius $|\Delta z|$, centered at the unit mass, which just touches the face plane (Figure 3). Use a central projection to map the planar face onto this 'tangent sphere' and find the area of its image. The differential area element on the tangent sphere, always positive, is $\mathrm{d}A = |\Delta z|^3/r^3\,\mathrm{d}S$ (Appendix A). The positive area of the image on the sphere is

$$A_{\text{image}} = \iint\limits_{\substack{\text{image}\\ \text{on sphere}}} \mathrm{d}A = \iint\limits_{\substack{\text{planar}\\ \text{face}}} |\Delta z|^3/r^3\,\mathrm{d}S$$

Lemma 1 is

$$\iint\limits_{\substack{\text{planar}\\ \text{face}}} \Delta z^3/r^3\,\mathrm{d}S = \operatorname{sign}(\Delta z)\cdot A_{\text{image}}$$

where $\operatorname{sign}(\Delta z)$ is $+1$, -1, or 0 as Δz is greater than, less than, or equal to, zero.

With this lemma, a planar face's contribution to the volume gravitational potential can be developed quickly:

$$
\begin{aligned}
2U_{\substack{\text{planar}\\ \text{face}}} &= \iint\limits_{\substack{\text{planar}\\ \text{face}}} \hat{\mathrm{n}}_f \bullet \hat{\mathrm{r}}\, \mathrm{d}S \\
&= \iint\limits_{\substack{\text{planar}\\ \text{face}}} \hat{\mathrm{k}} \bullet \left(\hat{\mathrm{i}}\,\frac{\Delta x}{r} + \hat{\mathrm{j}}\,\frac{\Delta y}{r} + \hat{\mathrm{k}}\,\frac{\Delta z}{r} \right) \mathrm{d}S \\
&= \iint\limits_{\substack{\text{planar}\\ \text{face}}} \frac{\Delta z}{r}\, \mathrm{d}S \qquad (3.1) \\
&= \iint\limits_{\substack{\text{planar}\\ \text{face}}} \left(\frac{\Delta z}{r} + \frac{\Delta z^3}{r^3} - \frac{\Delta z^3}{r^3} \right) \mathrm{d}S \\
&= \Delta z \iint\limits_{\substack{\text{planar}\\ \text{face}}} \left(\frac{1}{r} + \frac{\Delta z^2}{r^3} \right) \mathrm{d}S - \iint\limits_{\substack{\text{planar}\\ \text{face}}} \frac{\Delta z^3}{r^3}\, \mathrm{d}S \\
&= \Delta z \iint\limits_{\substack{\text{planar}\\ \text{face}}} \left(\frac{r^2 + \Delta z^2}{r^3} \right) \mathrm{d}S - \operatorname{sign}(\Delta z) \cdot A_{\text{image}} \\
&= \Delta z \iint\limits_{\substack{\text{planar}\\ \text{face}}} \left(\frac{r^2 - \Delta x^2}{r^3} + \frac{r^2 - \Delta y^2}{r^3} \right) \mathrm{d}S - \operatorname{sign}(\Delta z) \cdot A_{\text{image}} \\
&= \Delta z \iint\limits_{\substack{\text{planar}\\ \text{face}}} \left(\frac{\partial}{\partial \Delta x}\,\frac{\Delta x}{r} + \frac{\partial}{\partial \Delta y}\,\frac{\Delta y}{r} \right) \mathrm{d}S - \operatorname{sign}(\Delta z) \cdot A_{\text{image}} \\
&= \Delta z \oint_{\substack{\text{planar face}\\ \text{boundary}}} \left(\frac{\Delta x}{r}\, \mathrm{d}\Delta y - \frac{\Delta y}{r}\, \mathrm{d}\Delta x \right) - \operatorname{sign}(\Delta z) \cdot A_{\text{image}} \qquad (3.2)
\end{aligned}
$$

where the last step is an application of Green's Theorem. The contribution of one planar face to its volume's potential involves a line integral counterclockwise around the boundary of the face and the area of the face projected onto the tangent sphere described for Lemma 1. Counterclockwise is in the sense of the right-hand rule and $\hat{\mathrm{n}}_f$.

3.2. Convex Planar Polygon

In the preceding development we assumed only that the face is planar. We now assume the face has straight sides and is convex; that it is a convex planar polygon. Straight sides allow us to evaluate the line integral easily. Convexity simplifies the expression for the area of the face image.

Here are definitions we use subsequently. Let $P_1, P_2, \ldots$ denote the vertices of the polygonal face, taken in counterclockwise order. Relative to the unit mass, the coordinates of these vertices are $P_1 = (\Delta x_1, \Delta y_1, \Delta z)$, $P_2 = (\Delta x_2, \Delta y_2, \Delta z)$, and so on. Let $r_1, r_2, \ldots$ denote the distances between the unit mass and points $P_1, P_2, \ldots$. Let r_{12} denote the constant distance between points P_1 and P_2, with similar definitions for r_{23} and so on. The angle α_{12} is measured between the Δx axis and the edge connecting P_1 and P_2; we define it in terms of $\cos\alpha_{12} = (\Delta x_2 - \Delta x_1)/r_{12} = (\xi_2 - \xi_1)/r_{12}$ and $\sin\alpha_{12} = (\Delta y_2 - \Delta y_1)/r_{12} = (\eta_2 - \eta_1)/r_{12}$, with similar definitions for α_{23} and so on. Finally, it is convenient to define certain determinant expressions: $\det_{12} = \Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1$ and similarly for $\det_{23}$, etc.

3.2.1. *Line integral*

We compute the line integral around the polygon boundary (taken from the first term of Equation (3.2)) by integrating along each edge and summing:

$$\oint_{\substack{\text{polygon}\\ \text{boundary}}} \left(\frac{\Delta x}{r}\,\mathrm{d}\Delta y - \frac{\Delta y}{r}\,\mathrm{d}\Delta x\right) = \sum_{\substack{\text{polygon}\\ \text{edges}}} \left[\int_{\text{edge}} \left(\frac{\Delta x}{r}\,\mathrm{d}\Delta y - \frac{\Delta y}{r}\,\mathrm{d}\Delta x\right)\right]$$

To evaluate

$$\int_{\text{edge}} \left(\frac{\Delta x}{r}\,\mathrm{d}\Delta y - \frac{\Delta y}{r}\,\mathrm{d}\Delta x\right)$$

for the typical edge connecting points P_1 and P_2, let s parameterize the distance along the path of integration; $0 \le s \le r_{12}$. Coordinates Δx and Δy become $\Delta x = \Delta x(s) = \Delta x_1 + s\cos\alpha_{12}$ and $\Delta y = \Delta y(s) = \Delta y_1 + s\sin\alpha_{12}$. Also, $\mathrm{d}\Delta x = \mathrm{d}s\cos\alpha_{12}$ and $\mathrm{d}\Delta y = \mathrm{d}s\sin\alpha_{12}$. The edge integral becomes

$$\int_{P_1}^{P_2} \left(\frac{\Delta x}{r}\,\mathrm{d}\Delta y - \frac{\Delta y}{r}\,\mathrm{d}\Delta x\right) = \int_0^{r_{12}} \frac{\Delta x\sin\alpha_{12} - \Delta y\cos\alpha_{12}}{r}\,\mathrm{d}s$$

$$= \int_0^{r_{12}} \frac{(\Delta x_1 + s\cos\alpha_{12})\sin\alpha_{12} - (\Delta y_1 + s\sin\alpha_{12})\cos\alpha_{12}}{r}\,\mathrm{d}s$$

$$= \int_0^{r_{12}} \frac{(\Delta x_1\sin\alpha_{12} - \Delta y_1\cos\alpha_{12})\,\mathrm{d}s}{\sqrt{s^2 + 2(\Delta x_1\cos\alpha_{12} + \Delta y_1\sin\alpha_{12})\,s + (\Delta x_1^2 + \Delta y_1^2 + \Delta z^2)}}$$

$$= (\Delta x_1 \sin \alpha_{12} - \Delta y_1 \cos \alpha_{12}) \ln(r + s + \Delta x_1 \cos \alpha_{12} + \Delta y_1 \sin \alpha_{12})\Big|_{s=0, r=r_1}^{s=r_{12}, r=r_2}$$

$$= (\Delta x_1 \sin \alpha_{12} - \Delta y_1 \cos \alpha_{12}) \ln \left(\frac{r_2 + r_{12} + \Delta x_1 \cos \alpha_{12} + \Delta y_1 \sin \alpha_{12}}{r_1 + \Delta x_1 \cos \alpha_{12} + \Delta y_1 \sin \alpha_{12}} \right)$$

$$= (\Delta x_1 \sin \alpha_{12} - \Delta y_1 \cos \alpha_{12}) \ln \left(\frac{r_2 + \Delta x_2 \cos \alpha_{12} + \Delta y_2 \sin \alpha_{12}}{r_1 + \Delta x_1 \cos \alpha_{12} + \Delta y_1 \sin \alpha_{12}} \right)$$

$$= -\Delta y' \ln \left(\frac{r_2 + \Delta x'_2}{r_1 + \Delta x'_1} \right)$$

where the primed coordinates locate the two points in an 'edge' coordinate system rotated so its $\Delta x'$ axis is aligned with the edge. That is, $\Delta x'_1 = \Delta x_1 \cos \alpha_{12} + \Delta y_1 \sin \alpha_{12}$, $\Delta x'_2 = \Delta x_2 \cos \alpha_{12} + \Delta y_2 \sin \alpha_{12}$, and $\Delta y' = \Delta y'_1 = \Delta y'_2 = \Delta y_2 \cos \alpha_{12} - \Delta x_2 \sin \alpha_{12}$. Note that in this edge coordinate system, $\Delta x'_2 - \Delta x'_1 = r_{12}$.

MacMillan (1930, §43) simplifies the ln argument further:

$$\frac{r_2 + \Delta x'_2}{r_1 + \Delta x'_1} = \frac{2r_2 + (\Delta x'_2 + \Delta x'_1) + (\Delta x'_2 - \Delta x'_1)}{2r_1 + (\Delta x'_2 + \Delta x'_1) - (\Delta x'_2 - \Delta x'_1)} \cdot \frac{(\Delta x'_2 - \Delta x'_1)}{(\Delta x'_2 - \Delta x'_1)}$$

$$= \frac{2(\Delta x'_2 - \Delta x'_1) r_2 + (\Delta x'^2_2 - \Delta x'^2_1) + (\Delta x'_2 - \Delta x'_1)^2}{2(\Delta x'_2 - \Delta x'_1) r_1 + (\Delta x'^2_2 - \Delta x'^2_1) - (\Delta x'_2 - \Delta x'_1)^2}$$

$$= \frac{2r_{12} r_2 + (r_2^2 - r_1^2) + r_{12}^2}{2r_{12} r_1 + (r_2^2 - r_1^2) - r_{12}^2}$$

$$= \frac{(r_2 + r_{12})^2 - r_1^2}{r_2^2 - (r_1 - r_{12})^2}$$

$$= \frac{(r_2 + r_{12} + r_1)(r_2 + r_{12} - r_1)}{(r_2 + r_1 - r_{12})(r_2 - r_1 + r_{12})}$$

$$= \frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} .$$

With this simplification, the line integral between points P_1 and P_2 becomes

$$\int_{P_1}^{P_2} \left(\frac{\Delta x}{r} \mathrm{d}\Delta y - \frac{\Delta y}{r} \mathrm{d}\Delta x \right) = -\Delta y' \ln \left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right)$$

$$= \frac{\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1}{r_{12}} \ln\left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}}\right)$$

$$= \det_{12} L_{12} \tag{3.3}$$

where we define $L_{12} = \ln[(r_1 + r_2 + r_{12})/(r_1 + r_2 - r_{12})]/r_{12}$, with similar definitions for L_{23} and so on. L_{12} is expressed intrinsically in terms of distances and is independent of any coordinate system.

In a rigid polyhedron, the edge length r_{12} is fixed; a computer subroutine need not recompute the value each time the subroutine is called. Further, any edge is shared between two faces. We need not calculate L_{12} for both.

3.2.2. Area of a spherical polygon

Now we evaluate the other term in Equation (3.2) which involves the area of the polygon projected onto the tangent sphere described for Lemma 1.

The image of a face polygon projected on its tangent sphere is a spherical polygon whose edges are great circular arcs. On a sphere of radius R, the area of such a spherical polygon is $R^2[\Sigma \text{ vertex angles} - (n-2)\pi]$, where n is the number of edges.

Here we present expressions for one vertex angle S_2 of a spherical polygon in terms of the corresponding planar polygon. We stipulate that the planar polygon be convex because the spherical polygon's interior angles will be confined to the range $[0, \pi)$.

Let P_1, P_2, and P_3 be three consecutive vertices of the planar polygon, and S_1, S_2, and S_3 be the corresponding points on the tangent sphere (Figure 3). In Appendix B we derive this expression for the spherical vertex angle S_2:

$$S_2 = \arctan \frac{|\Delta z|\,[\eta_1(\xi_3 - \xi_2) + \eta_2(\xi_1 - \xi_3) + \eta_3(\xi_2 - \xi_1)]}{-\{\det_{12}\det_{23} + \Delta z^2[(\xi_3 - \xi_2)(\xi_2 - \xi_1) + (\eta_3 - \eta_2)(\eta_2 - \eta_1)]\}/r_2}.$$

The bracketed factors in the numerator and denominator are invariant for a rigid face. The bracketed numerator factor $[\eta_1(\xi_3 - \xi_2) + \eta_2(\xi_1 - \xi_3) + \eta_3(\xi_2 - \xi_1)]$ is twice the area of the planar triangle defined by P_1, P_2, and P_3.

Since we stipulate that the polygon face is convex, the angle S_2 must lie in quadrants I or II. Since the arctangent's numerator is never negative (Appendix B), the sign of the denominator indicates whether S_2 lies in quadrant I (denominator positive) or II (negative).

Recall that we actually need the expression $\text{sign}(\Delta z) \cdot A_{\text{image}}$. We can automatically incorporate the $\text{sign}(\Delta z)$ factor by substituting Δz for $|\Delta z|$ in the arctangent expression, with the understanding that arctan must now return angles in the range $(-\pi, +\pi)$:

$$S_2 = \arctan \frac{\Delta z\,[\eta_1(\xi_3 - \xi_2) + \eta_2(\xi_1 - \xi_3) + \eta_3(\xi_2 - \xi_1)]}{-\{\det_{12}\det_{23} + \Delta z^2[(\xi_3 - \xi_2)(\xi_2 - \xi_1) + (\eta_3 - \eta_2)(\eta_2 - \eta_1)]\}/r_2}. \tag{3.4}$$

We compute all the vertex angles of the spherical polygon by substituting the coordinates of each triplet of consecutive planar vertices. Then

$$\iint_{\substack{\text{planar}\\\text{face}}} \Delta z^3/r^3 \, \mathrm{d}S = \operatorname{sign}(\Delta z) \cdot A_{\text{image}}$$

$$= \Delta z^2 [S_1 + S_2 + \ldots + S_n - (n-2)\operatorname{sign}(\Delta z)\,\pi] \,.$$

For a triangular face there are only three vertices, so the numerators of the three arctan expressions S_1, S_2, and S_3 are equal.

3.3. Triangular Face

Equations (3.3) and (3.4) are reasonably simple subexpressions for the contribution of a convex polygonal face to its polyhedron's gravitational potential. Here we present the entire contribution to the potential due to one triangular face. We restate our definitions here for your convenience.

First, the triangular face's vertex coordinates must be rotated from the body coordinate system to a face-specific, right-handed Cartesian coordinate system whose $\hat{\mathbf{k}}$ axis is aligned with the face's outward-pointing surface normal $\hat{\mathbf{n}}_f$. Here is one way to find the rotation matrix. Suppose the three vertices have components $P_1 = (X_1, Y_1, Z_1)$, $P_2 = (X_2, Y_2, Z_2)$, and $P_3 = (X_3, Y_3, Z_3)$ in the body coordinate system. Form two vector differences $\mathbf{R}_{12} = P_2 - P_1$ and $\mathbf{R}_{23} = P_3 - P_2$. The outward-pointing surface normal vector is the normalized cross product of the two, in this order: $\hat{\mathbf{n}}_f = \hat{\mathbf{k}} = (\mathbf{R}_{12} \times \mathbf{R}_{23})/\|\mathbf{R}_{12} \times \mathbf{R}_{23}\|$. We can normalize one of the vector differences and adopt it as another coordinate direction, say $\hat{\mathbf{i}} = \mathbf{R}_{12}/\|\mathbf{R}_{12}\|$, and find the last coordinate direction by another cross product: $\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$. The components of these basis vectors (which are expressed in the body coordinate system) form the rotation matrix from body to face:

$$\begin{bmatrix} \xi \\ \eta \\ \zeta_f \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_X & \hat{\mathbf{i}}_Y & \hat{\mathbf{i}}_Z \\ \hat{\mathbf{j}}_X & \hat{\mathbf{j}}_Y & \hat{\mathbf{j}}_Z \\ \hat{\mathbf{k}}_X & \hat{\mathbf{k}}_Y & \hat{\mathbf{k}}_Z \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} .$$

In the face coordinate system, the three vertices have fixed coordinates (ξ_1, η_1, ζ_f), (ξ_2, η_2, ζ_f), and (ξ_3, η_3, ζ_f) taken in counterclockwise order according to the right-hand rule and $\hat{\mathbf{n}}_f$. Let r_{12}, r_{23}, and r_{31} represent the fixed lengths of the edges. All of these coordinates and distances can be computed once and saved.

The unit mass has coordinates (x, y, z) in the face coordinate system. The coordinate differences from the unit mass to the first vertex are $\Delta x_1 = \xi_1 - x$, $\Delta y_1 = \eta_1 - y$, and similarly for the other two vertices. All three vertices share the same $\Delta z = \zeta_f - z$. Symbols r_1, r_2, and r_3 represent distances between the unit mass and the three vertices.

For notation, we define three determinant, three logarithm, and three arctangent expressions:

$$\det_{12} = \Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1; \quad \det_{23} = \Delta x_2 \Delta y_3 - \Delta x_3 \Delta y_2;$$

$$\det_{31} = \Delta x_3 \Delta y_1 - \Delta x_1 \Delta y_3$$

$$L_{12} = \frac{1}{r_{12}} \ln \left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right); \quad L_{23} = \frac{1}{r_{23}} \ln \left(\frac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}} \right);$$

$$L_{31} = \frac{1}{r_{31}} \ln \left(\frac{r_3 + r_1 + r_{31}}{r_3 + r_1 - r_{31}} \right)$$

$$S_1 = \arctan \frac{\Delta z[\xi_1(\eta_2 - \eta_3) + \xi_2(\eta_3 - \eta_1) + \xi_3(\eta_1 - \eta_2)]}{-\{\det_{31}\det_{12} + \Delta z^2[(\xi_2 - \xi_1)(\xi_1 - \xi_3) + (\eta_2 - \eta_1)(\eta_1 - \eta_3)]\}/r_1}$$

$$S_2 = \arctan \frac{\Delta z[\xi_1(\eta_2 - \eta_3) + \xi_2(\eta_3 - \eta_1) + \xi_3(\eta_1 - \eta_2)]}{-\{\det_{12}\det_{23} + \Delta z^2[(\xi_3 - \xi_2)(\xi_2 - \xi_1) + (\eta_3 - \eta_2)(\eta_2 - \eta_1)]\}/r_2}$$

$$S_3 = \arctan \frac{\Delta z[\xi_1(\eta_2 - \eta_3) + \xi_2(\eta_3 - \eta_1) + \xi_3(\eta_1 - \eta_2)]}{-\{\det_{23}\det_{31} + \Delta z^2[(\xi_1 - \xi_3)(\xi_3 - \xi_2) + (\eta_1 - \eta_3)(\eta_3 - \eta_2)]\}/r_3} .$$

The numerators of the three arctangent expressions are equal. The numerator and denominator of each arctangent expression have their own signs and together those signs determine the quadrant of the result, which must range through $(-\pi, +\pi)$.

The bracketed factors in each arctangent expression are constants for a rigid triangle. They too can be computed once and saved.

A triangular face's contribution to the gravitational potential of its polyhedron is

$$U_{\text{triangle}} = \frac{\Delta z}{2} (\det_{12} L_{12} + \det_{23} L_{23} + \det_{31} L_{31})$$
$$- \frac{\Delta z^2}{2} (S_1 + S_2 + S_3 - \operatorname{sign}(\Delta z)\,\pi) .$$

The unit mass cannot be inside the body or on its surface without violating a precondition of the Gauss Divergence Theorem. If the unit mass were to lie exactly within some edge, then the L expression for that edge contains a division by zero.

4. Acceleration Contribution due to a Triangular Face

The acceleration experienced by the unit mass at (x, y, z) is found by taking the gradient of the potential. Here we derive the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ acceleration components due to one triangular face, starting from Equation (3.1).

The $\hat{\mathbf{i}}$ component of acceleration due to one triangular face is

$$\frac{\partial}{\partial x} U_{\text{triangle}} = \frac{\partial}{\partial x} \left[\frac{1}{2} \iint\limits_{\text{triangle}} \frac{\Delta z}{r} \, \mathrm{d}S \right] = -\frac{\partial}{\partial \Delta x} \left[\frac{1}{2} \iint\limits_{\text{triangle}} \frac{\Delta z}{r} \, \mathrm{d}S \right]$$

$$= -\frac{\Delta z}{2} \iint_{\text{triangle}} \left[\frac{\partial}{\partial \Delta x} \left(\frac{1}{r} \right) + \frac{\partial}{\partial \Delta y} (0) \right] \mathrm{d}S$$

$$= -\frac{\Delta z}{2} \oint_{\substack{\text{triangle}\\ \text{boundary}}} \left[\frac{1}{r} \mathrm{d}\Delta y - 0 \,\mathrm{d}\Delta x \right]$$

$$= -\frac{\Delta z}{2} \sum_{\substack{\text{triangle}\\ \text{edges}}} \left[\int_{\text{edge}} \frac{1}{r} \mathrm{d}\Delta y \right]$$

$$= -\frac{\Delta z}{2} \left[\sin \alpha_{12} \int_{P_1}^{P_2} \frac{\mathrm{d}s}{r} + \sin \alpha_{23} \int_{P_2}^{P_3} \frac{\mathrm{d}s}{r} + \sin \alpha_{31} \int_{P_3}^{P_1} \frac{\mathrm{d}s}{r} \right]$$

$$= -\frac{\Delta z}{2} \left[\begin{array}{l} \dfrac{\eta_2 - \eta_1}{r_{12}} \ln \left(\dfrac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right) \\ + \dfrac{\eta_3 - \eta_2}{r_{23}} \ln \left(\dfrac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}} \right) \\ + \dfrac{\eta_1 - \eta_3}{r_{31}} \ln \left(\dfrac{r_3 + r_1 + r_{31}}{r_3 + r_1 - r_{31}} \right) \end{array} \right]$$

$$= -\frac{\Delta z}{2} \left[(\eta_2 - \eta_1) L_{12} + (\eta_3 - \eta_2) L_{23} + (\eta_1 - \eta_3) L_{31} \right] .$$

If we arrange the face coordinate system so that, say, the edge between points 1 and 2 parallels the $\hat{\mathbf{i}}$ axis, then $\eta_1 = \eta_2$ and we need accumulate one fewer term:

$$\frac{\partial}{\partial x} U_{\substack{\text{special}\\ \text{triangle}}} = -\frac{\Delta z}{2} \left[0 + (\eta_3 - \eta_2) L_{23} + (\eta_1 - \eta_3) L_{31} \right] .$$

This simplification ($\eta_2 - \eta_1 = 0$) also affects S_1, S_2, and S_3.

In a similar way, the $\hat{\mathbf{j}}$ component is found to be:

$$\frac{\partial}{\partial y} U_{\text{triangle}} = +\frac{\Delta z}{2} \left[(\xi_2 - \xi_1) L_{12} + (\xi_3 - \xi_2) L_{23} + (\xi_1 - \xi_3) L_{31} \right] .$$

The $\hat{\mathbf{k}}$ component requires a different approach:

$$\frac{\partial}{\partial z} U_{\text{triangle}} = \frac{\partial}{\partial z} \left[\tfrac{1}{2} \iint_{\text{triangle}} \frac{\Delta z}{r} \mathrm{d}S \right] = -\frac{\partial}{\partial \Delta z} \left[\tfrac{1}{2} \iint_{\text{triangle}} \frac{\Delta z}{r} \mathrm{d}S \right]$$

$$= -\tfrac{1}{2} \iint_{\text{triangle}} \frac{r^2 - \Delta z^2}{r^3} \mathrm{d}S$$

$$= -\tfrac{1}{2} \iint\limits_{\text{triangle}} \left(\frac{1}{r} + \frac{\Delta z^2}{r^3} - 2\frac{\Delta z^2}{r^3} \right) \mathrm{d}S$$

$$= -\tfrac{1}{2} \iint\limits_{\text{triangle}} \left(\frac{1}{r} + \frac{\Delta z^2}{r^3} \right) \mathrm{d}S + \frac{1}{\Delta z} \iint\limits_{\text{triangle}} \frac{\Delta z^3}{r^3} \mathrm{d}S$$

$$= -\tfrac{1}{2} \left[\det_{12} L_{12} + \det_{23} L_{23} + \det_{31} L_{31} \right]$$

$$+\Delta z [S_1 + S_2 + S_3 - \operatorname{sign}(\Delta z)\,\pi] \,.$$

The acceleration components are expressed in the face-specific coordinate system. They must be rotated to a common body coordinate system before being summed over all faces.

In summary, we have derived closed-form expressions for the gravitational potential and acceleration components experienced by a unit mass due to a constant-density polyhedron. Next we apply the technique to approximate the gravitational potential of the Martian satellite Phobos.

5. Equipotential Surfaces of Phobos

We use coordinates published in Turner (1978) to model the inner Martian satellite Phobos as a polyhedron containing 146 vertices and 288 triangular faces (Figure 2). The volume of this model is slightly less than 5400 km^3, and the centroid is at (–0.23, 0.07, –0.20) km relative to the origin used by Turner. Duxbury (1989) develops a spherical-harmonic representation of Phobos' surface and finds its volume to be 5533.1 km^3. Duxbury's model does not incorporate the large crater Stickney, whereas Turner's model does. Simonelli *et al.* (1993) express the surface of Phobos numerically as the radial distances in a number of preselected (latitude, longitude) directions. The volume of their high-resolution model is 5740 $\pm$ 190 km^3.

Figures 4a and b show equipotential surfaces derived from two Phobos models. Both models are based on surface coordinates given in (Turner, 1978, Table III), and include the large crater Stickney. The two figures depict the potential due solely to the models' gravitation; they omit the rotational potential and third-body potential due to Mars. Each equipotential surface passes through the point (15,0,0) km (approximately 1.5 km above the sub-Mars point). To enhance irregularities, the figures depict the surfaces' radial altitude relative to fitted ellipsoids.

In Figure 4a, the topography of Phobos is modeled as a polyhedron. Gravitational potential is evaluated from this constant-density polyhedron using the formulas of this paper. The fitted ellipsoid has semiaxes (15.3, 14.0, 13.5) km. For comparison, the equipotential surface in Figure 4b is generated from Stokes' coefficients in (Martinec *et al.*, 1989, Table IV). These authors derive their Stokes' coefficients

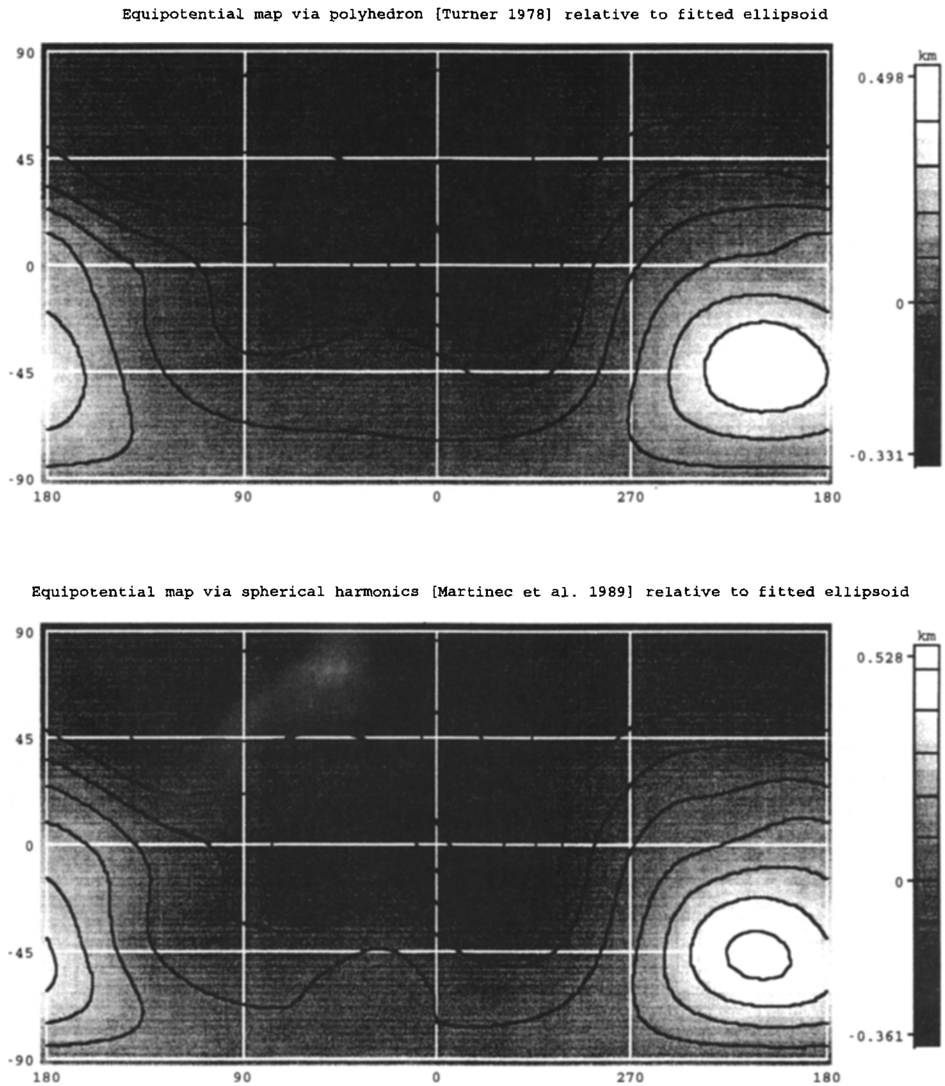

Fig. 4. Equipotential surfaces of Phobos' gravitation relative to fitted ellipsoids, omitting rotation and third-body effects. The surfaces pass through the point (15,0,0) km which is approximately 1.5 km above the sub-Mars point at 0° latitude and 0° longitude. Contours are at integer multiples of 0.1 km. Both surfaces are based on surface coordinates in (Turner, 1978, Table III). In part (a), the coordinates are made into a polyhedron and the equipotential surface is generated using this paper's techniques. Radial altitudes are relative to a fitted ellipsoid with semiaxes (15.3, 14.0, 13.5) km. In part (b), the equipotential surface is generated from a 6×6 spherical-harmonic field (Martinec *et al.*, 1989, Table IV) derived from the (Turner, 1978, Table III) surface coordinates. Radial altitudes are relative to an ellipsoid with semiaxes (15.3, 13.9, 13.4) km.

from surface coordinates in (Turner, 1978, Table III). Like us, they assume the model has constant density. The fitted ellipsoid has semiaxes (15.3, 13.9, 13.4) km.

Visually, Figures 4a and b compare well. Both show the northern hemisphere as being lower than the southern, but this is because the centers of the fitted ellipsoids are at the origins of surface coordinates, not at the centroids of the models. The extreme altitudes occur in the same areas in the two maps, the high at about 230°W, 45°S and the low near 340°W, 25°N. The large crater Stickney is centered approximately 60°W, 10°S and both equipotential surfaces show a depression there. We conclude that the polyhedral technique produces much the same results as a conventional spherical-harmonic approach.

6. Relation to Other Work

Strakhov *et al.* (1990, §1) state, "The direct gravimetric and magnetometric problems for homogeneous polyhedrons are classical. They were first studied in the 19th century (Meller and Sludsky), but it wasn't until the 1960s–1980s that special attention was devoted to them", and cite eighteen references. We have not yet located these 19th century works of Meller and Sludsky.

MacMillan (1930, §43) derives a closed-form expression for the potential of a homogeneous, rectangular parallelepiped by evaluating a triple integral. It was from this work that we gained the insight of projecting a face onto a tangent sphere. For verification, we have specialized our expressions and found exact correspondence with MacMillan's.

Timmer *et al.* (1980) evaluate the surface area, volume, centroid, and moments of inertia of solid objects using the same theorems as we do (Divergence, Green's), but in the context of computer-aided design. They model intersecting objects such as cylinders, cones, and spheres, and use Gaussian quadrature approximations because of the difficulty in expressing surface boundaries. Our result are in closed form because we use polyhedra exclusively.

Waldvogel (1979) expresses the gravitational potential of a general homogeneous polyhedron in closed form. The potential at one vertex of a standard element, a rectangular simplex, is found by evaluating a triple integral. The potential of a general polyhedron is found by adding and subtracting appropriately sized simplices. Each edge of a given face requires two simplices for a total of four transcendental function evaluations (arctan and arctanh), twice the number required by the technique presented in this paper.

Barnett (1976) and Okabe (1979) use the three-dimensional Gauss Divergence Theorem and its two-dimensional form or other means to evaluate the gradient and second partials of a polyhedron's potential. Pohànka (1988) states those authors should not claim to have valid expressions when the unit mass is inside the body or on its surface. The Divergence Theorem should not be used with a singular integrand.

We confess to using manual techniques to form a polyhedron from a collection of surface points. Computer algorithms for this task are described in Cavendish *et al.* (1985) and Hoppe *et al.* (1992).

7. Conclusions and Future Work

We have derived closed-form expressions for the exterior gravitational potential and acceleration components of a homogeneous polyhedron whose surface consists of planar triangles. The expressions contain two kinds of terms, logarithms associated with the polyhedron's edges and arctangents associated with interior vertex angles of the faces. To demonstrate the utility of the technique, we depict an equipotential surface of the Martian satellite Phobos. The large crater Stickney is present in the model of the physical surface and its effect is discernible in the equipotential surface. The equipotential surface derived from the polyhedral technique compares well with one produced using a conventional spherical-harmonic expansion.

One advantage we see in the polyhedral technique is that the quantities which parameterize the potential and acceleration expressions (the polyhedron's vertices, edges, and faces) are local in scope. Incorporating a surface feature into the model affects only the faces in the vicinity of the feature. This is in contrast to a spherical-harmonic surface representation where the quantities have global scope; incorporating a surface feature generally affects every coefficient.

The polyhedron expressions are an approximation to reality since real bodies are not polyhedra and contain density irregularities. The expressions seem unsuitable for analytic studies since they do not separate into the Keplerian term and disturbing function. But they can certainly be used to propagate trajectories in ejecta studies or mission design.

It might be fruitful to study orbits around simple polyhedra such as the Platonic solids. A tetrahedron seems to affect orbits in a way similar to the spherical harmonics J_3 and $J_{3,3}$.

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Appendix A. Differential Area Element on the Tangent Sphere

We use the following results from differential geometry (Kreyszig, 1959, §35–36). Let $\mathbf{S}(u_1, u_2)$ define a surface embedded in three-dimensional space. The directions of u_1 and u_2 are not necessarily orthogonal. For notation we define

$$g_{11} = \frac{\partial \mathbf{S}}{\partial u_1} \bullet \frac{\partial \mathbf{S}}{\partial u_1}\,, \quad g_{12} = g_{21} = \frac{\partial \mathbf{S}}{\partial u_1} \bullet \frac{\partial \mathbf{S}}{\partial u_2}\,,$$

and

$$g_{22} = \frac{\partial \mathbf{S}}{\partial u_2} \bullet \frac{\partial \mathbf{S}}{\partial u_2}$$

$$g = g_{11}g_{22} - g_{12}g_{21} = \det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} .$$

Then the differential area element on the surface is $\mathrm{d}A = \sqrt{g}\,\mathrm{d}u_1\,\mathrm{d}u_2$, and the angle ω between coordinate directions u_1 and u_2 is $\cos\omega = g_{12}/\sqrt{g_{11}g_{22}}$.

The arccos function is very sensitive to argument inaccuracy when the angle is near a multiple of π, so we use $\tan\omega = \sqrt{g}/g_{12}$ instead. An angle evaluated using arccos lies in quadrants I or II, so we require this of arctan as well. Since the numerator of $\tan\omega$, $\sqrt{g}$, is never negative, it is the sign of the denominator g_{12} which indicates quadrant I ($g_{12} > 0$) or II ($g_{12} < 0$).

In this appendix we use such expressions to evaluate the differential area element on the tangent sphere. Intermediate results are then used in the next appendix to evaluate a typical vertex angle of a spherical polygon. In both cases the surface $\mathbf{S}(u_1, u_2)$ is the tangent sphere described in Section 3.1. All results can be expressed in terms of quantities on the face plane.

Differential Area Element

Let $\mathbf{P} = \Delta x\hat{\mathbf{i}} + \Delta y\hat{\mathbf{j}} + \Delta z\hat{\mathbf{k}}$ denote the location of an arbitrary point on the face plane. Coordinates are measured relative to the unit mass, and Δz is the constant, signed perpendicular distance from the unit mass to the face plane. Let $\mathbf{S}$ denote the corresponding point projected on a sphere of radius R centered at the unit mass:

$$\mathbf{S} = \mathbf{S}(\Delta x, \Delta y) = R\left[\frac{\Delta x}{r}\hat{\mathbf{i}} + \frac{\Delta y}{r}\hat{\mathbf{j}} + \frac{\Delta z}{r}\hat{\mathbf{k}}\right]$$

where $r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. For the particular case of the tangent sphere, the radius $R = |\Delta z|$. However, we postpone this substitution until later.

The partials of $\mathbf{S}$ are easily found to be

$$\begin{aligned} \frac{\partial \mathbf{S}}{\partial \Delta x} &= \frac{R}{r^3}\left[(r^2 - \Delta x^2)\,\hat{\mathbf{i}} - \Delta x\,\Delta y\,\hat{\mathbf{j}} - \Delta x\,\Delta z\,\hat{\mathbf{k}}\right] \\ \frac{\partial \mathbf{S}}{\partial \Delta y} &= \frac{R}{r^3}\left[- \Delta x\,\Delta y\,\hat{\mathbf{i}} + (r^2 - \Delta y^2)\,\hat{\mathbf{j}} - \Delta y\,\Delta z\,\hat{\mathbf{k}}\right] . \end{aligned} \tag{A.1}$$

Now we evaluate the quantities g_{11}, g_{22}, g_{12}, and g, from which we can determine dA. The derivation of g_{22} parallels g_{11} and we omit intermediate steps:

$$g_{11} = \frac{\partial \mathbf{S}}{\partial \Delta x} \bullet \frac{\partial \mathbf{S}}{\partial \Delta x} = \frac{R^2}{r^6}\left[(r^2 - \Delta x^2)^2 + (-\Delta x \Delta y)^2 + (-\Delta x \Delta z)^2\right]$$

$$= \frac{R^2}{r^6}\left[r^4 - 2r^2 \Delta x^2 + \Delta x^4 + \Delta x^2 \Delta y^2 + \Delta x^2 \Delta z^2\right]$$

$$= \frac{R^2}{r^6}\left[r^4 - 2r^2 \Delta x^2 + \Delta x^2(\Delta x^2 + \Delta y^2 + \Delta z^2)\right] = \frac{R^2}{r^6}\left[r^4 - r^2 \Delta x^2\right]$$

$$= \frac{R^2}{r^4}\left[r^2 - \Delta x^2\right] \tag{A.2}$$

$$g_{22} = \frac{\partial \mathbf{S}}{\partial \Delta y} \bullet \frac{\partial \mathbf{S}}{\partial \Delta y} = \frac{R^2}{r^4}\left[r^2 - \Delta y^2\right] \tag{A.3}$$

$$g_{12} = \frac{\partial \mathbf{S}}{\partial \Delta x} \bullet \frac{\partial \mathbf{S}}{\partial \Delta y} = \frac{R^2}{r^6}\begin{bmatrix} (r^2 - \Delta x^2)(-\Delta x\, \Delta y) + (-\Delta x\, \Delta y)(r^2 - \Delta y^2) \\ +(-\Delta x\, \Delta z)(-\Delta y\, \Delta z) \end{bmatrix}$$

$$= \frac{R^2}{r^6}\left[-2r^2\, \Delta x\, \Delta y + \Delta x\, \Delta y(\Delta x^2 + \Delta y^2 + \Delta z^2)\right]$$

$$= \frac{R^2}{r^4}\left[-\Delta x\, \Delta y\right] \tag{A.4}$$

$$g = g_{11} g_{22} - g_{12}^2 = \frac{R^4}{r^8}\left[(r^2 - \Delta x^2)(r^2 - \Delta y^2) - (-\Delta x\, \Delta y)^2\right]$$

$$= \frac{R^4}{r^8}\left[r^4 - r^2(\Delta x^2 + \Delta y^2) + \Delta x^2\, \Delta y^2 - \Delta x^2\, \Delta y^2\right]$$

$$= \frac{R^4}{r^6}\left[r^2 - (\Delta x^2 + \Delta y^2)\right]$$

$$= \frac{R^4}{r^6}\, \Delta z^2 \,.$$

When we substitute $R = |\Delta z|$, we find the differential area element on the tangent sphere is $\mathrm{d}A = \sqrt{g}\, \mathrm{d}\Delta x\, \mathrm{d}\Delta y = \sqrt{\Delta z^6 / r^6}\, \mathrm{d}\Delta x\, \mathrm{d}\Delta y = |\Delta z|^3 / r^3\, \mathrm{d}\Delta x\, \mathrm{d}\Delta y$. Note that we must use the absolute value since Δz is signed. This is the result we use in section 3.1.

Appendix B. Vertex Angle of a Spherical Polygon

In this appendix we use intermediate expressions from Appendix A to evaluate a typical vertex angle S_2 of a spherical polygon in terms of the corresponding planar vertex $\mathbf{P}_2 = \Delta x_2\,\hat{\mathbf{i}}+\Delta y_2\,\hat{\mathbf{j}}+\Delta z\,\hat{\mathbf{k}}$ and its two neighbors $\mathbf{P}_1 = \Delta x_1\,\hat{\mathbf{i}}+\Delta y_1\,\hat{\mathbf{j}}+\Delta z\,\hat{\mathbf{k}}$ and $\mathbf{P}_3 = \Delta x_3\,\hat{\mathbf{i}} + \Delta y_3\,\hat{\mathbf{j}} + \Delta z\,\hat{\mathbf{k}}$ (Figure 3).

There are many preliminaries. For brevity, we introduce $c_{12} = \cos\alpha_{12} = (\Delta x_2 - \Delta x_1)/r_{12} = (\xi_2 - \xi_1)/r_{12}$, $s_{12} = \sin\alpha_{12} = (\Delta y_2 - \Delta y_1)/r_{12} = (\eta_2 - \eta_1)/r_{12}$, and corresponding definitions for c_{23} and s_{23}. The angles α_{12} and α_{23} are between the Δx axis and the edges $\overline{P_1P_2}$ and $\overline{P_2P_3}$.

In the face plane, we set up a non-orthogonal coordinate system $\mathbf{P}(u_1, u_2)$ such that we move from point P_2 to P_1 or P_3 along the two non-orthogonal directions: $\mathbf{P}(0,0) = \mathbf{P}_2$, $\mathbf{P}(r_{12},0) = \mathbf{P}_1$, and $\mathbf{P}(0,r_{23}) = \mathbf{P}_3$. The general point is

$$\begin{aligned}\mathbf{P}(u_1,u_2) &= (\Delta x_2 - u_1 c_{12} + u_2 c_{23})\,\hat{\mathbf{i}} + (\Delta y_2 - u_1 s_{12} + u_2 s_{23})\,\hat{\mathbf{j}} + \Delta z\,\hat{\mathbf{k}} \\ &= \Delta x(u_1,u_2)\,\hat{\mathbf{i}} + \Delta y(u_1,u_2)\,\hat{\mathbf{j}} + \Delta z\,\hat{\mathbf{k}}\,.\end{aligned} \tag{B.1}$$

We are more interested in the partial derivatives at the spherical point S_2 than its coordinates. We evaluate these partials by employing the chain rule:

$$\frac{\partial \mathbf{S}_2}{\partial u_1} = \frac{\partial \mathbf{S}_2}{\partial \Delta x}\frac{\partial \Delta x}{\partial u_1} + \frac{\partial \mathbf{S}_2}{\partial \Delta y}\frac{\partial \Delta y}{\partial u_1}\,, \quad \frac{\partial \mathbf{S}_2}{\partial u_2} = \frac{\partial \mathbf{S}_2}{\partial \Delta x}\frac{\partial \Delta x}{\partial u_2} + \frac{\partial \mathbf{S}_2}{\partial \Delta y}\frac{\partial \Delta y}{\partial u_2}$$

Expressions for $\partial\mathbf{S}/\partial\Delta x$, $\partial\mathbf{S}/\partial\Delta y$, g_{11}, g_{22}, and g_{12}, appear in Appendix A, Equations (A.1–A.4). To specialize them for the specific point S_2, we need only relabel the quantities Δx, Δy, and r to read Δx_2, Δy_2, and r_2:

$$\frac{\partial \mathbf{S}_2}{\partial \Delta x} = \frac{R}{r_2^3}\left[(r_2^2 - \Delta x_2^2)\,\hat{\mathbf{i}} - \Delta x_2\,\Delta y_2\,\hat{\mathbf{j}} - \Delta x_2\,\Delta z_2\,\hat{\mathbf{k}}\right]$$

$$\frac{\partial \mathbf{S}_2}{\partial \Delta y} = \frac{R}{r_2^3}\left[-\Delta x_2\,\Delta y_2\,\hat{\mathbf{i}} + (r_2^2 - \Delta y_2^2)\,\hat{\mathbf{j}} - \Delta y_2\,\Delta z_2\,\hat{\mathbf{k}}\right]$$

$$g_{11} = \frac{R^2}{r_2^4}\left(r_2^2 - \Delta x_2^2\right)\,, \quad g_{22} = \frac{R^2}{r_2^4}\left(r_2^2 - \Delta y_2^2\right)\,, \quad g_{12} = \frac{R^2}{r_2^4}\left(-\Delta x_2\,\Delta y_2\right)\,.$$

From Equation (B.1) we have $\Delta x(u_1,u_2) = \Delta x_2 - u_1 c_{12} + u_2 c_{23}$ and $\Delta y(u_1,u_2) = \Delta y_2 - u_1 s_{12} + u_2 s_{23}$, so the partials are

$$\frac{\partial \Delta x}{\partial u_1} = -c_{12}\,, \quad \frac{\partial \Delta x}{\partial u_2} = c_{23}$$

$$\frac{\partial \Delta y}{\partial u_1} = -s_{12}\,, \quad \frac{\partial \Delta y}{\partial u_2} = s_{23}\,.$$

Certain expressions with geometric meanings occur often in the derivations below. For brevity, we define coordinates of P_2 expressed in two different primed coordinate systems, one rotated so the x' axis is aligned with the edge $\overline{P_1P_2}$, and the other's x'' axis aligned with the edge $\overline{P_2P_3}$ (Figure B1, parts a–c):

$$\Delta x_2' = \Delta x_2 c_{12} + \Delta y_2 s_{12}\ , \quad \Delta x_2'' = \Delta x_2 c_{23} + \Delta y_2 s_{23}\ ,$$

$$\Delta y_2' = \Delta y_2 c_{12} - \Delta x_2 s_{12}\ , \quad \Delta y_2'' = \Delta y_2 c_{23} - \Delta x_2 s_{23}\ ,$$

$$r_2^2 = \Delta x_2'^2 + \Delta y_2'^2 + \Delta z^2 = \Delta x_2''^2 + \Delta y_2''^2 + \Delta z^2\ .$$

The exterior angle at P_2, viz. $\alpha_{23} - \alpha_{12}$, often appears. Since we stipulate a convex polygon face (Section 3.2), this exterior angle is always in quadrants I or II.

Two lemmas complete the preliminaries:

Lemma B1

$$\begin{aligned}\Delta x_2'\,\Delta x_2'' + \Delta y_2'\,\Delta y_2'' &= (\Delta x_2 c_{12} + \Delta y_2 s_{12})(\Delta x_2 c_{23} + \Delta y_2 s_{23})\\ &\quad +(\Delta y_2 c_{12} - \Delta x_2 s_{12})(\Delta y_2 c_{23} - \Delta x_2 s_{23})\\ &= \Delta x_2^2 c_{12} c_{23} + \Delta x_2 \Delta y_2 (c_{12} s_{23} + s_{12} c_{23}) + \Delta y_2^2 s_{12} s_{23}\\ &\quad +\Delta y_2^2 c_{12} c_{23} - \Delta x_2 \Delta y_2 (c_{12} s_{23} + s_{12} c_{23}) + \Delta x_2^2 s_{12} s_{23}\\ &= \Delta x_2^2 (c_{12} c_{23} + s_{12} s_{23}) + \Delta y_2^2 (c_{12} c_{23} + s_{12} s_{23})\\ &= (\Delta x_2^2 + \Delta y_2^2)\ \cos(\alpha_{23} - \alpha_{12})\ .\end{aligned}$$

We use this lemma as $\Delta x_2'\,\Delta x_2'' - (\Delta x_2^2 + \Delta y_2^2)\ \cos(\alpha_{23} - \alpha_{12}) = -\Delta y_2'\,\Delta y_2''$.

Lemma B2

Let d be a certain distance shown in Figure B1(d). The first two equalities below hold because of the Law of Cosines for Planar Triangles. We take the second equality and continue:

$$\begin{aligned}d^2 &= \Delta x_2'^2 + \Delta x_2''^2 - 2\Delta x_2'\,\Delta x_2''\ \cos(\alpha_{23} - \alpha_{12})\\ &= \Delta y_2'^2 + \Delta y_2''^2 - 2\Delta y_2'\,\Delta y_2''\ \cos(\alpha_{23} - \alpha_{12})\\ &= (\Delta y_2 c_{12} - \Delta x_2 s_{12})^2 + (\Delta y_2 c_{23} - \Delta x_2 s_{23})^2\\ &\quad -2(\Delta y_2 c_{12} - \Delta x_2 s_{12})(\Delta y_2 c_{23} - \Delta x_2 s_{23})(c_{12} c_{23} + s_{12} s_{23})\end{aligned}$$

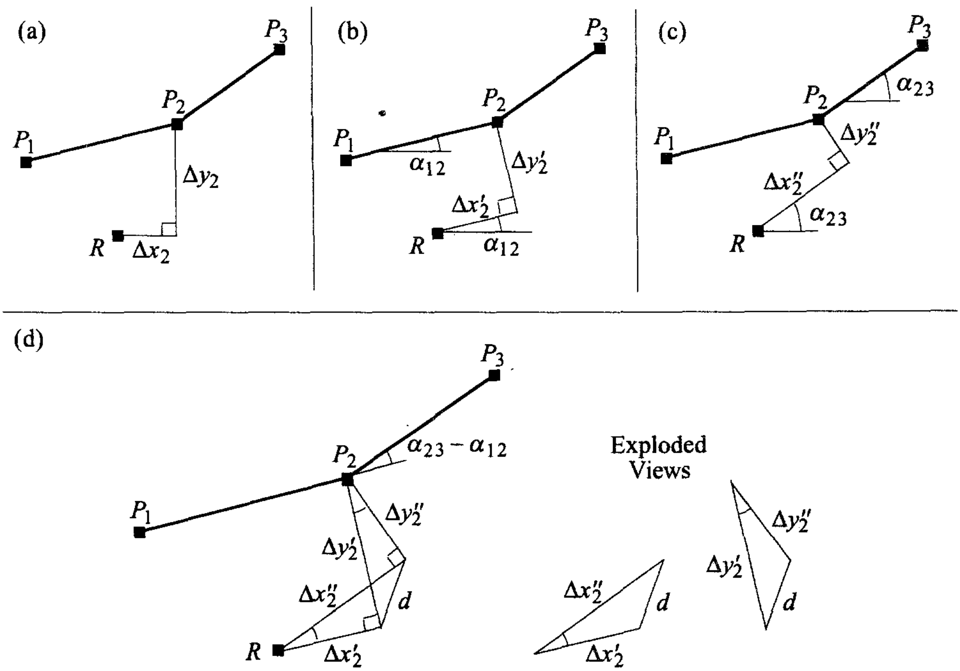

Fig. B1. (Part a) Relative to point R (the location of the unit mass projected onto the face plane), coordinates of P_2 are $(\Delta x_2, \Delta y_2)$ expressed in the usual coordinate system. We also use coordinates of P_2 in rotated coordinate systems with x-axes paralleling edge $\overline{P_1P_2}$ (part b) and $\overline{P_2P_3}$ (part c). Part (d) shows a certain distance d is derived easily from the law of cosines for planar triangles. The three angles drawn with arcs all equal $\alpha_{23} - \alpha_{12}$, the exterior vertex angle at P_2.

$$= \Delta y_2^2 c_{12}^2 - 2\Delta x_2 \Delta y_2 c_{12} s_{12} + \Delta x_2^2 s_{12}^2 + \Delta y_2^2 c_{23}^2 - 2\Delta x_2 \Delta y_2 c_{23} s_{23} + \Delta x_2^2 s_{23}^2$$

$$-2[\Delta y_2^2 c_{12} c_{23} - \Delta x_2 \Delta y_2 (s_{12} c_{23} + c_{12} s_{23}) + \Delta x_2^2 s_{12} s_{23}](c_{12} c_{23} + s_{12} s_{23})$$

$$= \Delta x_2^2 [s_{12}^2 + s_{23}^2 - 2 s_{12} s_{23} (c_{12} c_{23} + s_{12} s_{23})]$$

$$+\Delta y_2^2 [c_{12}^2 + c_{23}^2 - 2 c_{12} c_{23} (c_{12} c_{23} + s_{12} s_{23})]$$

$$-2\Delta x_2 \Delta y_2 [c_{12} s_{12} + c_{23} s_{23} - (s_{12} c_{23} + c_{12} s_{23})(c_{12} c_{23} + s_{12} s_{23})]$$

$$= \Delta x_2^2 [s_{12}^2 + s_{23}^2 - 2 c_{12} s_{12} c_{23} s_{23} - 2 s_{12}^2 s_{23}^2]$$

$$+\Delta y_2^2 [c_{12}^2 + c_{23}^2 - 2 c_{12}^2 c_{23}^2 - 2 c_{12} s_{12} c_{23} s_{23}]$$

$$-2\Delta x_2 \Delta y_2 [c_{12} s_{12} + c_{23} s_{23} - c_{12} s_{12} c_{23}^2 - s_{12}^2 c_{23} s_{23} - c_{12}^2 c_{23} s_{23} - c_{12} s_{12} s_{23}^2]$$

$$= \Delta x_2^2\Big[s_{12}^2(1 - s_{23}^2) + s_{23}^2(1 - s_{12}^2) - 2c_{12}s_{12}c_{23}s_{23}\Big]$$

$$+\Delta y_2^2\Big[c_{12}^2(1 - c_{23}^2) + c_{23}^2(1 - c_{12}^2) - 2c_{12}s_{12}c_{23}s_{23}\Big]$$

$$-2\Delta x_2 \Delta y_2[c_{12}s_{12} + c_{23}s_{23} - c_{12}s_{12} - c_{23}s_{23}]$$

$$= \Delta x_2^2[s_{12}^2c_{23}^2 - 2c_{12}s_{12}c_{23}s_{23} + s_{23}^2c_{12}^2]$$

$$+\Delta y_2^2[c_{12}^2s_{23}^2 - 2c_{12}s_{12}c_{23}s_{23} + c_{23}^2s_{12}^2]$$

$$= \Delta x_2^2[s_{23}c_{12} - s_{12}c_{23}]^2 + \Delta y_2^2[s_{23}c_{12} - s_{12}c_{23}]^2$$

$$= (\Delta x_2^2 + \Delta y_2^2)\ \sin^2(\alpha_{23} - \alpha_{12})\ .$$

We use the lemma in this form: $\Delta y_2'^2 + \Delta y_2''^2 - 2\Delta y_2' \Delta y_2''\ \cos(\alpha_{23} - \alpha_{12}) = (\Delta x_2^2 + \Delta y_2^2)\ \sin^2(\alpha_{23} - \alpha_{12})$.

Spherical Vertex Angle

With the preliminaries finished, we evaluate the spherical vertex angle S_2 as $\tan S_2 = \sqrt{G}/G_{12}$. We capitalize G_{11}, G_{22}, G_{12}, and G in this appendix to distinguish them from g_{11}, g_{22}, and g_{12} which we continue to use from Appendix A. The derivation of G_{22} parallels G_{11} and we omit intermediate steps:

$$G_{11} = \frac{\partial \mathbf{S}_2}{\partial u_1} \bullet \frac{\partial \mathbf{S}_2}{\partial u_1} = \left[\frac{\partial \mathbf{S}_2}{\partial \Delta x}\frac{\partial \Delta x}{\partial u_1} + \frac{\partial \mathbf{S}_2}{\partial \Delta y}\frac{\partial \Delta y}{\partial u_1}\right] \bullet \left[\frac{\partial \mathbf{S}_2}{\partial \Delta x}\frac{\partial \Delta x}{\partial u_1} + \frac{\partial \mathbf{S}_2}{\partial \Delta y}\frac{\partial \Delta y}{\partial u_1}\right]$$

$$= \left(\frac{\partial \Delta x}{\partial u_1}\right)^2 \frac{\partial \mathbf{S}_2}{\partial \Delta x} \bullet \frac{\partial \mathbf{S}_2}{\partial \Delta x} + 2\left(\frac{\partial \Delta x}{\partial u_1}\frac{\partial \Delta y}{\partial u_1}\right) \frac{\partial \mathbf{S}_2}{\partial \Delta x} \bullet \frac{\partial \mathbf{S}_2}{\partial \Delta y}$$

$$+ \left(\frac{\partial \Delta y}{\partial u_1}\right)^2 \frac{\partial \mathbf{S}_2}{\partial \Delta y} \bullet \frac{\partial \mathbf{S}_2}{\partial \Delta y}$$

$$= c_{12}^2 g_{11} + 2c_{12}s_{12}g_{12} + s_{12}^2 g_{22}$$

$$= \frac{R^2}{r_2^4}\ \Big[c_{12}^2(r_2^2 - \Delta x_2^2) + 2c_{12}s_{12}(-\Delta x_2\,\Delta y_2) + s_{12}^2(r_2^2 - \Delta y_2^2)\Big]$$

$$= \frac{R^2}{r_2^4}\ [r_2^2c_{12}^2 - \Delta x_2^2c_{12}^2 - 2\Delta x_2\Delta y_2 c_{12}s_{12} + r_2^2s_{12}^2 - \Delta y_2^2 s_{12}^2]$$

$$= \frac{R^2}{r_2^4}\left[r_2^2 - (\Delta x_2 c_{12} + \Delta y_2 s_{12})^2\right] = \frac{R^2}{r_2^4}\left[r_2^2 - \Delta x_2'^2\right]$$

$$= \frac{R^2}{r_2^4}\left[\Delta y_2'^2 + \Delta z^2\right]$$

$$G_{22} = \frac{\partial \mathbf{S}_2}{\partial u_2} \bullet \frac{\partial \mathbf{S}_2}{\partial u_2} = \frac{R^2}{r_2^4}\left[\Delta y_2''^2 + \Delta z^2\right]$$

$$G_{12} = \frac{\partial \mathbf{S}_2}{\partial u_1} \bullet \frac{\partial \mathbf{S}_2}{\partial u_2} = \left[\frac{\partial \mathbf{S}_2}{\partial \Delta x}\frac{\partial \Delta x}{\partial u_1} + \frac{\partial \mathbf{S}_2}{\partial \Delta y}\frac{\partial \Delta y}{\partial u_1}\right] \bullet \left[\frac{\partial \mathbf{S}_2}{\partial \Delta x}\frac{\partial \Delta x}{\partial u_2} + \frac{\partial \mathbf{S}_2}{\partial \Delta y}\frac{\partial \Delta y}{\partial u_2}\right]$$

$$= \left(\frac{\partial \Delta x}{\partial u_1}\frac{\partial \Delta x}{\partial u_2}\right)\frac{\partial \mathbf{S}_2}{\partial \Delta x} \bullet \frac{\partial \mathbf{S}_2}{\partial \Delta x} + \left(\frac{\partial \Delta x}{\partial u_1}\frac{\partial \Delta y}{\partial u_2} + \frac{\partial \Delta x}{\partial u_2}\frac{\partial \Delta y}{\partial u_1}\right)\frac{\partial \mathbf{S}_2}{\partial \Delta x} \bullet \frac{\partial \mathbf{S}_2}{\partial \Delta y}$$

$$+ \left(\frac{\partial \Delta y}{\partial u_1}\frac{\partial \Delta y}{\partial u_2}\right)\frac{\partial \mathbf{S}_2}{\partial \Delta y} \bullet \frac{\partial \mathbf{S}_2}{\partial \Delta y}$$

$$= -c_{12}c_{23}g_{11} + (-c_{12}s_{23} - c_{23}s_{12})\,g_{12} - s_{12}s_{23}g_{22}$$

$$= \frac{R^2}{r_2^4}\Big[-c_{12}c_{23}(r_2^2 - \Delta x_2^2) - (c_{12}s_{23} + c_{23}s_{12})(-\Delta x_2\,\Delta y_2)$$

$$-s_{12}s_{23}(r_2^2 - \Delta y_2^2)\Big]$$

$$= \frac{R^2}{r_2^4}\begin{bmatrix} -r_2^2 c_{12}c_{23} + \Delta x_2^2 c_{12}c_{23} + \Delta x_2\,\Delta y_2(c_{12}s_{23} + c_{23}s_{12}) - r_2^2 s_{12}s_{23} \\ +\Delta y_2^2 s_{12}s_{23} \end{bmatrix}$$

$$= \frac{R^2}{r_2^4}\left[-r_2^2(c_{12}c_{23} + s_{12}s_{23}) + (\Delta x_2 c_{12} + \Delta y_2 s_{12})(\Delta x_2 c_{23} + \Delta y_2 s_{23})\right]$$

$$= \frac{R^2}{r_2^4}\left[\Delta x_2'\,\Delta x_2'' - r_2^2\,\cos(\alpha_{23} - \alpha_{12})\right]$$

$$= \frac{R^2}{r_2^4}\left[\Delta x_2'\,\Delta x_2'' - (\Delta x_2^2 + \Delta y_2^2 + \Delta z_2)\,\cos(\alpha_{23} - \alpha_{12})\right]$$

$$= \frac{R^2}{r_2^4}\left[-\Delta y_2'\,\Delta y_2'' - \Delta z^2\,\cos(\alpha_{23} - \alpha_{12})\right].$$

We used Lemma B1 in the last step because the ultimate result of this appendix then contains the $\det_{ij}$ expressions from Section 3.2: $\det_{12} = \Delta x_1\,\Delta y_2 - \Delta x_2\,\Delta y_1$

and $\det_{23} = \Delta x_2\, \Delta y_3 - \Delta x_3\, \Delta y_2$. A computer subroutine needs to calculate them anyway for other terms.

From G_{11}, G_{12}, and G_{22}, we calculate G, using Lemma B2 in the fourth-to-last step:

$$
\begin{aligned}
G &= G_{11}G_{22} - G_{12}^2 \\
&= \left[\frac{R^2}{r_2^4}\,(\Delta y_2'^2 + \Delta z^2)\right]\,\left[\frac{R^2}{r_2^4}\,(\Delta y_2''^2 + \Delta z^2)\right] \\
&\quad -\frac{R^4}{r_2^8}\,\Big[-\Delta y_2'\,\Delta y_2'' - \Delta z^2\,\cos(\alpha_{23} - \alpha_{12})\Big]^2 \\
&= \frac{R^4}{r_2^8}\left[\begin{array}{l} \Delta y_2'^2\,\Delta y_2''^2 + \Delta z^2(\Delta y_2'^2 + \Delta y_2''^2) + \Delta z^4 \\ -\Delta y_2'^2\,\Delta y_2''^2 - 2\Delta y_2'\,\Delta y_2''\,\Delta z^2\,\cos(\alpha_{23} - \alpha_{12}) \\ -\Delta z^4\,\cos^2(\alpha_{23} - \alpha_{12}) \end{array}\right] \\
&= \frac{R^4}{r_2^8}\left\{\begin{array}{l} \Delta z^2\Big[\Delta y_2'^2 + \Delta y_2''^2 - 2\Delta y_2'\,\Delta y_2''\,\cos(\alpha_{23} - \alpha_{12})\Big] \\ +\Delta z^4\Big[1 - \cos^2(\alpha_{23} - \alpha_{12})\Big] \end{array}\right\} \\
&= \frac{R^4}{r_2^8}\Big\{\Delta z^2(\Delta x_2^2 + \Delta y_2^2)\,\sin^2(\alpha_{23} - \alpha_{12}) + \Delta z^4\,\sin^2(\alpha_{23} - \alpha_{12})\Big\} \\
&= \frac{R^4\,\Delta z^2}{r_2^8}\,(\Delta x_2^2 + \Delta y_2^2 + \Delta z^2)\,\sin^2(\alpha_{23} - \alpha_{12}) \\
&= \frac{R^4\,\Delta z^2}{r_2^6}\,\sin^2(\alpha_{23} - \alpha_{12})\,.
\end{aligned}
$$

The spherical vertex angle S_2 can be found from $\tan S_2 = \sqrt{G}/G_{12}$ where the numerator is non-negative and the quadrant (I, II) is determined by the sign of the denominator. Since we stipulate only convex polygons for faces, the exterior vertex angle $\alpha_{23} - \alpha_{12}$ is in quadrants I and II and $\sin(\alpha_{23} - \alpha_{12}) \geq 0$. Thus it is safe to remove it from beneath the radical. However, we must use the absolute value of Δz since it is signed.

$$
\begin{aligned}
\tan S_2 &= \sqrt{G}/G_{12} \\
&= \sqrt{\frac{R^4\,\Delta z^2\,\sin^2(\alpha_{23} - \alpha_{12})}{r_2^6}}\Bigg/\left\{\frac{R^2}{r_2^4}\,\Big[-\Delta y_2'\,\Delta y_2''\right.
\end{aligned}
$$

$$\left. -\Delta z^2 \cos(\alpha_{23} - \alpha_{12})\right] \Bigg\}$$

$$= \frac{|\Delta z|\, r_2 \sin(\alpha_{23} - \alpha_{12})}{-\Delta y_2' \,\Delta y_2'' - \Delta z^2 \cos(\alpha_{23} - \alpha_{12})}.$$

R, the radius of the sphere, has disappeared as it should. The vertex angle is independent of the radius.

Now we insert the definitions of $\sin(\alpha_{23} - \alpha_{12})$, $\cos(\alpha_{23} - \alpha_{12})$, $\Delta y_2'$, and $\Delta y_2''$ into $\tan S_2$. When it suits us, we expand factors such as c_{12} as either $(\Delta x_2 - \Delta x_1)/r_{12}$ or $(\xi_2 - \xi_1)/r_{12}$.

$$\tan S_2 = \frac{|\Delta z|\, r_2 \sin(\alpha_{23} - \alpha_{12})}{-\Delta y_2' \,\Delta y_2'' - \Delta z^2 \cos(\alpha_{23} - \alpha_{12})}$$

$$= \frac{|\Delta z|\, r_2 (s_{23}c_{12} - c_{23}s_{12})}{-(\Delta y_2 c_{12} - \Delta x_2 s_{12})(\Delta y_2 c_{23} - \Delta x_2 s_{23}) - \Delta z^2 (c_{23}c_{12} + s_{23}s_{12})}$$

$$= \frac{|\Delta z|\, r_2 \left(\frac{\eta_3 - \eta_2}{r_{23}} \frac{\xi_2 - \xi_1}{r_{12}} - \frac{\xi_3 - \xi_2}{r_{23}} \frac{\eta_2 - \eta_1}{r_{12}}\right)}{\left[\begin{array}{l} -\left(\Delta y_2 \frac{\Delta x_2 - \Delta x_1}{r_{12}} - \Delta x_2 \frac{\Delta y_2 - \Delta y_1}{r_{12}}\right) \\ \times \left(\Delta y_2 \frac{\Delta x_3 - \Delta x_2}{r_{23}} - \Delta x_2 \frac{\Delta y_3 - \Delta y_2}{r_{23}}\right) \\ -\Delta z^2 \left(\frac{\xi_3 - \xi_2}{r_{23}} \frac{\xi_2 - \xi_1}{r_{12}} + \frac{\eta_3 - \eta_2}{r_{23}} \frac{\eta_2 - \eta_1}{r_{12}}\right) \end{array}\right]}$$

$$= \frac{|\Delta z|\, r_2 [(\eta_3 - \eta_2)(\xi_2 - \xi_1) - (\xi_3 - \xi_2)(\eta_2 - \eta_1)]}{\left\{\begin{array}{l} -[\Delta y_2(\Delta x_2 - \Delta x_1) - \Delta x_2(\Delta y_2 - \Delta y_1)] \\ \times \Delta y_2(\Delta x_3 - \Delta x_2) - \Delta x_2(\Delta y_3 - \Delta y_2)] \\ -\Delta z^2[(\xi_3 - \xi_2)(\xi_2 - \xi_1) + (\eta_3 - \eta_2)(\eta_2 - \eta_1)] \end{array}\right\}}$$

$$= \frac{|\Delta z|\, r_2 [\eta_1(\xi_3 - \eta_2) + \eta_2(\xi_1 - \xi_3) + \eta_3(\xi_2 - \eta_1)]}{\left\{\begin{array}{l} -(\Delta x_2 \,\Delta y_1 - \Delta x_1 \,\Delta y_2)(\Delta x_3 \,\Delta y_2 - \Delta x_2 \,\Delta y_3) \\ -\Delta z^2[(\xi_3 - \xi_2)(\xi_2 - \xi_1) + (\eta_3 - \eta_2)(\eta_2 - \eta_1)] \end{array}\right\}}$$

$$= \frac{|\Delta z|\, [\eta_1(\xi_3 - \xi_2) + \eta_2(\xi_1 - \xi_3) + \eta_3(\xi_2 - \xi_1)]}{-\{\det_{12}\det_{23} + \Delta z^2[(\xi_3 - \xi_2)(\xi_2 - \xi_1) + (\eta_3 - \eta_2)(\eta_2 - \eta_1)]\}/r_2}.$$

We use this expression in Section 3.2.2 of the paper. The positive distance r_2 has been moved from the numerator to the denominator for esthetic reasons. Doing so does not affect the quadrant returned by arctan.

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