NON INTEGRABILITY OF THE J_2 PROBLEM

MAYLIS IRIGOYEN
Université Paris 2, 92 rue d'Assas, 75006 Paris, France

and

CARLES SIMÓ
Dept. de Matemàtica Aplicada i Anàlisi, Univ. de Barcelona, Gran Via 585, 08007 Barcelona, Spain

(Received 17 March, 1992; accepted 17 August, 1992)

Abstract. We consider the motion of a massless particle around an oblate planet, keeping only in the expression of the perturbing potential the second degree zonal harmonic. We prove the analytical non integrability of this problem, by using Ziglin's theorem and the Yoshida criterion for homogeneous potentials.

Key words: Non integrability, second degree zonal harmonic.

1. Introduction

Consider the three degrees of freedom family of Hamiltonians defined by

$$\mathcal{H} = \tfrac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) - \frac{\mu}{r} + \frac{1}{r^3}\left(\alpha\left(\frac{z}{r}\right)^2 + \beta\right), \tag{1}$$

where $r^2 = x^2 + y^2 + z^2$, and α and β are real parameters with $\beta \neq 0$. We are interested, in particular, in the case: $\alpha = (3/2)a_e^2\,\mu\,J_2$ and $\beta = -(1/2)a_e^2\,\mu\,J_2$. This corresponds to the motion of an artificial satellite around an oblate planet, keeping only the J_2 term in the expansion of the perturbing potential in spherical harmonics. The coefficient μ is the gravitational coefficient ($\mu = GM$, where G is the gravitational constant of Newton and M is the mass of the planet) and a_e is the equatorial radius of the planet. It is immediate that J_2 is adimensional. The value for the Earth is close to 1.082×10^{-3}.

The numerical evidence seems to show that (1) is non integrable. However, the size of the stochastic zones in the case of the Earth and for reallistic orbits (i.e., $r > a_e$ for all time) is so small that the lack of integrability can be neglected for all practical purposes (Simó, 1991). Of course, the numerical experiments to visualize the non integrability are carried out, either for values of J_2 much larger than the one of the Earth, or allowing r to be quite small.

We shall prove here that (1), with $\alpha/\beta = -3$, has no other meromorphic integrals than the classical ones (energy and z component of the angular momentum).

2. The Spatial Problem with Fixed z Component of the Angular Momentum

We can write $\mathcal{H}$ in cylindrical coordinates $(\rho, \theta, z, p_\rho, p_\theta, p_z)$, and we obtain:

Celestial Mechanics and Dynamical Astronomy **55:** 281–287, 1993.

$$\mathcal{H} = \tfrac{1}{2}\left(p_\rho^2 + \frac{p_\theta^2}{\rho^2} + p_z^2\right) - \frac{\mu}{r} + \frac{1}{r^3}\left(\alpha\left(\frac{z}{r}\right)^2 + \beta\right), \tag{2}$$

where $r^2 = \rho^2 + z^2$.

The first integral, $p_\theta = c$, allows to split the couple (θ, p_θ) and to reduce (2) to a two degrees of freedom Hamiltonian, with reduced Hamiltonian:

$$\tilde{\mathcal{H}}(\rho, z, p_\rho, p_z) = \tfrac{1}{2}\left(p_\rho^2 + p_z^2\right) - \frac{\mu}{r} + \frac{c^2}{2\rho^2} + \frac{1}{r^3}\left(\alpha\left(\frac{z}{r}\right)^2 + \beta\right). \tag{3}$$

In the equatorial plane, $z = 0$, the reduced system has one degree of freedom, and the first integral, $\tilde{\mathcal{H}} = h$, defines the solutions, Γ_h, by:

$$p_\rho^2 = 2h + \frac{2\mu}{\rho} - \frac{c^2}{\rho^2} - \frac{2\beta}{\rho^3}, \tag{4}$$

where $\dot{\rho} = p_\rho$.

The change of time $\mathrm{d}t = \rho^2\,\mathrm{d}s$, shows that the solutions Γ_h are defined by Jacobi elliptic functions (Whittaker and Watson, 1927), $\rho(s)$, where s is a complex variable, because (4) can be written as:

$$s = \int \left(2h\rho^4 + 2\mu\rho^3 - c^2\rho^2 - 2\beta\rho\right)^{-1/2}\mathrm{d}\rho\,. \tag{5}$$

These solutions Γ_h are also solutions of the spatial problem defined by (3). We want to show that this system has not other meromorphic integral (independent of $\tilde{\mathcal{H}}$) in a neighbourhood of Γ_h. To this end, we shall apply Ziglin's theorem (Ziglin, 1983) and the Yoshida criterion concerning homogeneous potentials (Yoshida, 1987). First, we recall these results.

3. Ziglin's Theorem

We shall state this Theorem in the context of the Hamiltonian systems with two degrees of freedom. It gives necessary conditions to be satisfied by the solutions of the linearized equations along a family of particular solutions, assuming that the Hamiltonian system has a second meromorphic integral. The proof can be found in (Ziglin, 1983) or (Ito, 1985).

Theorem (Ziglin). Assume that a Hamiltonian system has a family of particular solutions, Γ_h, parametrized by elliptic functions of a complex time, and depending analytically on a real parameter $h \in (h_1, h_2)$. Let G be the monodromy group of the normal variational equation associated to the solution Γ_h. We say that $g \in G$ is non resonant if no one of its eigenvalues is a root of unity. If the Hamiltonian system has a meromorphic integral, F, functionally independent of $\mathcal{H}$, in a neighbourhood of Γ_h, and G contains a non resonant element, g_1, then, for any $g_2 \in G$, the commutator $g^* = g_2^{-1}\, g_1^{-1}\, g_2\, g_1$ satisfies: either $g^* = Id$, or $g^* = g_1^2$.

It is enough that these necessary conditions be not satisfied by a Hamiltonian system to ensure that is not analytically integrable. This happens, in particular, if one can find two different non resonant monodromy matrices, g_1 and g_2, such that they do not commute (Yoshida *et al.*, 1988). We shall apply this to the Hamiltonian (3).

4. Change of Scale

We perform the change of scale $(\rho, z, t) \mapsto (\varphi, \psi, \tau)$ defined by

$$\begin{cases} \rho = -(-h)^{-1/3}\, \beta^{1/3}\, \varphi\,, \\ z = -(-h)^{-1/3}\, \beta^{1/3}\, \psi\,, \\ t = 3^{-1/2}(-h)^{-5/6}\, \beta^{1/3}\, \tau\,, \end{cases} \tag{6}$$

always assuming, $h < 0$.

Let $\varepsilon = -(-h)^{-1/3}$. Then the Hamiltonian (3) becomes

$$K_\varepsilon(\varphi, \psi, p_\varphi, p_\psi) = -\tfrac{1}{6}\,(p_\varphi^2 + p_\psi^2) - \frac{\mu\varepsilon^2}{R\beta^{1/3}} + \frac{c^2\varepsilon}{2\varphi^2\,\beta^{2/3}} + \frac{1}{R^3}\left(1 + \frac{\alpha}{\beta}\left(\frac{\psi}{R}\right)^2\right), \tag{7}$$

where $R^2 = \varphi^2 + \psi^2$ and $p_\varphi = -3\,\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}$, $p_\psi = -3\,\frac{\mathrm{d}\psi}{\mathrm{d}\tau}$. The study of (3) on the level $\tilde{\mathcal{H}} = h$, is equivalent to the study of (7) on the level $K_\varepsilon = 1$.

The family of particular solutions Γ_h, defined by (4), becomes the family of solutions of (7) defined by

$$-\tfrac{3}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 = 1 - \frac{1}{\varphi^3} - \frac{c^2}{2\beta^{2/3}\,\varphi^2}\,\varepsilon + \frac{\mu}{\beta^{1/3}\,\varphi}\,\varepsilon^2\,. \tag{8}$$

The variational equations along Γ_h in the ρ and z directions are uncoupled. The normal variational equation, i.e., in the z direction, is given, in the original variables, by

$$\ddot{\zeta} + \left(\frac{\mu}{\rho^3} + \frac{2\alpha - 3\beta}{\rho^5}\right)\zeta = 0\,. \tag{9}$$

This normal variational equation can also be written in the variables φ, ψ, τ

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + \left(\left(1 - \frac{2\alpha}{3\beta}\right)\frac{1}{\varphi^5} - \frac{\mu}{3\beta^{1/3}\,\varphi^3}\,\varepsilon^2\right)\eta = 0\,, \tag{10}$$

where $\varphi = \varphi(\tau)$ is defined by (8).

We shall study the monodromy group of (10), first for the limit problem that one obtains by letting h tend to $-\infty$, and later for finite values of the energy.

5. Study of the Limit Problem: $h \to -\infty$

When $\varepsilon = -(-h)^{-1/3} \to 0$ we reach the limit Hamiltonian

$$K_0(\varphi, \psi, p_\varphi, p_\psi) = -\tfrac{1}{6}(p_\varphi^2 + p_\psi^2) + \frac{1}{R^3}\left(1 + \frac{\alpha}{\beta}\left(\frac{\psi}{R}\right)^2\right), \tag{7'}$$

with the first integral $K_0 = 1$.

The family of particular solutions Γ_h of (7) becomes $\Gamma_{-\infty}$, which are the solutions of (7′) defined by

$$-\tfrac{3}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 = 1 - \frac{1}{\varphi^3}. \tag{8'}$$

Along these solutions $\Gamma_{-\infty}$, the normal variational equation is written as

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + \left(1 - \frac{2\alpha}{3\beta}\right)\frac{\eta}{\varphi^5} = 0. \tag{10'}$$

In this way, we find exactly the problem, studied by Yoshida (Yoshida, 1987), defined by a Hamiltonian system with a homogeneous potential of degree $k = -3$.

For those Hamiltonians, Yoshida established a criterion of analytic non integrability (based on Ziglin's Theorem) which can be applied here. We describe it first.

Consider the Hamiltonian system defined by

$$\mathcal{H}(q_1, q_2, p_1, p_2) = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{11}$$

where V is a homogeneous function of degree $k \in \mathbb{Z}^*$. Then (11) admits solutions of the form $\vec{q} = \vec{c}\ \Phi(t)$, where $\Phi(t)$ is determined from the first integral

$$\frac{k}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)^2 = 1 - \Phi^k.$$

Along these particular solutions, the normal variational equation has the form $\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} + \lambda(\Phi(t))^{k-2}\eta = 0$.

6. Yoshida's Non Integrability Criterion

We only consider $k \le -3$ (this is enough for our purposes). If the coefficient λ belongs to

$$S_k = (1, \infty) \cup \left(\bigcup_{j \in \mathbb{N}} \left(-\frac{j(j+1)\,|k|}{2} + j + 1,\ \frac{-j(j-1)\,|k|}{2} - j + 1\right)\right), \tag{12}$$

then (11) has no analytic first integral independent of $\mathcal{H}$.

The proof of this criterion (cf. Yoshida, 1987; Nahon) consists, first, in the study of the family of particular solutions characterized by $\Phi(t)$. This function of $t \in C$ defines a Riemann surface, Γ. To every loop, γ, on Γ with base point w_0, we can associate the symplectic matrix g which accounts for the evolution of the fundamental solutions of the normal variational equation along γ. The matrix g depends only on the homotopy class of γ. The set of all these matrices forms the monodromy group, G, of the normal variational equation.

To study G, Yoshida remarks that the normal variational equation can be transformed, by means of the change of independent variable $z = (\Phi(t))^k$, into the Gauss hypergeometric equation

$$z(1-z)\,\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} + (c - (a+b+1)\,z)\,\frac{\mathrm{d}\eta}{\mathrm{d}z} - ab\eta = 0\,, \tag{13}$$

where $a + b = 1/2 - 1/k$, $ab = -\lambda/2k$, $c = 1 - 1/k$.

But the monodromy group of the Gauss equation (13) is explicitly known (cf. Hille, 1976; Ince, 1956; Plemelj, 1964; Churchill and Rod, 1988). This group is generated by two matrices, g_1 and g_2, which can be chosen in a suitable way (Yoshida, 1987):

$$g_1 = \begin{pmatrix} 1+\Omega AB & B(2+\Omega AB) \\ A(\Omega - 1 - \Omega AB) & 1 + (\Omega - 2)AB - \Omega(AB)^2 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 1 + (2\Omega - 1)AB - \Omega(AB)^2 & \Omega B(2 - AB) \\ A(1 - \Omega^{-1} - AB) & 1 - AB \end{pmatrix},$$

where $\Omega = e^{2\pi i/k}$, $A = 1 - \Omega^{-1}\,e^{-2\pi ai}$, $B = 1 - \Omega^{-1}\,e^{-2\pi bi}$.

These explicit expressions allow to establish the following result:

$$\{\lambda \in S_k\} \Longleftrightarrow \{\mathrm{tr}\,g_1 > 2,\ \mathrm{tr}\,g_2 > 2\} \Longrightarrow \{g_1 \text{ and } g_2 \text{ are non resonant and, in this case, they do not commute}\}.$$

Theorem 1. *If* $1 - \frac{2\alpha}{3\beta} \in S_{-3}$, *as given by (12)* ($S_{-3} = \ldots \cup (-39,-34) \cup (-25,-21) \cup (-14,-11) \cup (-6,-4) \cup (-1,0) \cup (1,\infty)$), *then the Hamiltonian system (1) in the limit case when* $h \to -\infty$, *has no analytic first integral independent of* $\mathcal{H}$ *and* p_θ.

Corollary. *In the* J_2 *case, as* $1 - \frac{2\alpha}{3\beta} = 3$, *the same result holds.*

Remark 1. The Theorem 1 follows from Yoshida criterion, with $k = -3$, applied to the limit problem $\varepsilon = h^{-1/3} = 0$. Physically this means that we are considering the motion near the center of the planet but keeping the expression of the potential. Indeed, from (4), if $h \to -\infty$, one has $\rho \to 0$ (we recall that $\beta < 0$ in the J_2

problem). To establish the corollary, the actual value of J_2 is irrelevant because only the quotient α/β is needed.

Remark 2. As noted in (Yoshida, 1987) the sufficient condition for non integrability is $\cos\left(\frac{\pi}{k}\left((k-2)^2+8\lambda k\right)^{1/2}\right) \neq -1 - \cos\frac{2\pi}{k} + \cos\left(2\pi\frac{p}{q}\right), \forall \frac{p}{q} \in Q$ $\left(\cos\left(\frac{\pi}{3}(25-24\lambda)^{1/2}\right) \neq -\frac{1}{2} + \cos\left(2\pi\frac{p}{q}\right), \forall \frac{p}{q} \in Q \text{ if } k=-3\right)$. This condition is not only satisfied in S_k but for all λ except a countable set.

7. Non Integrability for Finite Values of h

When $\varepsilon \neq 0$ it does not seem feasible to know explicitly the monodromy group of (10). However, one can show that Theorem 1 is still true for finite values of h. Hence the Corollary also holds.

Theorem 2. *There exists a value of the energy, $h_0 < 0$, such that for $h \in (-\infty, h_0)$ the Hamiltonian system (1) has no first independent of $\mathcal{H}$ and p_θ, and analytical, provided* $1 - \frac{2\alpha}{3\beta} \in S_{-3}$.

Proof. When we pass from $\varepsilon = 0$ to $\varepsilon \neq 0$, the family of particular solutions Γ_h is defined by (8) instead of (8′). The change $d\tau = \varphi^2 ds$ shows that, in both cases, the solutions are described by Jacobi elliptic functions. As the second member of (8) is analytic with respect to ε, the two simple poles of $\varphi(s)$ are as close as desired to the poles of the limit problem, provided $|h|$ is large enough. The fundamental group of the Riemann surface Γ_h is, then, the same as the one of $\Gamma_{-\infty}$.

Therefore, one can define, for $\varepsilon \neq 0$ small enough, the generators $g_1(\varepsilon)$ and $g_2(\varepsilon)$ of the monodromy group of the Equation (10) by using the same loops as used in the case $\varepsilon = 0$. As the coefficient of η in (10) is also analytic in ε along the loops, the matrices $g_1(\varepsilon)$ and $g_2(\varepsilon)$ are analytic in ε. Hence, if $1 - \frac{2\alpha}{3\beta} \in S_{-3}$, there exists $\varepsilon_0 < 0$ such that the conditions: $\mathrm{tr}\, g_1(\varepsilon) > 2$, $\mathrm{tr}\, g_2(\varepsilon) > 2$, and $g_1(\varepsilon)\, g_2(\varepsilon) - g_2(\varepsilon)\, g_1(\varepsilon) \neq 0$ hold if $\varepsilon \in (\varepsilon_0, 0)$. We can define $h_0 < 0$ such that $h_0 = \varepsilon_0^{-3}$, and the proof of Theorem 2 is complete. □

As a consequence, the J_2 problem has no other global analytic integral, valid for all levels of energy, beyond the total energy, p_θ and functions of that two ones.

Acknowledgements

The authors are indebted to A. Milani who suggested the application of Ziglin's method to the J_2 problem. The second author has been partially supported by a CICYT Grant number ESP91–403.

References

Churchill, R.C. and Rod, D.L.: 1988, 'Geometrical Aspects of Ziglin's Non-Integrability Theorem', *J. of Diff. Eq.* **76**, 91–114.

Hille, E.: 1976, *Ordinary Differential Equations in the Complex Domain*, J. Wiley and Sons, New York.

Ince, E.L.: 1956, *Ordinary Differential Equations*, Dover Publ., New York.

Ito, H.: 1985, 'Non-integrability of Hénon-Heiles system and a Theorem of Ziglin', *Kodai Math. J.* **8**, 120–138.

Nahon, F.: 'Le groupe de monodromie de l'équation de Hill et de l'équation de Gauss associée, unpublished notes, Univ. of Paris VI.

Plemelj, J.: 1964, *Problem in the Sense of Riemann and Klein*, Wiley-Interscience, New York.

Simó, C.: 1991, 'Measuring the Lack of Integrability of the J_2 Problem for Earth Satellites', in Roy, A.E. (ed.), *Predictability, Stability and Chaos in N-Body Dynamical Systems*, Plenum Press, pp. 305–309.

Whittaker, E.T. and Watson, G.N.: 1927, *A Course of Modern Analysis*, Cambridge Univ. Press.

Yoshida, H.: 1987, 'A Criterion for the Non-Existence of an Additional Integral in Hamiltonian Systems with a Homogeneous Potential', *Physica* **29D**, 128–142.

Yoshida, H., Ramani, A., and Grammaticos, B.: 1988, 'Non-Integrability of the Fourth-Order Truncated Toda Hamiltonian', *Physica* **30D**, 151–163.

Ziglin, S.L.: 1983, 'Branching of Solutions and Non-Existence of First Integrals in Hamiltonian Mechanics', *Funct. Anal. Appl.* **16**, 181–189, and **17**, 6–17.