ATTITUDE DYNAMICS OF A RIGID BODY ON A KEPLERIAN ORBIT: A SIMPLIFICATION

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Abstract. An infinitesimal contact transformation is proposed to simplify at first order the Hamiltonian representing the attitude of a triaxial rigid body on a Keplerian orbit around a mass point. The simplified problem reduces to the Euler-Poinsot model, but with moments of inertia depending on time through the longitude in orbit. Should the orbit be circular, the moments of inertia would be constant.

Key words: Attitude dynamics, Lie series, simplifications.

About the attitude of a rigid body revolving around another body, this note considers a simple case: the attracting body is taken to be a mass point, the center of mass of the rigid body is moving on a fixed Keplerian orbit; nevertheless, no assumption is made concerning the moments of inertia of the rigid body. In spite of these assumptions, the problem is not an elementary one: its general solution cannot be expressed in terms of elementary functions including the elliptic functions (Hughes, 1986).

Although it is well known that both motions, orbital and rotational, are coupled, under certain assumptions, mainly that the orbital distances are much bigger than the dimensions of the rigid body and that its spinning is much faster than the orbital mean motion, it is usual to neglect the gravitational coupling of attitude to orbit, and therefore, the orbit is prescribed prior to solving for attitude. In this context, we shall assume that the orbit is a fixed elliptic orbit. Then, according to Cochran (1972), the Hamiltonian is decomposed as the sum

$$\mathcal{H} = \mathcal{H}_E + \varepsilon\mathcal{H}_C,$$

where $\mathcal{H}_E$ stands for the Hamiltonian of a rigid body in free rotation, whereas $\mathcal{H}_C$ contains the coupled terms. The small parameter ε is the quotient of the orbital mean motion of the center of mass by a reference value of the rigid body's rotational angular velocity (Cochran, 1972, p. 128).

The Hamiltonian is formulated in Serret–Andoyer variables for the attitude motion. These variables $(\ell,\ g,\ h,\ L,\ G,\ H)$ are defined as usual (Serret, 1866; Andoyer, 1923). In this note, we use two angles δ and σ given by $\cos\delta = H/G$, $\cos\sigma = L/G$.

Without loss of generality, we shall assume that the principal moments of inertia of the rigid body (I_1, I_2, I_3) are in the relation $I_1 \leq I_2 \leq I_3$.

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The Hamiltonian of the Euler–Poinsot motion of a rigid body in torque free rotation ($\mathcal{H}_E$) in these variables is:

$$\mathcal{H}_E = \left(\frac{\sin^2 \ell}{2I_1} + \frac{\cos^2 \ell}{2I_2}\right)(G^2 - L^2) + \frac{1}{2I_3}L^2. \tag{1}$$

This problem is integrable, and ways of solving it are well known. We intend to show in some cases how a perturbed Euler–Poinsot problem can be readily reduced to the unperturbed model.

Since the orbit is suppossed to be Keplerian, we choose for fixed frame $Os_1s_2s_3$ the frame determined by the orbit and the orbital angular momentum (Os_3) with the axis Os_1 in the direction of the pericenter. This is why we may take for the perturbation $\mathcal{H}_C$ the developments in Fourier series in three arguments, namely ℓ, g and $f-h$ with coefficients that are functions of the canonical momenta given by Kinoshita (1972, Formulas 58 and 59), where now $I=0$.

Once the perturbation function is expanded in Fourier series, we proceed to construct an infinitesimal contact transformation (Whittaker, 1904)

$$(\ell, g, h, L, G, H) \longmapsto (\ell', g', h', L', G', H'),$$

so that the new Hamiltonian $\mathcal{H}' = \mathcal{H}'_E + \varepsilon\mathcal{H}'_C$ will be appropriate for our purposes.

The problem here considered is a particular case of the one considered by Kinoshita (1972). This author obtains a first order perturbation theory for the orbital rotational motion of a rigid body attracted by a mass point. By means of a canonical transformation he reduces the Hamiltonian to a new one depending only on the action variables. That is to say, he normalizes the Hamiltonian. The transformation we propose is not a normalization, in the sense that the new Hamiltonian still contains one angular variable (namely ℓ'), but belongs to the what recently has been named as *simplification* (Deprit and Miller, 1989).

The generating function $\mathcal{W}$ of the transformation and the first order $\mathcal{H}'_C$ in the transformed Hamiltonian must satisfy the partial differential identity

$$(\mathcal{W}, \mathcal{H}'_E) + \mathcal{H}'_C = \mathcal{H}_C, \tag{2}$$

where $(-,-)$ denotes the Poisson bracket, and $\mathcal{H}'_E$ is the unperturbed Hamiltonian $\mathcal{H}_E$ expressed in the new set of variables. To solve this PDE, we select for $\mathcal{H}'_C$ the average of $\mathcal{H}_C$ over the two angles g and h. Then (2) becomes a partial differential equation in the unknown $\mathcal{W}$.

To solve it, we limit ourselves to the regions in phase space where all orbits of the unperturbed problem are periodic, that is to say, to the connected domain containing a stable equilibrium (a rotation about the axis of greatest inertia or of smallest inertia). There the variables (ℓ, g, L) may be expressed explicitly as functions of the independent variable t, in terms of elliptic functions. Assuming that these expressions have been substituted into $\mathcal{H}'_E$, we recognize that

$$(\mathcal{W}, \mathcal{H}'_E) = \frac{d\mathcal{W}}{dt}, \qquad \text{and} \qquad \mathcal{W} = \int [\mathcal{H}_C - \mathcal{H}'_C]\, dt,$$

which means that the generator of the infinitesimal contact transformation can be obtained by quadratures. Of course, these quadratures are not elementary at all, since functions of Keplerian problem and elliptic functions are involved; ways in performing these quadratures are by expansions of the functions involved in Fourier series of t, or by means of specific algebraic packages which handle this kind of function.

Kinoshita chooses for the new order one the averaging of the disturbing function over all the angular variables, whereas in our case we average over all the variables except ℓ. This is why we could use his formulas to obtain the generating function of our transformation. The averaged Hamiltonian is

$$\mathcal{H}'_C = \frac{1}{16r^3}\Big\{ \quad (I_1 + I_2 - 2I_3)(1 - 3\cos^2\delta')(1 - 3\cos^2\sigma') \\ -3(I_1 - I_2)(1 - 3\cos^2\delta')\sin^2\sigma'\cos(2\ell')\Big\}. \tag{3}$$

For our purpose, since we are not interested at this moment in obtaining the equations of the transformation, but we are concerned solely in analyzing the qualitative behavior of the transformed Hamiltonian $\mathcal{H}'$, we do not need to deal with the very complicated problem of performing the quadrature along the solution of the unperturbed problem.
The relation among the old variables and the new ones is given by the equations

$$(\ell, g, h) = (\ell', g', h') + \frac{\partial \mathcal{W}}{\partial(L', G', H')}$$
$$(L, G, H) = (L', G', H') + \frac{\partial \mathcal{W}}{\partial(\ell', g', h')}.$$

It is worth recalling once again that the new Hamiltonian still contains the angular variable ℓ'; in other words, the Hamiltonian has not been normalized, but it is an *intermediary* of the original one.

The angles g', h' being cyclic, its conjugate momenta G', H' are integrals of the motion. Thus, the averaged first order is reduced to one degree of freedom, and therefore it is integrable.

It is not much the integrability character that makes this intermediary interesting, but the fact that it represents the rotation of a body with time–varying moments of inertia. A few elementary manipulations will make that point very clear. Indeed, after performing some elementary algebraic manipulations, and dropping those terms which do not depend explicitly on the variables ℓ' and L', the perturbation (3) may be expressed as:

$$\mathcal{H}_C = \frac{3}{8r^3}(1 - 3\cos^2\delta')\Big[(I_1 - I_2)\sin^2\ell'\sin^2\sigma' + (I_3 - I_2)\cos^2\sigma'\Big].$$

Keeping into account that $\sin^2\sigma' = (G'^2 - L'^2)/G'^2$ and that $\cos^2\sigma' = L'^2/G'^2$, the previous expression is equivalent to

$$\mathcal{H}_C = (A\sin^2\ell' + B\cos^2\ell')(G'^2 - L'^2) + CL'^2, \tag{4}$$

where

$$A = \frac{3}{8G'^2 r^3}(1 - 3\cos^2\delta')(I_1 - I_2),$$

$$B = 0,$$

$$C = \frac{3}{8G'^2 r^3}(1 - 3\cos^2\delta')(I_3 - I_2).$$

The expression (4) matches with the unperturbed Hamiltonian (1), and therefore, putting all terms together, the Hamiltonian $\mathcal{H}' = \mathcal{H}'_E + \varepsilon\mathcal{H}'_C$ is written, eventually, as:

$$\mathcal{H}' = \frac{1}{2}\left(\frac{\sin^2\ell'}{I_1^*} + \frac{\cos^2\ell'}{I_2^*}\right)(G'^2 - L'^2) + \frac{1}{2I_3^*}L'^2. \tag{5}$$

It has exactly the same form as the Hamiltonian of a rigid body in torque-free motion (1); the pseudo-moments of inertia I_i^* are

$$\frac{1}{I_1^*} = \frac{1}{I_1} + 2\varepsilon A, \qquad \frac{1}{I_2^*} = \frac{1}{I_2}, \qquad \frac{1}{I_3^*} = \frac{1}{I_3} + 2\varepsilon C. \tag{6}$$

It is worth noting that in spite of the appearance, the pseudo– moments of inertia I_i^* varies with time, because they contain the radius vector r. Time dependence of the moments of inertia is quite common when we consider attitude dynamics of deformable bodies, such as flexible platforms, satellites with damping or rotors, and, of course, rotation of the Earth, just to mention a few examples.

When the orbit is circular, the quantities A and B are constant, and of oposite signs, and therefore, the pseudo-moments of inertia I_i^* are time independent. In this case, the phase flow of the intermediary is identical to the phase flow of the unperturbed problem (1). The result of the perturbation on the unperturbed problem, is a slight increase (decrease), of the moments of inertia I_1, I_3 depending on the sign of $1 - 3\cos^2\delta'$, whereas the moment of inertia I_2 is not affected.

Let it be remarked also that, whatever the orbit would be, for the particular value of the inclination angle $\delta' = \arccos(1/\sqrt{3})(= 54\overset{\circ}{.}7356)$, the constant $(1 - 3\cos^2\delta') = 0$, and therefore, the pseudo–moments of inertia coincide with the original ones, and the perturbation disappears from the averaged Hamiltonian. The same value has been obtained by Chernousko (1972) when he considers a near-spherical rigid body moving in a Keplerian orbit.

Since we have not computed the generating function, we cannot afirm that it will not contain secular terms. Should this be the case, though, our intermediary would still be a valid approximation of the original system, but only for a small span of time. To be practical, let us say that the intermediary is worth considering, provided it is valid over the short intervals separating two maneuvers to control the attitude of a spacecraft.

Conclusions

After the problem of the attitude dynamics of a tri-axial rigid body moving in a Keplerian orbit has been formulated in Serret-Andoyer variables, and after it has been averaged over the variables g and h, we recognize that the intermediary Hamiltonian has exactly the same expression that the unperturbed Hamiltonian, i.e., a tri-axial rigid body in torque free motion, but with moments of inertia varying with the time.

This simplification is very important when the orbit is circular; indeed, the moments of inertia of the reduced Hamiltonian are constant and the solution may be obtained as in the classical Euler–Poinsot case.

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