LUNI-SOLAR EFFECTS OF GEOSYNCHRONOUS ORBITS AT THE CRITICAL INCLINATION

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Abstract. The luni-solar effects of a geosynchronous artificial satellite orbiting near the critical inclination is investigated. To tackle this four-degrees-of-freedom problem, a preliminary exploration separately analyzing each harmonic formed by a combination of the satellite longitude of the node and the Moon longitude of the node is opportune. This study demonstrates that the dynamics induced by these harmonics does not show resonance phenomena. In a second approach, the number of degrees of freedom is halved by averaging the total Hamiltonian over the two non-resonant angular variables. A semi-numerical method can now be applied as was done when considering solely the inhomogeneity of the geopotential (see Delhaise et Henrard, 1992). Approximate surfaces of section are constructed in the plane of the inclination and argument of perigee. The main effects of the Sun and Moon attractions compared to the terrestrial attraction alone are a strong increase in the amplitude of libration in inclination (from 0.6° to 3.2°) and a decrease of the corresponding libration period (from the order of 200 years to the order of 20 years).

Key words: artificial satellite theory– geosynchronous orbits– critical inclination–luni-solar effects– surface of section

1. Introduction

The published solutions of the main problem of artificial satellite theory (Brouwer 1959, Garfinkel 1959, Kozai 1959) contain an intrinsic singularity at the so-called critical inclination ($\cos^2 i = 1/5$). For these values of the inclination the secular J_2 effect in the argument of perigee is zero. The dynamics of this one-degree of freedom resonance problem has been extensively studied by many authors using various methods and variables. A detailed review of these investigations is given by Jupp (1988).

The perturbation effects on geosynchronous satellites orbiting near the critical inclination have however rarely been under investigation. In fact, this system requires a treatment totally different from the critical inclination problem. We deal with a double resonance problem: a mean motion commensurability between the satellite's mean motion and the Earth's rotation rate, combined with the critical inclination. In a preliminary study (Delhaise et Henrard, 1992), only the gravitational perturbations arising from an aspherical Earth were taken into consideration. The secular dynamics of this two-degrees-of-freedom system was explored in a global way. More specifically the 12-hr Molniya-type and the 24-hr Tundra-type orbits, both located at the critical inclination, were analyzed.

In this paper, the perturbation model is expanded through inclusion of the luni-solar gravitational effects. For the Molniya and Tundra type orbits which travel out to several Earth radii, the effects of the gravitational attractions of Sun and Moon are

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appreciable. The eccentricity can increase sufficiently to cause premature decay of the satellite by forcing the perigee down to the Earth surface. This will be illustrated for the case of the Molniya orbit.

First, the Hamiltonian of this system is developed. The lunar orbital elements are preferably referred with respect to an ecliptic reference system while the satellite's orbital elements are still related to the Earth equator. Obviously, taking into account the terrestrial together with the luni-solar perturbation effects compounds the difficulties of the problem. After averaging over the fast angular variables, the problem still has four degrees of freedom. Due to this complexity, the terrestrial and lunar effects are at first considered alone. A preliminary exploration separately analyzing each harmonic defined as a combination of the longitude of the ascending node of the satellite (x_4) and the longitude of the Moon (x_5), is appropriate for a subsequent simplification of the system.

In a second approach, the number of degrees of freedom is halved by averaging the total Hamiltonian over the two non-resonant angular variables x_4 and x_5. The resulting two-degrees-of-freedom Hamiltonian can then be studied by means of the semi-numerical method (Henrard, 1990). Approximate surfaces of section are constructed in the plane of the inclination and argument of perigee. Such figures yield a global view of the secular dynamics of such type of orbits. Finally, the solar effects are added to the Hamiltonian system.

2. Formulation of the Total Hamiltonian

The part of the Hamiltonian taking into account the oblateness of the Earth as well as the tesseral harmonics, is averaged over the mean anomaly of the satellite. Up to the second order in the second zonal harmonic J_2, it takes on the following form (see Brouwer, 1959 and Sochilina, 1982):

$$\begin{aligned}
\mathcal{H}_{\text{Earth}} &= -\frac{\mu^2}{2L^2} - \omega_e H \\
&- \frac{\mu^4}{L^6}\frac{R_e^2 J_2}{4}\left(3\frac{H^2}{G^2}-1\right)\left(\frac{L}{G}\right)^3 \\
&- \frac{\mu^6}{L^{10}}\frac{R_e^4 J_2^2}{4}\left[A(L,G,H)\cos 2\omega + C(L,G,H)\right] \\
&- \sum_{l,m,p,q} B_{l,m,p,q}(a,e,i) \begin{matrix}\cos\\ \sin\end{matrix}\{\theta_{l,m,p,q}\} \begin{matrix}\text{for } l-m \text{ even}\\ \text{for } l-m \text{ odd,}\end{matrix}
\end{aligned} \tag{1}$$

where μ is the product of the gravitational constant and the Earth's mass and

$$\theta_{l,m,p,q} = m\left(\frac{\alpha}{\beta}\ell + h - \lambda_{lm}\right) + (l-2p)\omega. \tag{2}$$

The angular variable $\frac{\alpha}{\beta}\ell + h$ is the critical argument associated to the resonance in mean motion where the satellite performs β revolutions while the Earth rotates

α times. The variable λ_{lm} is the longitude of major axis of symmetry of the (l, m) spherical harmonic. The set of variables $(\ell = M, g = \omega, h, L, G, H)$ is the usual set of the Delaunay variables with respect to a frame rotating with the Earth, which implies:

$$h = \Omega - \omega_e t, \tag{3}$$

where Ω is the right ascension of the ascending node in an inertial frame and t is the time. The coefficients A, B_{lmpq} and C are the same as in Delhaise et Henrard (1992). It is noteworthy that these coefficients are not expanded in powers of eccentricity or inclination permitting, for example, a very large value of the eccentricity. The term $\omega_e H$, where ω_e is the Earth's mean angular velocity, is added to the initial Hamiltonian in order to remove its explicit time dependence.

The indices of the summation l, m, p, q verify the following relation:

$$l - 2p + q = \frac{\alpha}{\beta} m, \tag{4}$$

so that the short periodic terms disappear of the Hamiltonian leaving only long period ones for commensurate or near-commensurate orbits. The main critical tesseral harmonics which are retained in this study are listed in Delhaise et Henrard (1992).

The units have been chosen such that the gravitational constant G, the mass of the Earth M_e and the mean equatorial radius of the Earth R_e are unity.

A third body gravitational force acting on the spacecraft can be described in a reference system defined at the Earth centre by the following potential:

$$V = \frac{-GM_k}{\rho_k} + GM_k \frac{\mathbf{r}_k \cdot \mathbf{r}}{r_k^3}. \tag{5}$$

The index k refers to the third body. The vectors $\mathbf{r}$ and $\mathbf{r}_k$ are the coordinates of the spacecraft and of the third body, respectively, $\boldsymbol{\rho}_k = \mathbf{r} - \mathbf{r}_k$, GM_k is the Newtonian gravitational constant times the third body's mass. It is worth noting that the accelerating terms due to the Earth rotation around the Sun are neglected in this study. Kaula (1962) expanded this function in terms of the equatorial Keplerian elements of the satellite and of the third body:

$$\begin{aligned} V = & -\mu_k \sum_{l=2}^{\infty} \frac{a^l}{a_k^{l+1}} \sum_{m=0}^{l} k_m \frac{(l-m)!}{(l+m)!} \sum_{p=0}^{l} F_{lmp}(i) \sum_{s=0}^{l} F_{lms}(i_k) \\ & \sum_{q=-\infty}^{\infty} H_{lpq}(e) \sum_{j=-\infty}^{\infty} G_{lsj}(e_k) \cos\left[(l-2p)\omega + (l-2p+q)M - \right. \\ & \left. - (l-2s)\,\omega_k - (l-2s+j)\,M_k + m\,(\Omega - \Omega_k)\right], \end{aligned} \tag{6}$$

where $k_o = 1$ and $k_m = 2$ for $m \neq 0$. The eccentricity functions H_{lpq} and G_{lsj} can be related to the Hansen function $X_k^{n,m}$ (see Giacaglia, 1976). The inclination functions $F_{lmp}(i)$ are given in Kaula (1966). The form (6) is convenient for inclusion of the solar effects with the Earth gravitational effects.

Kozai (1973) noted that it is more desirable to develop a theory relying on Keplerian ephemerides for the Moon referred to the ecliptic plane. This renders the Moon's inclination roughly constant and the lunar longitude of the ascending node can be approximated by a single linear function of time. The orbital elements of the satellite should still be referred to the Earth equator. Lane (1989) assumes an ecliptic reference system for the lunar ephemerides while the satellite elements are related to the Earth equator. After several geometrical rotations, the following form of the lunar potential is obtained:

$$
\begin{aligned}
V = & -\sum_{l=2}^{\infty}\sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=0}^{l}\sum_{j=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}(-1)^{k_1}\,\mu_k\,\frac{\varepsilon_m\,\varepsilon_s\,(l-s)!}{2\,a_k\,(l+m)!}\left(\frac{a}{a_k}\right)^l \\
& F_{lmp}(i)\,F_{lsq}(i_k)\,G_{lqr}(e_k)\,H_{lpj}(e) \\
& \Big\{(-1)^{k_2}\,U_l^{m,-s}(\varepsilon)\,\cos\left[\theta_{lmpj}\;+\;\theta'_{lsqr}\;-\;b_s\,\pi\right] \\
& \;(-1)^{k_3}\,U_l^{m,+s}(\varepsilon)\,\cos\left[\theta_{lmpj}\;-\;\theta'_{lsqr}\;-\;b_s\,\pi\right]\Big\}\,,
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
\varepsilon_m &= \begin{cases} 1 & \text{if } m=0, \\ 2 & \text{if } m\neq 0, \end{cases} \\
\varepsilon_s &= \begin{cases} 1 & \text{if } s=0, \\ 2 & \text{if } s\neq 0, \end{cases} \\
k_1 &= \Big\|\frac{m}{2}\Big\|, \text{ the largest integer part of } \frac{m}{2}, \\
k_2 &= t\,(m+s-1)+1, \\
k_3 &= t\,(m+s), \\
t &= 0 \text{ if } l-1 \text{ is even and } t=1 \text{ if } l-1 \text{ is odd}, \\
b_s &= 0 \text{ if } s \text{ is even and } b_s = 1/2 \text{ if } s \text{ is odd}, \\
\theta_{lmpj} &= (l-2p)\,\omega\;+\;(l-2p+j)\,M\;+\;m\,\Omega, && (8) \\
\theta'_{lsqr} &= (l-2q)\,\omega_k\;+\;(l-2q+r)\,M_k\;+\;s\,(\Omega_k-\pi/2), && (9)
\end{aligned}
$$

and

$$
\begin{aligned}
U_l^{m,s}(\varepsilon) = & (-1)^{m-s}\sum_{r=\max[0,-(m+s)]}^{\min[l-s,l-m]}(-1)^{l-m-r}\binom{l+m}{m+s+r}\binom{l-m}{r} \\
& \times\;\cos^{m+s+2r}\left(\frac{\varepsilon}{2}\right)\,\sin^{-m-s+2l-2r}\left(\frac{\varepsilon}{2}\right),
\end{aligned}
\tag{10}
$$

with $\varepsilon = 23.4^{\circ}\,27'\,08.26''$, the ecliptic inclination at Epoch J.D. 2415020.0. The total Hamiltonian is the result of the summation of the geopotential (1), the solar potential (6) and the lunar potential as given by (7).

3. Inclusion of the Moon's Gravitational Effects

Due to the complexity of the system the terrestrial and lunar gravitational effects are at first taken into account alone. The solar perturbations are included into the model subsequently. This study is here limited to the analysis of the Molniya-type orbits. Identical methods could of course be applied for the Tundra-type orbits too.

3.1. SIMPLIFYING ASSUMPTIONS

The following simplifications and assumptions are introduced in the approximate study of the secular dynamics of the Molniya-type orbits subjected to the terrestrial and lunar attractions.

1. The summation over the index "l" of the lunar potential (7) is truncated to include just the second harmonic (where the first is absent). We thus have:
$$l = 2. \tag{11}$$
2. The Hamiltonian is averaged over the satellite mean anomaly up to the first order. The indices (l, p, j) verify:
$$l - 2p + j = 0. \tag{12}$$
3. The Hamiltonian is averaged over the Moon mean anomaly up to the first order. The indices (l, q, r) verify:
$$l - 2q + r = 0. \tag{13}$$
4. The Moon's orbital eccentricity is assumed as zero so
$$l - 2q = 0. \tag{14}$$
Note that assuming the Moon orbit as circular has nearly the same effect as averaging the Hamiltonian over the Moon argument of perigee at the first order since
$$G_{210}(e_M) = \left(1 - e_M^2\right)^{-3/2} = 1.00438 \simeq G_{210}(0). \tag{15}$$

The part of the Hamiltonian taking into account the lunar effects is then reduced to the following form:

$$\begin{aligned}\mathcal{H}_M = & - \sum_{m=0}^{2} \sum_{s=0}^{2} \sum_{p=0}^{2} D_M(a, e, i, a_M, i_M) \\ & \left\{(-1)^{k_2}\, U_2^{m,-s}(\varepsilon) \cos\left[(2-2p)\omega + m\Omega + s(\Omega_M - \frac{\pi}{2}) - b_s\pi\right]\right. \\ & \left.(-1)^{k_3} U_2^{m,+s}(\varepsilon) \cos\left[(2-2p)\omega + m\Omega - s(\Omega_M - \frac{\pi}{2}) - b_s\pi\right]\right\}\end{aligned} \tag{16}$$

where

$$D_M = \mu_M (-1)^{k_1} \frac{\varepsilon_m\, \varepsilon_s\, (2-s)!}{2\, a_M\, (2+m)!} \left(\frac{a}{a_M}\right)^2 F_{2,m,p}(i)\, F_{2,s,1}(i_M)\, H_{2,p,2p-2}(e) \tag{17}$$

where the index M refers to the Moon's parameter.

3.2. PRELIMINARY EXPLORATION

We are confronted with a three-degrees-of-freedom problem, the Hamiltonian of which is time-dependent through the two angular variables $g_e = \omega_e t$ and Ω_M. These can be approximated by linear functions of time:

$$\begin{aligned} g_e(t) &= g_e(0) - 5.8834 \cdot 10^{-2}\, t, \\ \Omega_M(t) &= \Omega_M(0) - 8.6298 \cdot 10^{-6}\, t. \end{aligned} \tag{18}$$

The time unit is given with the units defined in section 2 and is of the order of $13^{\mathrm{m}}44^{\mathrm{sec}}$.

The phase space is extended by two additional degrees of freedom in order to render the Hamiltonian autonomous. The set of the canonical variables is defined by:

$$\begin{array}{lll} \ell = M & L & = \sqrt{\mu a}, \\ g = \omega & G & = L\sqrt{(1-e^2)}, \\ \Omega & H & = G \cos i, \\ g_e = \omega_e t & \Lambda_e, & \\ \Omega_M & \Lambda_M. & \end{array} \tag{19}$$

Note that the phase space is in fact restricted to four degrees of freedom inasmuch as the Hamiltonian is averaged over the satellite's mean anomaly. The frequency of Ω (of order of $2 \cdot 10^{-5}$ due to the J_2 secular effects) and especially Ω_M (equals to $8.6 \cdot 10^{-6}$) being relatively small, it is interesting to analyze the effect produced by each harmonic of the type $m\Omega + s\Omega_M$, for $m \in \{0, 2\}$ and for $s \in \{-2, 2\}$, taking also into account the dynamics stemming from (x_3, y_3), simultaneously. Following Delaunay's idea (1867), this study is performed sequentially for each value of the indices m and s. The phase space corresponding to such a harmonic could present two types of dynamics: a resonant or a non-resonant one. A resonant topology including libration and circulation zones separated by a critical curve would preclude all averaging over the above mentioned harmonics. In this case, a more sophisticated method like the successive elimination of perturbation harmonics (Morbidelli, (1991)) could be applied. This would still meet with considerable difficulties due to the high number of degrees of freedom. In the other case if the topology induced by each of the harmonics presents only circulation with a reasonable amplitude of variations, a classical averaging technique can be applied to rid the Hamiltonian of these harmonics up to the desired order.

To proceed to the sequential analysis of the harmonics, a canonical transformation is performed to obtain a suitable set of variables containing the two angular variables x_3 ($x_3 = \frac{\alpha}{\beta}\ell + h$) and x_4 ($x_4 = m\Omega + s\Omega_M$). Three different transformations are performed depending on the values taken on by the indices m and s:

1. **for $m \neq 0$ and $s \neq 0$:**

$$\begin{aligned} x_1 &= \ell & y_1 &= L - \tfrac{\alpha}{\beta} H + \tfrac{\alpha}{\beta}\tfrac{m}{s}\Lambda_M, \\ x_2 &= g & y_2 &= G, \\ x_3 &= \tfrac{\alpha}{\beta}\ell + \Omega - g_e & y_3 &= H - \tfrac{m}{s}\Lambda_M, \\ x_4 &= m\,\Omega + s\,\Omega_M & y_4 &= \tfrac{1}{s}\Lambda_M, \\ x_5 &= g_e & y_5 &= \Lambda_e + H - \tfrac{m}{s}\Lambda_M. \end{aligned} \tag{20}$$

2. **for $m = 0$ and $s \neq 0$:**

$$\begin{aligned} x_1 &= \ell & y_1 &= L - \tfrac{\alpha}{\beta} H, \\ x_2 &= g & y_2 &= G, \\ x_3 &= \tfrac{\alpha}{\beta}\ell + \Omega - g_e & y_3 &= H, \\ x_4 &= \Omega_M & y_4 &= \Lambda_M \\ x_5 &= g_e & y_5 &= \Lambda_e + H. \end{aligned} \tag{21}$$

3. **for $m \neq 0$ and $s = 0$:**

$$\begin{aligned} x_1 &= \ell & y_1 &= L - \tfrac{\alpha}{\beta}\Lambda_e, \\ x_2 &= g & y_2 &= G, \\ x_3 &= \tfrac{\alpha}{\beta}\ell + \Omega - g_e & y_3 &= -\Lambda_e, \\ x_4 &= \Omega & y_4 &= H + \Lambda_e \\ x_5 &= \Omega_M & y_5 &= \Lambda_M. \end{aligned} \tag{22}$$

The total Hamiltonian taking into account the terrestrial (see (1)) and lunar attractions (given in (16)), is expanded up to the quadratic terms in $y_i - y_i^*$ around their values at the equilibrium y_i^*. The values y_i^* are computed at the critical inclination ($i^* = 63.4^o$) and at the value of the semi-major axis (a^*) which corresponds to the resonance in mean motion, i.e. such that $dx_3/dt = 0$ at the first order in J_2. The value of the eccentricity is arbitrarily chosen as $e^* = 0.7222$ (see Delhaise et Henrard, 1992).

For the indices m and s fixed to a particular value, this Hamiltonian takes on the form:

$$\begin{aligned} \mathcal{H} = {} & \left[s\,\dot{\Omega}_M + m\,\dot{\Omega}\Big|_{J_2} \right] y_4 + \frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial y_2^2}\, y_2^2 + \frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial y_3^2}\, y_3^2 \\ & + \frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial y_4^2}\, y_4^2 + \frac{\partial^2\mathcal{H}}{\partial y_2 y_3}\, y_2\, y_3 + \frac{\partial^2\mathcal{H}}{\partial y_2 y_4}\, y_2\, y_4 + \frac{\partial^2\mathcal{H}}{\partial y_3 y_4}\, y_3\, y_4 \\ & - \sum_{l,r,p,q} B_{lrpq}(a^*, e^*, i^*) \begin{matrix}\cos \\ \sin\end{matrix} \{ r(x_3 - \lambda_{lr}) + (l-2p)x_2 \} \begin{matrix}\text{for } l-r \text{ even} \\ \text{for } l-r \text{ odd,}\end{matrix} \\ & - \frac{\mu^6}{L^{*10}} \frac{R_e^4 J_2^2}{4} A(y_i^*) \cos 2x_2 \\ & - \sum_p D^* \, (-1)^{k_i}\, U_2^{m,s}(\varepsilon) \cos\left[(2-2p)x_2 + x_4 + s\frac{\pi}{2} - b_s\pi \right], \end{aligned} \tag{23}$$

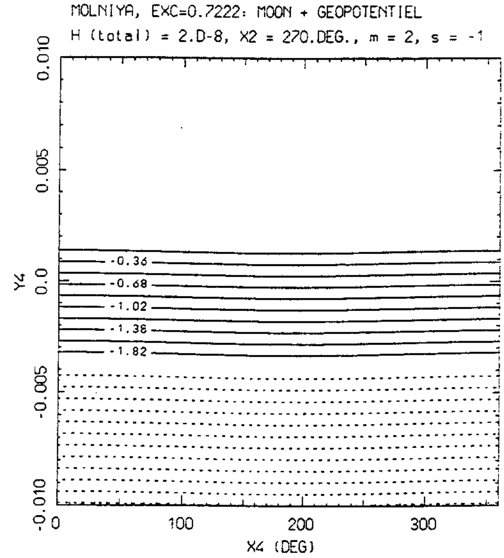

Fig. 1. Level curves of the pseudo-integral $\overline{J}$ illustrating the dynamics of the Molniya orbit in the plane (x_4, y_4) issued from the specific harmonics x_4 defined by $m = 2$ and $s = -1$. The full lines represent the cases when the point (x_3, y_3) is in libration mode and the dotted lines when it is in circulation mode. The results obtained for the harmonics: (m, s) = (1,1), (1,-2), (1,2), (2,-2), (2,1), (2,2) are similar to those represented here.

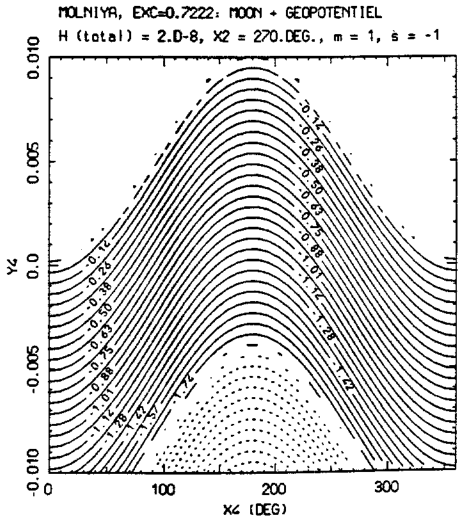

Fig. 2. Level curves of the pseudo-integral $\overline{J}$ illustrating the dynamics of the Molniya orbit in the plane (x_4, y_4) for the harmonics x_4 defined by $m = 1$ and $s = -1$. The results obtained for the harmonics: $m = 0$ and $s = 0$ are similar to those represented here.

where all partial derivatives are evaluated at $y_i = y_i^*$. Furthermore to shorten the notation, $y_i - y_i^*$ is simply written as y_i. The Hamiltonian can be decomposed into three parts as follows:

$$H_o(x_2, x_3, y_3, y_4) + H_1(x_2, x_4) + H_1'(y_2, y_3, y_4), \tag{24}$$

such that the semi-numerical method can be applied to $H_0 + H_1$ in the same way as performed in Delhaise et Henrard (1992). The terms of $H_1{}'$ containing y_2 are not considered in this preliminary approach to enable an examination of the dynamics of the two-degrees-of-freedom problem $H_0 + H_1$ in the field $< x_4, y_4 >$ with x_2 taken as parameter. The perturbation method applied here takes into account the full distortion of the invariant tori of the separable Hamiltonian H_0 which is resonant, instead of expanding it around the equilibrium point as is the case for a local analysis. This aim is achieved by introducing numerically a set of convenient action angle variables (J, ψ) in the three different topological regions of the phase space of the separable Hamiltonian H_0. The transformation from (x_3, y_3) to (J, ψ) has to be extended to a two-degrees of freedom canonical transformation in order to compute H_1 in the new set of variables. The total Hamiltonian is then averaged at the first order over the fast angular variable ψ. A supplementary quasi-integral results from this averaging. Indeed $\bar{J}$ now remains constant up to the first order. The action angle variables (J, ψ) and the pseudo-integral $\bar{J}$ are computed in a semi-numerical way (see Henrard, 1990): a grid of points is defined in the plane (x_4, y_4) while $x_3 = x_3^*$ defines the Poincaré surface of section and y_3 is computed such that $H_0 + H_1 = \text{const.}$. The level curves of the pseudo-integral $\bar{J}$ denote the trajectories on such Poincaré section. They are computed for the case of the Molniya-type orbits and are represented in the plane (x_4, y_4) in Figs (1) and (2) for two different harmonics x_4. The values of the constant energy level and of the parameter x_2 equal $2 \cdot 10^{-8}$ and 270^o, respectively. From these figures, it can be deduced that the harmonics producing the most significant variations on y_4 are given by $(m, s) = (1, -1)$, $m = 0$ and $s = 0$. All other harmonics generate practically no variations over their conjugated momentum y_4.

The main conclusion emanating from this preliminary approach is that no libration zone is induced by any harmonic $m\Omega + s\Omega_M$. The amplitudes of variations of their corresponding conjugated momentum are non-negligible but not excessively large. This justifies further simplifications of the model by averaging over the two non-resonant angular variables Ω and Ω_M. The averaging process will be done up to the second order to take into account the effects of these harmonics as far as possible.

3.3. REDUCTION TO A TWO DEGREES OF FREEDOM SYSTEM

In this section, the secular dynamics of the Molniya-type orbits subjecting terrestrial and lunar gravitational perturbations is studied in a global way. The Hamiltonian will be averaged over the two non-resonant variables Ω and Ω_M. The dimension of the phase space is thus reduced from 8 to 4. Approximate surfaces of section are constructed in the plane of the inclination and argument of perigee. The qualitative and some quantitative aspects illustrating the global secular dynamics are deduced from these figures.

The following set of canonical variables is convenient for this study:

$$\begin{aligned} x_1 &= \ell & y_1 &= L + \tfrac{\alpha}{\beta}\Lambda_e, \\ x_2 &= g & y_2 &= G, \\ x_3 &= \tfrac{\alpha}{\beta}\ell + \Omega - g_e & y_3 &= -\Lambda_e, \\ x_4 &= \Omega & y_4 &= H + \Lambda_e, \\ x_5 &= \Omega_M & y_5 &= \Lambda_M. \end{aligned} \tag{25}$$

The relations describing the dependence of the set (a, e, i) with respect to the variables (y_1, y_2, y_3) are the following:

$$\begin{aligned} a &= \frac{1}{\mu}\left(y_1 + \frac{\alpha}{\beta}y_3\right)^2, \\ e &= \sqrt{1 - \frac{y_2^2}{\left(y_1 + \frac{\alpha}{\beta}y_3\right)^2}}, \\ i &= \arccos\left(\frac{y_3 + y_4}{y_2}\right). \end{aligned} \tag{26}$$

In this study the full Hamiltonian $\mathcal{H}(x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5)$, taking into account all harmonics is expanded into a Taylor series over the variables y_i around y_i^* up to the third order for the non-trigonometric terms and up to the first and second orders for the trigonometric terms derived from the inhomogeneity of the geopotential and from the Moon's gravitation, respectively. The higher degree of accuracy compared to the study considering solely the inhomogeneity of the geopotential (see Delhaise et Henrard, 1992) is necessary due to the larger considered interval in inclination and consequently in y_2. The selected extremal inclination values are:

$$\begin{aligned} i_{\text{min}} &= 59.835^o, \\ \text{and} \quad i_{\text{max}} &= 67.035^o. \end{aligned} \tag{27}$$

In this case, the maximum difference between y_2 and y_2^* equals 0.180. The corresponding interval in e_{mean} is:

$$\begin{aligned} e_{\text{min}} &\simeq 0.609, \\ e_{\text{max}} &\simeq 0.788. \end{aligned} \tag{28}$$

Note that the value of e_{mean} has the same meaning as that given in Delhaise et Henrard (1992):

$$e_{\text{mean}} = \sqrt{1 - \frac{(y_3^*/\cos i)^2}{\left(y_1 + \frac{\alpha}{\beta} y_3^*\right)^2}}. \tag{29}$$

The form of the expanded full Hamiltonian is given in Delhaise (1992, p. 138). This Hamiltonian has four degrees of freedom. To halve the number of degrees of freedom of the phase space, the Hamiltonian is averaged over the two non-resonant variables x_4 and x_5. This course of action is supported by the results obtained in the preliminary analysis. As was demonstrated in section 5.3, the variations of the corresponding conjugated momentum y_4 produced by certain harmonics of the type $m\Omega + s\Omega_M$ are quite significant. For this reason the Hamiltonian is averaged up to the second order. The technique of Lie transforms is applied. The coefficients y_i are small quantities (of order of 10^{-1} for y_2 and 10^{-2} for the others). The coefficients of the trigonometric terms are of the order of y_i^2 or below. It is then natural to organize the inputs of the Lie triangle as follows:

– linear terms in y_i in $H_0^{(0)}$.
– quadratic terms in y_i and trigonometric terms independent of y_i in $H_1^{(0)}$.
– cubic terms in y_i and trigonometric terms dependent on y_i in $H_2^{(0)}$.

However, in our particular case the coefficient of y_3^2 is so large that the magnitude of this quadratic term is about that of the linear terms in y_4 and y_5. That is the reason why the libration period in x_3 turns out to be the same order as the circulation period in y_4 and y_5. Therefore, the quadratic term in y_3^2 must be included in $H_0^{(0)}$. In the summation of the terms of the kind $B_{i,j}\cos(ix_2 + jx_3)$, the term independent of x_2 is retained in $H_0^{(0)}$, which thus takes the form:

$$H_0^{(0)} = \omega_4 y_4 + \omega_5 y_5 + \frac{1}{2}\frac{\partial^2 \mathcal{H}}{\partial y_3^2} y_3^2 + B_{0,j}\cos(jx_3) \tag{30}$$

The Lie algorithm to average the Hamiltonian over x_4 and x_5 can not be directly applied with such a choice of $H_0^{(0)}$. To overcome this problem, suitable action angle variables (φ_3, I_3) are implicitly introduced by the canonical variable transformation:

$$\begin{aligned} x_3 &= X_3(\varphi_3, I_3), \\ y_3 &= Y_3(\varphi_3, I_3), \end{aligned} \tag{31}$$

So that the term $\frac{1}{2}\frac{\partial^2 \mathcal{H}}{\partial y_3^2} y_3^2 + B_{0,j}\cos(jx_3)$ can be rewritten as $K_3(I_3)$. Note that the functions X_3 and Y_3 are not known explicitly. Furthermore, the function K_3 is expanded locally around I_3^*, thus giving

$$K_3^* = \omega_3 \hat{I}_3 + \alpha \hat{I}_3^2 + \beta \hat{I}_3^3 + \dots, \tag{32}$$

where $\hat{I}_3 = I_3 - I_3^*$ is considered small. Following this transformation, the inputs of the Lie triangle may now be written as:

$$
\begin{aligned}
H_0^{(0)} &= \omega_4\, y_4 \,+\, \omega_5\, y_5 \,+\, \omega_3\, \hat{I}_3, \\
H_1^{(0)} &= \alpha\, \hat{I}_3^2 + \frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial y_2^2}\, y_2^2 + \frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial y_4^2}\, y_4^2 + \frac{\partial^2\mathcal{H}}{\partial y_2 y_3}\, y_2\, Y_3(\varphi_3, I_3) + \frac{\partial^2\mathcal{H}}{\partial y_2 y_4}\, y_2\, y_4 \\
&\quad + \frac{\partial^2\mathcal{H}}{\partial y_3 y_4}\, Y_3(\varphi_3, I_3)\, y_4 - \sum_{lmpq} B_{lmpq} \cos\theta_{lmpq} + A\, \cos 2x_2 \qquad (33) \\
&\quad - \sum_{mps} D_M\, \cos\left((2-2p)x_2 \,+\, m\, x_4 \,+\, s\,(x_5 - \frac{\pi}{2}) - b_s\pi\right), \\
H_2^{(0)} &= \text{the remaining terms of the Hamiltonian with } y_3 \text{ replaced by} \\
&\quad\; Y_3(\varphi_3, I_3) \text{ and } x_3 \text{ by } X_3(\varphi_3, I_3).
\end{aligned}
$$

With these settings, the Lie algorithm can be applied, in principle, up to an arbitrary order. The computations are performed here up to the second order to compute the mean values: $H_0^{(1)}$ and $H_0^{(2)}$ and the generating function W_1. This is feasible even if the functions X_3, Y_3 and K_3 are not given explicitly. The calculation of the Lie algorithm beyond the second order requires the explicit knowledge of x_3 and y_3 as function of the action angle variables (φ_3, I_3).

The averaged Hamiltonian turns out to be a two degrees of freedom one:

$$\overline{\mathcal{H}} \;=\; \overline{H}_o(\overline{x}_2, \overline{x}_3, \overline{y}_3, \overline{y}_4) \;+\; \overline{H}_1(\overline{x}_2, \overline{y}_2, \overline{x}_3, \overline{y}_3, \overline{y}_4). \qquad (34)$$

The semi-numerical method (Henrard, 1990) can thus be applied. The technique computing the level curves of the pseudo-integral $\overline{J}$ denoting the trajectories in the field of the inclination and argument of perigee for a constant energy level is the same as described in Delhaise et Henrard (1992). The surface of section is defined by $x_3 = x_3^*$ verifying:

$$\frac{dy_3}{dt} \;=\; \frac{-\partial \overline{H}_o}{\partial x_3} \;=\; 0, \qquad (35)$$

with x_2 fixed at its initial value. The results illustrating the global secular dynamics of the Molniya type orbits are displayed through Figures (3) and (4) for energy levels equal: 0 and $-3 \cdot 10^{-7}$, respectively. The continuous lines represent the cases where the angle x_3 librates, the dotted lines the cases where it circulates. This allows to locate the separatrix curve for the resonance of the first degree of freedom in the graphs. As will be verified by numerical integration, chaotic motion can be expected in the regions where the level curves cross this critical curve. The contributions of the Moon's attraction on the orbit evolution can be deduced from these figures. The lunar effects strongly increase the amplitude of libration

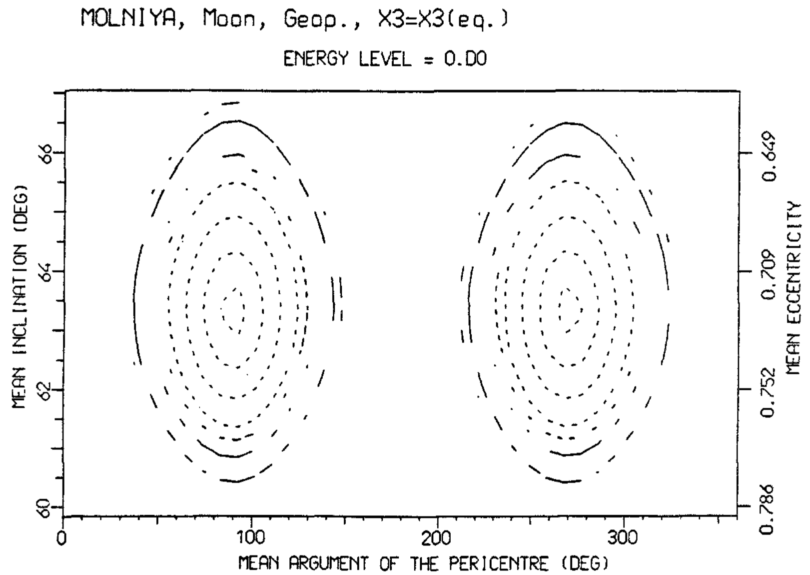

Fig. 3. Level curves of the pseudo-integral $\overline{J}$ for $\mathcal{H} = 0$ for the Molniya orbit. Terrestrial and lunar effects are taken into account. The continuous lines represent the cases where the angle x_3 librates, the dotted lines the cases where it circulates. Here, the continuous lines are "cut in pieces" since we are either close to the separatrix or close to the stable equilibrium point in the y_3, x_3 degree of freedom, which are both singularities for our action angle variables.

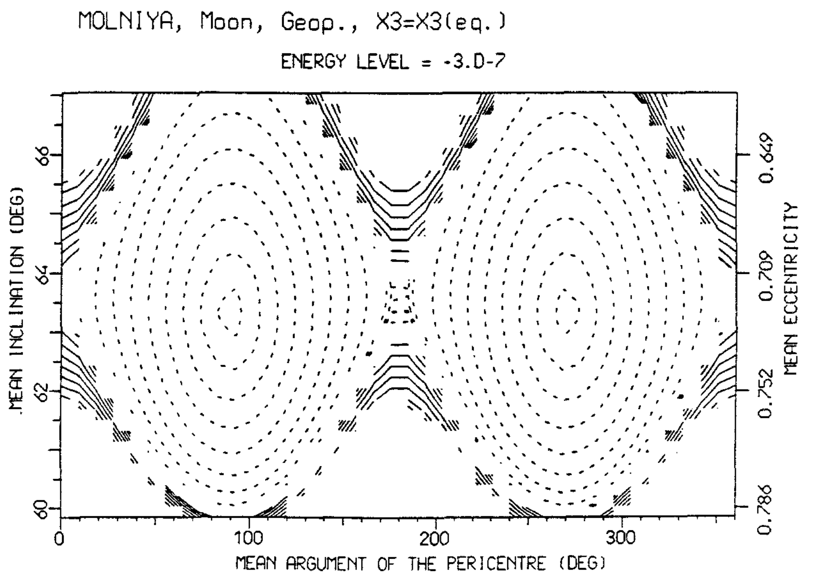

Fig. 4. Level curves of the pseudo-integral $\overline{J}$ for $\mathcal{H} = -3 \cdot 10^{-7}$ for the Molniya orbit. Terrestrial and lunar effects are taken into account.

in inclination, which reaches a value as high as 3.2° while the geopotential effects alone cause a maximum libration amplitude of around 0.6°. Another significant effect of the Moon's gravitation is the decrease of the libration period in (x_2, y_2) (numerically computed). This ranges between 25 and 50 years but attains more than 200 years when solely the inhomogeneity of the geopotential is considered. The value of the critical inclination is slightly different when the lunar gravitational perturbations are added to the model. Figures like (4) clearly show that this value is a function of x_2. The equilibrium value in inclination is maximal (63.8°) for $x_2 = 0°$ and 180°, the unstable equilibria, and minimal (63.2°) for $x_2 = 90°$ and 270°, the stable equilibria.

In conclusion, including the Moon's gravitational effects to the model boils down to adding a term of the form $\alpha \cos kx_2$ with a factor α approximately 10 times larger than when considering only the effects of the geopotential.

4. Inclusion of the Sun's Gravitational Effects

In this section the solar effects are included in the model. Therefore, all dominant gravitational effects from the Earth, Sun and Moon are taken into consideration.

4.1. SIMPLIFYING ASSUMPTIONS

The parameters of the apparent Sun orbit related to the Earth equator are given in Appendix 1. The form of the Hamiltonian taking into account the solar perturbations is given in equation (6). The following simplifications are introduced in this part of the Hamiltonian:

1. The summation over the index "l" of the solar potential (7) is truncated to include just the second harmonic.
$$l = 2. \tag{36}$$
2. The Hamiltonian is averaged over the satellite and Earth mean anomaly up to the first order. Therefore the indices (l, p, q) and (l, s, j) verify:
$$l - 2p + q = 0,$$
$$l - 2s + j = 0. \tag{37}$$
3. The longitude of the ascending node Ω_s is assumed as invariant over the time interval of interest.
4. The Earth orbit is assumed being circular, so
$$l - 2s = 0. \tag{38}$$

4.2. RESULTS

Assuming Ω_s constant, no supplementary variable has to be added to the set of variables given in (25). The solar part of the Hamiltonian is also averaged over the satellite longitude of the ascending node x_4 up to the first order.

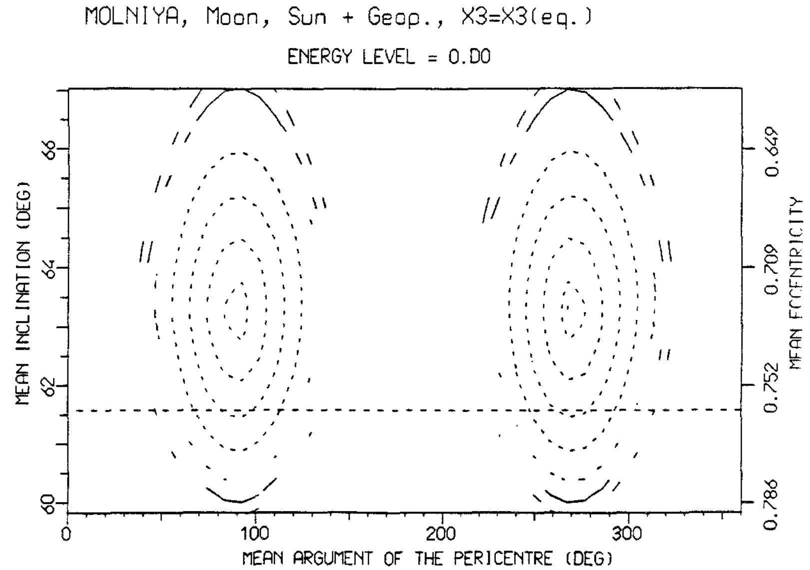

Fig. 5. Level curves of the pseudo-integral $\overline{J}$ for $\mathcal{H} = 0$ for the Molniya orbit. Terrestrial and luni-solar effects are taking into account. The continuous lines represent the cases where the angle x_3 librates, the dotted lines the cases where it circulates.

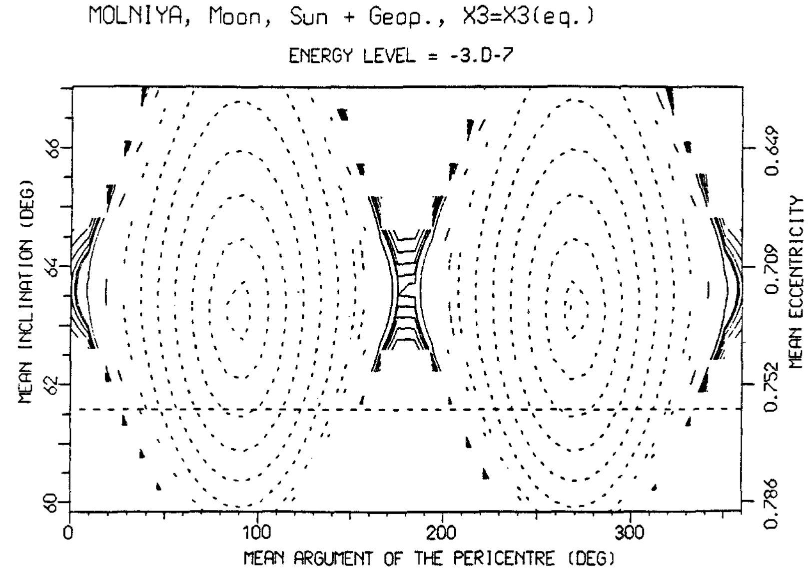

Fig. 6. Level curves of the pseudo-integral $\overline{J}$ for $\mathcal{H} = -3 \cdot 10^{-7}$ for the Molniya orbit. Terrestrial and luni-solar effects are taking into account.

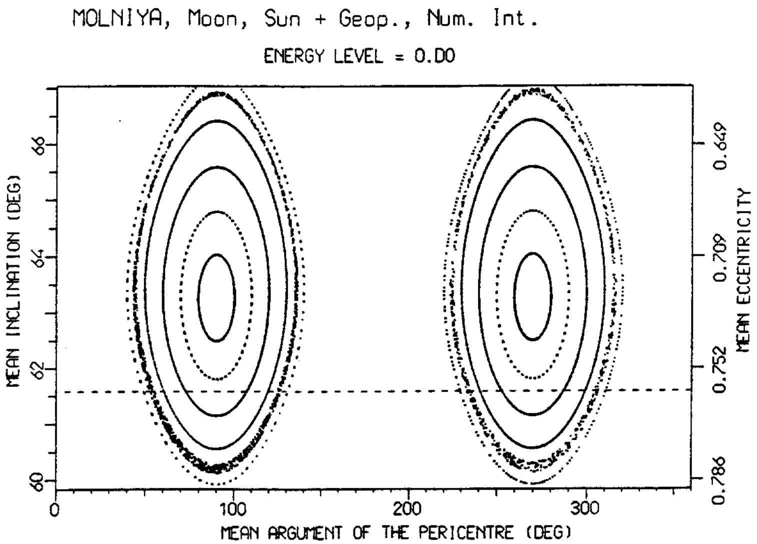

Fig. 7. Mapping resulting of the numerical integration of the Hamiltonian taking into account the terrestrial and luni-solar effects on a Molniya-type orbit. The energy level equals 0.

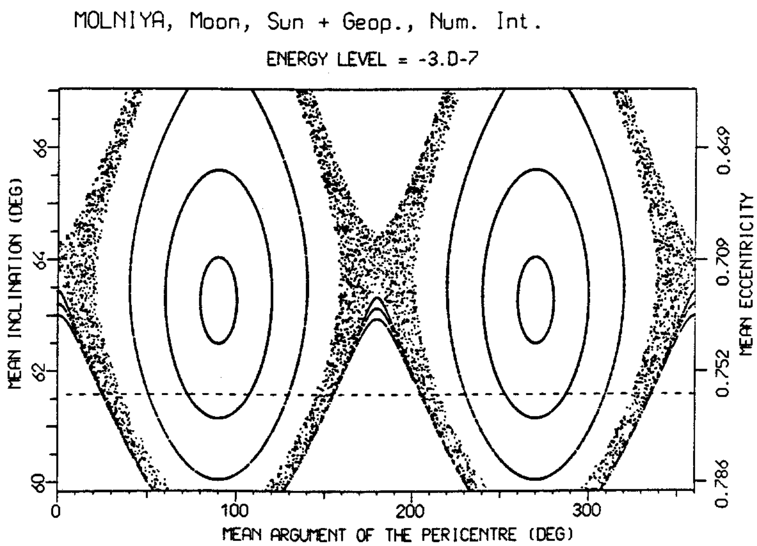

Fig. 8. Mapping resulting of the numerical integration of the Hamiltonian taking into account the terrestrial and luni-solar effects on a Molniya-type orbit. The energy level equals $-3 \cdot 10^{-7}$.

The contribution of the Sun to the total Hamiltonian is of the form:

$$\mathcal{H}_S = -\frac{\mu}{a_s} \sum_{p=o}^{2} \left(\frac{a}{a_s}\right)^2 F_{2,o,p}(i)\, F_{2,0,1}(i_s)\, H_{2,p,2p-l}(e)\, \cos[(2-2p)\,x_2]. \quad (39)$$

Each term of (39) is expanded around y_i^* up to the first order to obtain the following formulation:

$$[S_1 + S_2\, y_2 + S_3 y_3] \cos 2x_2 + S. \quad (40)$$

The semi-numerical method is again applied to yield the results of Figures (5) and (6) for the constant energy levels: 0 and $-3 \cdot 10^{-7}$, respectively. The Sun gravitation further increases the libration amplitude and shortens the libration period in (x_2, i). Close to the equilibrium point, the latter is reduced to about 20 years. The straight dashed line in Figures (5) and (6) represents the approximate limit where the eccentricity becomes so high that the perigee altitude descends below 100 km. The luni-solar effects are thus strong enough to reduce the satellite lifetime for particular initial conditions. Note that for a Tundra-type orbit, the initial perigee altitude being about 25000 km, there is no possibility of the luni-solar effects inducing a premature orbit decay.

5. Comparison with the Numerical Integrations

These results are confirmed by means of numerical integration of the differential equations derived from the averaged total Hamiltonian. The tests are performed for the constant energy levels 0 (see Fig. (7)) and $-3 \cdot 10^{-7}$ (see Fig. (8)). Comparing Figs (5) and (6) with Figs (7) and (8), we note that the results obtained by means of the semi-numerical method agree with those deriving of the numerical integration. As expected large chaotic regions appear along the separatrix curve of the resonance in (x_3, y_3).

6. Conclusion

The problem of a geosynchronous artificial satellite orbiting near the critical inclination is examined. In a preliminary study (Delhaise et Henrard, 1992), only the effects of the inhomogeneity of the geopotential were taken into consideration. This lead to a two-degrees-of-freedom problem. The perturbation model is now expanded through inclusion of the quite significant luni-solar gravitational effects. Obviously, this compounds the difficulties of the problem. Essentially, two additional degrees of freedom are introduced into the system. To tackle this four-degrees-of-freedom problem, a preliminary exploration separately analysing each harmonic stemming from the lunar potential is opportune. The considered harmonics are formed by a combination of the longitude of the ascending node of the satellite (x_4) and the longitude of the Moon (x_5). This study demonstrates that the topology induced by these harmonics does not show resonance phenomena (i.e. libration and circulation zones separated by a so-called critical curve). In all cases the dynamics remains that of a circulating angular variable. Three specific harmonics produce the most significant variations on their respective conjugated momenta; the others are of negligible influence. In a second approach, the number of degrees of freedom is halved by averaging the total Hamiltonian over the two non-resonant angular variables x_4 and x_5. As was demonstrated before, the variations produced by certain combinations of these two angles are significant. Therefore, the Hamiltonian is averaged up to the second order by the technique of Lie transforms. A semi-numerical method can now be applied as was done when considering solely the inhomogeneity of the geopotential. Approximate surfaces of section are constructed in the plane of the inclination and argument of perigee. The main effects of the Sun and Moon attractions compared to the terrestrial attraction alone are a strong increase in the amplitude of libration in inclination (from 0.6^o to 3.2^o) and a decrease of the corresponding libration period (from the order of 200 years to the order of 20 years).

Acknowledgements

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Appendix

A. Moon and Sun's parameters

The Moon's orbital parameters at Epoch J.D. 2415020.0 and referred to the ecliptic plane are the following:

$$\begin{aligned}
a_M &= 384399\,\text{km}, \\
e_M &= 0.0549, \\
i_M &= 5^o 8' 43.427'', \\
\dot{\Omega}_M &\simeq -0.05295^o/\text{days}, \\
& \quad \text{period} \simeq 18.6 \text{ years}, \\
\dot{\omega}_M &\simeq 0.1114^o/\text{days}, \\
& \quad \text{period} \simeq 8.85 \text{ years}, \\
\dot{M}_M &\simeq 12.8^o/\text{days}, \\
& \quad \text{period} \simeq 28 \text{ days}.
\end{aligned} \tag{41}$$

The parameters of the apparent Sun orbit related to the Earth equator are given by:

$$\begin{aligned} a_s &= 1495979 \cdot 10^2 \,\text{km}, \\ e_s &= 0.01673, \\ i_s &= 23.4^\circ 27' 08''.26 \\ \dot{\Omega}_s &\simeq 18''.8/\,\text{year}, \\ \dot{M}_s &\simeq 1.01^\circ/\,\text{days}, \\ &\quad \text{period} \simeq 1 \text{ year}. \end{aligned} \tag{42}$$

The motion of the argument of perigee with time is strongly nonlinear.

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