LINK BETWEEN THE RETROGRADE-PROGRADE NUTATIONS AND NUTATIONS IN OBLIQUITY AND LONGITUDE

P. DEFRAIGNE, V. DEHANT and P. PÂQUET
Royal Observatory of Belgium, Avenue Circulaire 3, B-1180 Brussels, Belgium

(Received: 7 December 1994; accepted: 26 July 1995)

Abstract. The nutations of the Earth can be seen as variations in longitude and in obliquity of the position of the Celestial Ephemeris Pole (CEP); these variations are given for each nutation frequency, and correspond to an elliptical motion of the CEP in the inertial space. Therefore the nutations can also be expressed as the sum of two circular motions of opposite direction (one prograde and one retrograde) with the same frequency. In the literature, the nutations are given in one or the other representation. Because the conventions are not always the same, we review here the mathematical expressions of both descriptions and we give the way to get one from the other.

Key words: Nutations, precession, non-rigid earth, reference frames.

1. Introduction

Due to the luni-solar attraction (and eventually the attraction of other planets) a torque is applied on the Earth which tends to rock the equator towards the ecliptic. Due to the Earth's rotation, this torque produces a 'precessional' motion of the Earth in space: the precession. The periodic changes of the relative positions of the Moon, Earth, and Sun (and eventually the other planets) imply additional, inertial, periodic motions of the Earth, called nutations.

The nutation coefficients for a deformable Earth are computed from the rigid Earth nutation amplitudes by convolution with the relative values of the amplitudes with respect to the rigid Earth results, at each nutation component. In the literature these relative values are known as the 'B_{ratio}' (Wahr, 1979, 1981); they express the ratios in the frequency domain between the nutation amplitude for a deformable Earth and for a rigid Earth. Rigid Earth theoretical nutations as well as observed nutations (obtained from VLBI and LLR observations) are expressed as variations in longitude and obliquity of the position of the Celestial Ephemeris Pole (CEP); taken together, these variations form a sum of elliptical motions at periods ranging from a few days to a few years in the inertial space. They result from the combination of two circular motions, one in the prograde direction and one in the retrograde direction, at the same period. The ratios as given by Wahr (1979, 1981), are corresponding to the normalized amplitudes of these circular motions. It is thus necessary to make the link between those two different ways of expressing nutations. Because in the literature it is very hard to find the conventions used by the different authors, the purpose of this paper is to clarify the mathematical expressions for both representations.

Let us consider a diurnal tidal forcing acting on the Earth at a retrograde (opposite to the Earth's rotation) quasi-diurnal frequency λ_j in the Earth's reference system. Let us write $\lambda_j = -\omega_j$ where ω_j is positive. The Earth will have a nutation at a frequency $\lambda'_j = \lambda_j + \Omega = -\Delta\omega_j = -\omega_j + \Omega$ in the inertial space where Ω is the Earth's rotation frequency (prograde). With this frequency λ_j, another frequency which is symmetrical with respect to the -1 $cycle/sidereal\ day$ (K_1) frequency is associated. With the frequency λ'_j (or $-\Delta\omega_j$), a frequency $-\lambda'_j$ (or $\Delta\omega_j$), symmetrical with respect to 0 $cycle/sidereal\ day$ (precession), is associated. So, a tidal frequency ω_j which is higher (respectively lower) than the Earth rotation frequency gives rise to a retrograde (respectively prograde) circular nutation in the inertial space.

Some authors work with the angular velocity $\Delta\omega_j$ as nutation frequency, i.e. with the frequency introduced here above and which is positive in a sense opposite to the prograde Earth's rotation; consequently, in that case, a positive frequency corresponds to a retrograde nutation and a negative frequency corresponds to a prograde nutation. If we really want to note that the angular velocities are in radians per time unit and knowing that positive angles are prograde, then the Earth's rotation rate must be $+\Omega$ if Ω is positive as before. Furthermore, the frequency in radians per time unit in the inertial space is $\lambda'_j = -\Delta\omega_j$. Positive λ'_j (negative $\Delta\omega_j$) correspond to prograde nutations and negative λ'_j (positive $\Delta\omega_j$) to retrograde nutations. Note that a prograde frequency does not automatically correspond to a linear combination of the Doodson variables (like in tide generating potential developments, see e.g. Roosbeek, 1995) or the Delaunay variables (like in rigid nutation developments, see e.g. Kinoshita and Souchay, 1990) with positive coefficients because one of the variables is the longitude of the Moon's nood which is retrograde. In this paper we shall work either with λ'_j, and thus take the convention that positive frequencies correspond to prograde nutations and negative frequencies to retrograde nutations, or with $\Delta\omega_j$ and thus take the opposite convention.

At each frequency (either prograde or retrograde), one can compute the product of a B_{ratio} and a *rigid Earth amplitude*; the combination of a couple of these products for opposite sign frequencies gives nutations in obliquity and in longitude. These nutations can also be retrieved from the prograde and retrograde nutations as we shall see.

2. Expression of the nutations in longitude and obliquity in function of the tidal forcing

The expression of the diurnal tidal forcing (as the tesseral part of the tidal potential $W_{21}R_{21}(\theta,\lambda) + \tilde{W}_{21}S_{21}(\theta,\lambda)$, where $R_{21}(\theta,\lambda) = P_{21}(\cos\theta)\cos m\lambda$ and $S_{21}(\theta,\lambda) = P_{21}(\cos\theta)\sin m\lambda$, and where P_{21} is the associated Legendre polynomial) in function of the tide generating potential (TGP) is given by

$$W_{21} = K_2 \sum_{j=0}^{\infty} A_{21j} \sin(\omega_j t + \alpha_j) \tag{1}$$

$$\tilde{W}_{21} = K_2 \sum_{j=0}^{\infty} A_{21j} \cos(\omega_j t + \alpha_j) \tag{2}$$

where $K_2 = \frac{2}{3}\frac{D}{a^2}$, and where D is the tidal Doodson constant (noted G in Melchior and Georis, 1968) and which is different for the Moon, the Sun and each of the planets; a is the mean radius of the Earth. The complex sum of the two first components of the volumetric external torque acting on Earth and associated to the diurnal tidal forcing can be expressed as:

$$\Gamma = \Gamma_1 + i\Gamma_2 \tag{3}$$

$$= \frac{3i\alpha A}{r^2}(W_{21} + i\tilde{W}_{21}) \tag{4}$$

$$= -\Omega^2 C E_{M-S} \left(\sum_{j=0}^{\infty} A_{21j}\, \mathrm{e}^{-i\omega_j t} + \sum_{j=0}^{\infty} A_{21-j}\, \mathrm{e}^{-i\omega_{-j} t} \right), \tag{5}$$

where $E_{M-S} = \frac{3\alpha A}{C\Omega a^2} K_2$; A and C are the principal inertia moments, Ω is the mean rotation, and α is the dynamical flattening of the Earth ($\alpha = (C - A)/A$). The position of the instantaneous rotation axis ($m = m_1 + im_2$) in a terrestrial frame is then given by the solution of Euler equations:

$$A\Omega\dot{m} - i(C - A)\Omega m = \Gamma. \tag{6}$$

This equation leads to

$$m = -\frac{i\Omega C}{A} E_{M-S} \left(\sum_{j=0}^{\infty} \frac{A_{21j}}{\frac{C}{A}\Omega + \Delta\omega_j} \mathrm{e}^{-i\omega_j t} + \sum_{j=0}^{\infty} \frac{A_{21-j}}{\frac{C}{A}\Omega + \Delta\omega_{-j}} \mathrm{e}^{-i\omega_{-j} t} \right). \tag{7}$$

From this expression, we can obtain the nutation of the Celestial Ephemeris Pole (CEP) by integration with respect to the time (Capitaine, 1982):

$$\Delta\epsilon_{\text{CEP}} + i \sin\epsilon_0 \Delta\psi'_{CEP} = -\int_{t_0}^{t} \Omega m\, \mathrm{e}^{i\Omega\tau}\, \mathrm{d}\tau \tag{8}$$

where $\Delta\psi'$ is the classical Euler angle (positive = prograde) for the precession, it is thus retrograde with time; the conventional definition of nutation in longitude corresponds to $\Delta\psi = -\Delta\psi'$ because nutations are considered to be added to the precessional angle. Substituting equation (7) in equation (8) leads to

$$\Delta\epsilon_{\mathrm{CEP}} + i\sin\epsilon_0\Delta\psi'_{\mathrm{CEP}} = -\frac{\Omega^2 C}{A}E_{M-S}\left(\sum_{j=0}^{\infty}\frac{A_{21j}}{\Delta\omega_j}\frac{1}{\frac{C}{A}\Omega+\Delta\omega_j}\,\mathrm{e}^{-i(\Delta\omega_j t+\beta_j)} + \sum_{j=0}^{\infty}\frac{A_{21-j}}{\Delta\omega_j}\frac{1}{\frac{C}{A}\Omega-\Delta\omega_j}\,\mathrm{e}^{i(\Delta\omega_j t+\beta_j)}\right), \quad (9)$$

using the fact that $\Delta\omega_{-j} = -\Delta\omega_j$, and $\beta_{-j} = -\beta_j$ which are the phases related to the initial conditions. This shows that to one tidal frequency ω_j (or λ_j) corresponds one nutation frequency $\Delta\omega_j$ (or λ'_j). These nutations, as derived from the external torque Γ, correspond to rigid Earth nutations. Nevertheless, the expression (9) can be extended for a non-rigid Earth (see Capitaine, 1982). If we note A_j the amplitude of the nutations for a non-rigid Earth, equation (9) is replaced by:

$$\Delta\epsilon_{\mathrm{CEP}} + i\sin\epsilon_0\Delta\psi'_{\mathrm{CEP}} = \sum_{j=0}^{\infty}\left(A_j\,\mathrm{e}^{-i(\Delta\omega_j t+\beta_j)} - A_{-j}\,\mathrm{e}^{i(\Delta\omega_j t+\beta_j)}\right) \quad (10)$$

$$= \sum_{j=0}^{\infty}\left(A_j\,\mathrm{e}^{i(\lambda'_j t+\phi_j)} - A_{-j}\,\mathrm{e}^{i(\lambda'_j t+\phi_j)}\right), \quad (11)$$

where $\phi_j = -\beta_j$. In the rigid case, the amplitudes A_j and A_{-j} can be related to the amplitudes of the tidal forcing by:

$$A_j = \frac{A_{21j}}{\lambda'_j}E_{M-S}\frac{\Omega^2 C}{A}\frac{1}{\frac{C}{A}\Omega-\lambda'_j} = -\frac{A_{21j}}{\Delta\omega_j}E_{M-S}\frac{\Omega^2 C}{A}\frac{1}{\frac{C}{A}\Omega+\Delta\omega_j}, \quad (12)$$

$$A_{-j} = \frac{A_{21-j}}{\lambda'_j}E_{M-S}\frac{\Omega^2 C}{A}\frac{1}{\frac{C}{A}\Omega+\lambda'_j} = -\frac{A_{21-j}}{\Delta\omega_j}E_{M-S}\frac{\Omega^2 C}{A}\frac{1}{\frac{C}{A}\Omega-\Delta\omega_j}. \quad (13)$$

From equations (10) and (11), we deduce the variations in longitude and in obliquity of the position of the CEP:

$$\Delta\epsilon_{\mathrm{CEP}} = \sum_{j=0}^{\infty}(A_j - A_{-j})\cos(\lambda'_j t+\phi_j) = \sum_{j=0}^{\infty}(A_j - A_{-j})\cos(\Delta\omega_j t+\beta_j) \quad (14)$$

$$\sin\epsilon_0\Delta\psi'_{\mathrm{CEP}} = \sum_{j=0}^{\infty}(A_j + A_{-j})\sin(\lambda'_j t+\phi_j)$$

$$= -\sum_{j=0}^{\infty}(A_j + A_{-j})\sin(\Delta\omega_j t+\beta_j). \quad (15)$$

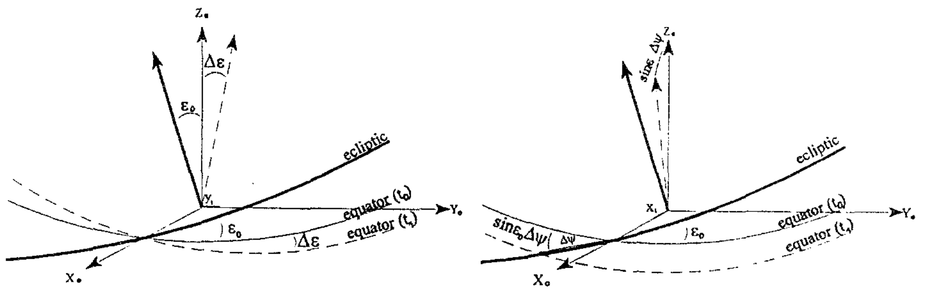

Fig. 1. Projections in the (x, y) plane of the new position of the CEP. Z_0 is the initial position and $\Delta\epsilon$ and $\sin\epsilon_0\Delta\psi$ are the variations in obliquity and in longtiude. We have $y_1 \simeq \Delta\epsilon$ and $x_1 \simeq \sin\epsilon_0\Delta\psi$.

These expressions show that the nutations of the Tisserand axis correspond to elliptical motions of the Tisserand pole (i.e. pole of the CEP) in the inertial space. The semi-minor and semi-major axes of these ellipses are given by $(A_j - A_{-j})$ and $(A_j + A_{-j})$, and the normal to the ellipses is directed towards the ecliptic pole.

3. Position of the CEP

The celestial ephemeris pole (CEP) position which determines the conventional nutation axis (it corresponds to the Tisserand axis) has a motion in space which can be expressed from the motion of the z-axis of the terrestrial frame: (see figure 1)

$$x_{\text{CEP}} = -\sin\epsilon_0\,\Delta\psi' = \sin\epsilon_0\Delta\psi \tag{16}$$

$$y_{\text{CEP}} = \Delta\epsilon \tag{17}$$

or

$$x_{\text{CEP}} + iy_{\text{CEP}} = \sin\epsilon_0\,\Delta\psi + i\Delta\epsilon. \tag{18}$$

From equations (14), (15), equation (18) gives:

$$x_{\text{CEP}} + iy_{\text{CEP}} = i\sum_{j=0}^{\infty}(A_j\,\mathrm{e}^{-i(\Delta\omega_j t+\beta_j)} - A_{-j}\,\mathrm{e}^{i(\Delta\omega_j t+\beta_j)}) \tag{19}$$

$$= i\sum_{j=0}^{\infty}(A_j\,\mathrm{e}^{i(\lambda'_j t+\phi_j)} - A_{-j}\,\mathrm{e}^{-i(\lambda'_j t+\phi_j)}). \tag{20}$$

The motion of the CEP in space is thus the sum of two spherical motions, one prograde and one retrograde, of the same angular velocity. The result is thus an elliptical motion at that frequency.

Wahr in his thesis (1979) works also with the retrograde and prograde nutations. He computes the toroidal mean displacement $\overline{\tau}_1^1$ at the surface of the Earth. It is easy to show that, if we express this displacement in a classical cartesian terrestrial coordinate system, we obtain :

$$\begin{aligned}\overline{\tau}_1^1 &= [\delta a(\hat{x} + i\hat{y})\, \mathrm{e}^{i(\omega_j t+\beta_j)}] \wedge \overline{r} \\ &= \left[\frac{i}{2\sqrt{2}}\,\frac{W_1^1}{r}\,(\hat{x} + i\hat{y})\, \mathrm{e}^{i(\omega_j t+\beta_j)}\right] \wedge \overline{r},\end{aligned} \tag{21}$$

where W_1^1 is the toroidal tangential displacement as defined by Smith (1974) or in Wahr's work (1979, 1981), $\hat{x}$ and $\hat{y}$ correspond to the unit director vectors of the terrestrial frame ($\hat{x}$ gives the direction of the equinox in the equatorial plane) and $\overline{r}$ corresponds to the initial position. The displacement $\overline{\tau}_1^1$ is then exactly a rotation around an axis in the equatorial plane, thus a nutation of the whole Earth. This is true for ω_j and ω_{-j}. The total effect due to this couple of frequencies, symmetrical around the frequency of 1 $cycle/sidereal\ day$, can be expressed in the terrestrial reference frame by :

$$[\delta a^+\, \mathrm{e}^{i(\omega_j t+\beta_j^+)} + \delta a^-\, \mathrm{e}^{i(\omega_{-j} t+\beta_j^-)}](\hat{x} + i\hat{y}) \wedge \overline{r}. \tag{22}$$

For $\beta_j^+ = \beta_j$ and $\beta_j^- = -\beta_j$, this gives in the inertial reference frame:

$$[\delta a^+\, \mathrm{e}^{i(\Delta\omega_j t+\beta_j)} + \delta a^-\, \mathrm{e}^{-i(\Delta\omega_j t+\beta_j)}](\hat{x} + i\hat{y}) \wedge \overline{r}. \tag{23}$$

In particular, this is true for the $\hat{z}$ axis, which points along the time averaged rotation vector $\overline{\Omega}$. So, for $\overline{r} = (0\ 0\ 1)^t$, and for a total toroidal displacement taken for the mean outer surface and free from body tide effects, we take the real part of (23) and get the position of the CEP:

$$-(\delta a^+ - \delta a^-)\sin(\Delta\omega_j t + \beta_j)\hat{x} - (\delta a^+ + \delta a^-)\cos(\Delta\omega_j t + \beta_j)\hat{y}. \tag{24}$$

This expression is given for positive $\Delta\omega_j$, so, if we allow the use of a negative frequency too, we have to write

$$\begin{aligned}&-\frac{\Delta\omega_j}{|\Delta\omega_j|}(\delta a^+ - \delta a^-)\sin(|\Delta\omega_j|\, t + \gamma_j)\hat{x} \\ &\quad - (\delta a^+ + \delta a^-)\cos(|\Delta\omega_j|\, t + \gamma_j)\hat{y},\end{aligned} \tag{25}$$

where $\gamma_j = \beta_j$ if $\Delta\omega_j$ is positive, and $\gamma_j = -\beta_j$ if $\Delta\omega_j$ is negative. If we now use the international convention that the nutation in longitude is measured positively in the retrograde direction (they are added to the precession angle), we then work with ψ rather than with the classical Euler angle $\psi'(= -\psi)$ used until now. By comparing (16) and (17) to (24) or (25), this yields:

$$\begin{aligned}\Delta\epsilon_{\rm CEP} &= -\left(\delta a^+ + \delta a^-\right)\cos(\Delta\omega_j t + \beta_j) \quad (26)\\ &= \Delta\epsilon\cos(\Delta\omega_j t + \beta_j), \quad (27)\end{aligned}$$

$$\begin{aligned}\sin\epsilon_0\Delta\psi_{\rm CEP} &= -(\delta a^+ - \delta a^-)\sin(\Delta\omega_j t + \beta_j) \quad (28)\\ &= \sin\epsilon_0\Delta\psi\sin(\Delta\omega_j t + \beta_j). \quad (29)\end{aligned}$$

From equations (14), (15), (26) and (28), one finds that:

$$\begin{aligned}A_j &= -\delta a^-\\ A_{-j} &= \delta a^+. \quad (30)\end{aligned}$$

So, we can express the nutation in obliquity and in longitude as the sum of two circular motions, one in the prograde and one in the retrograde directions:

$$\begin{aligned}&\Delta\epsilon_{\rm CEP} + i\sin\epsilon_0\Delta\psi_{\rm CEP}\\ &= -\sum_{j=0}^{\infty}\left(\delta a^+\ {\rm e}^{i(\Delta\omega_j t+\beta_j)} + \delta a^-\ {\rm e}^{-i(\Delta\omega_j t+\beta_j)}\right) \quad (31)\\ &= -\sum_{j=0}^{\infty}\left(\delta a^+\ {\rm e}^{-i(\lambda'_j t+\phi_j)} + \delta a^-\ {\rm e}^{i(\lambda'_j t+\phi_j)}\right), \quad (32)\end{aligned}$$

where for positive $\Delta\omega_j$ (negative λ'_j), δa^+ is the retrograde amplitude and δa^- is the prograde amplitude, and for negative $\Delta\omega_j$ (positive λ'_j), δa^+ is the prograde amplitude and δa^- is the retrograde amplitude. We then can write:

$$\begin{aligned}&\Delta\epsilon_{\rm CEP} + i\sin\epsilon_0\Delta\psi_{\rm CEP}\\ &= -\sum_{j=0}^{\infty}\left(A^{\rm retro}\ {\rm e}^{i(|\lambda'_j|t+\gamma_j)} + A^{\rm pro}\ {\rm e}^{-i(|\lambda'_j|t+\gamma_j)}\right) \quad (33)\\ &= -\sum_{j=0}^{\infty}\left(A^{\rm retro}\ {\rm e}^{i(|\Delta\omega_j|t+\gamma_j)} + A^{\rm pro}\ {\rm e}^{-i(|\Delta\omega_j|t+\gamma_j)}\right). \quad (34)\end{aligned}$$

Note that it could seem strange that a retrograde amplitude is in front of a counterclockwise (i.e. prograde) rotation, but in this conventional expression of the nutations, the retrograde amplitude ($A^{\rm retro}$) corresponds in fact to a retrograde motion of the CEP in space. The same remarks applies for prograde amplitude.

In practice, we do not compute directly the amplitudes of the nutations but rather the transfer functions i.e. coefficients in the frequency domain between the nutational response of a deformable Earth and of a rigid Earth. This will endup in what Wahr calls the 'B_{ratio}'. If δa_0 (noted η_0 in Wahr, 1979) is the amplitude of the rigid nutation and δa (noted η_s in Wahr, 1979), the amplitude of the nutations for a deformable Earth, the B_{ratio}'s are defined by:

$$B_{\text{ratio}} = \frac{\eta_s - \eta_0}{\eta_0} = \frac{\delta a - \delta a_0}{\delta a_0}. \tag{35}$$

In the frequency domain, (26) and (28) become:

$$\begin{aligned} \Delta\epsilon &= -(\delta a^+ + \delta a^-) \\ &= -(B_{\text{ratio}+}\delta a_0^+ + B_{\text{ratio}-}\delta a_0^-) - (\delta a_0^+ + \delta a_0^-), \end{aligned} \tag{36}$$

$$\begin{aligned} \sin\epsilon_0\Delta\psi &= -(\delta a^+ - \delta a^-) \\ &= -(B_{\text{ratio}+}\delta a_0^+ - B_{\text{ratio}-}\delta a_0^-) - (\delta a_0^+ - \delta a_0^-), \end{aligned} \tag{37}$$

where $B_{\text{ratio}+}$ and $B_{\text{ratio}-}$ are those given in the tables of Wahr's Ph. D. Thesis (1979, see also Wahr, 1981). The amplitudes of the rigid nutations δa_0^+ and δa_0^- can be found from the rigid nutation amplitudes in obliquity and in longitude by:

$$\delta a_0^+ = -\tfrac{1}{2}(\Delta\epsilon_0 + \sin\epsilon_0\Delta\psi_0), \tag{38}$$

$$\delta a_0^- = -\tfrac{1}{2}(\Delta\epsilon_0 - \sin\epsilon_0\Delta\psi_0). \tag{39}$$

This would give, combining (35), (36) and (37), (38) and (39):

$$\Delta\epsilon = B_{\text{ratio}+}(\tfrac{1}{2})(\Delta\epsilon_0 + \sin\epsilon_0\Delta\psi_0) + B_{\text{ratio}-}(\tfrac{1}{2})(\Delta\epsilon_0 - \sin\epsilon_0\Delta\psi_0) + \Delta\epsilon_0, \tag{40}$$

$$\begin{aligned} \sin\epsilon_0\Delta\psi &= B_{\text{ratio}+}(\tfrac{1}{2})(\Delta\epsilon_0 + \sin\epsilon_0\Delta\psi_0) \\ &\quad -B_{\text{ratio}-}(\tfrac{1}{2})(\Delta\epsilon_0 - \sin\epsilon_0\Delta\psi_0) + \sin\epsilon_0\Delta\psi_0. \end{aligned} \tag{41}$$

The nutations as defined from the toroidal mean surface displacement are nutations of an axis which has no diurnal rotation in a uniformly rotating frame. These are nutations of the CEP and are directly related to the Tisserand axis B.

4. In-phase and out-of-phase components

Let us now imagine there is some dissipation which causes phases to the prograde and retrograde spherical components of the nutations:

$$\Delta\epsilon + i\sin\epsilon_0\Delta\psi = -\left(\sum_{j=0}^{\infty} A^{\text{retro}}\, \mathrm{e}^{i(|\lambda'_j|t+\gamma_j+\alpha_j)} + \sum_{j=0}^{\infty} A^{\text{pro}}\, \mathrm{e}^{i(-(|\lambda'_j|t+\gamma_j)+\alpha_{-j})}\right) \tag{42}$$

or at the first order, supposing that these phases are small,

$$\Delta\epsilon + i\sin\epsilon_0\Delta\psi = -\left(\sum_{j=0}^{\infty}(A^{\text{retro}} + iA^{\text{retro}}\alpha_j)\, \mathrm{e}^{i(|\lambda'_j|t+\gamma_j)} + \sum_{j=0}^{\infty}(A^{\text{pro}} + iA^{\text{pro}}\alpha_{-j})\, \mathrm{e}^{-i(|\lambda'_j|t+\gamma_j)}\right) \tag{43}$$

or

$$\Delta\epsilon + i\sin\epsilon_0\Delta\psi = -\left(\sum_{j=0}^{\infty}(A_j^{\text{retro}\,ip} + iA_j^{\text{retro}\,op})\, \mathrm{e}^{i(|\lambda'_j|t+\gamma_j)} + (A_j^{\text{pro}\,ip} + iA_j^{\text{pro}\,op})\, \mathrm{e}^{-i(|\lambda'_j|t+\gamma_j)}\right) \tag{44}$$

$$= -\left(\sum_{j=0}^{\infty}(A_j^{\text{retro}\,ip} + iA_j^{\text{retro}\,op})\, \mathrm{e}^{i(|\Delta\omega_j|t+\gamma_j)} + (A_j^{\text{pro}\,ip} + iA_j^{\text{pro}\,op})\, \mathrm{e}^{-i(|\Delta\omega_j|t+\gamma_j)}\right) \tag{45}$$

where the superscript 'ip' means *in-phase*, and 'op', *out-of-phase*.

The dissipation introduced in the spherical motions (by the phases α_j and α_{-j}) must also be accounted for in the obliquity and longitude components of the nutations. This gives:

$$\Delta\epsilon = \sum_{j=0}^{\infty} A_{\omega j}^{\epsilon}\cos(\Delta\omega_j t + \beta_j + \alpha_j^{\epsilon}), \tag{46}$$

$$\sin\epsilon_0\Delta\psi = \sum_{j=0}^{\infty} A_{\omega j}^{\psi}\sin(\Delta\omega_j t + \beta_j + \alpha_j^{\psi}), \tag{47}$$

with $A^{\epsilon}_{\omega j} = A_j - A_{-j}$ and $A^{\psi}_{\omega j} = -(A_j + A_{-j})$. One usually writes at the first order:

$$\Delta\epsilon = \sum_{j=0}^{\infty} \underbrace{(A^{\epsilon}_{\omega j}\cos(\Delta\omega_j t + \beta_j)}_{in-phase-component} + \underbrace{(-A^{\epsilon}_{\omega j}\alpha^{\epsilon}_j)\sin(\Delta\omega_j t + \beta_j))}_{out-of-phase-component} \tag{48}$$

$$\sin\epsilon_0\Delta\psi = \sum_{j=0}^{\infty} \underbrace{(A^{\psi}_{\omega j}\sin(\Delta\omega_j t + \beta_j)}_{in-phase-component} + \underbrace{(A^{\psi}_{\omega j}\alpha^{\psi}_j)\cos(\Delta\omega_j t + \beta_j))}_{out-of-phase-component.} \tag{49}$$

Equations (48) and (49) can be written:

$$\Delta\epsilon = \sum_{j=0}^{\infty} \Delta\epsilon^{ip}_{\omega j}\cos(\Delta\omega_j t + \beta_j) + \Delta\epsilon^{op}_{\omega j}\sin(\Delta\omega_j t + \beta_j), \tag{50}$$

$$\Delta\psi = \sum_{j=0}^{\infty} \Delta\psi^{ip}_{\omega j}\sin(\Delta\omega_j t + \beta_j) + \Delta\psi^{op}_{\omega j}\cos(\Delta\omega_j t + \beta_j). \tag{51}$$

In order to include the possibility of having a negative frequency, we can write:

$$\Delta\epsilon = \sum_{j=0}^{\infty} \Delta\epsilon^{ip}_{\omega j}\cos(|\Delta\omega_j|t + \gamma_j) + \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\epsilon^{op}_{\omega j}\sin(|\Delta\omega_j|t + \gamma_j), \tag{52}$$

$$\Delta\psi = \sum_{j=0}^{\infty} \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi^{ip}_{\omega j}\sin(|\Delta\omega_j|t + \gamma_j) + \Delta\psi^{op}_{\omega j}\cos(|\Delta\omega_j|t + \gamma_j). \tag{53}$$

These expressions of the nutations can also be written for an inertial frequency λ'_j:

$$\Delta\epsilon = \sum_{j=0}^{\infty} \Delta\epsilon^{ip}_{\lambda j}\cos(\lambda'_j t + \phi_j) + \Delta\epsilon^{op}_{\lambda j}\sin(\lambda'_j t + \phi_j), \tag{54}$$

$$\Delta\psi = \sum_{j=0}^{\infty} \Delta\psi^{ip}_{\lambda j}\sin(\lambda'_j t + \phi_j) + \Delta\psi^{op}_{\lambda j}\cos(\lambda'_j t + \phi_j). \tag{55}$$

Again, in order to include the possibility of having a negative frequency, we can write:

$$\Delta\epsilon = \sum_{j=0}^{\infty} \Delta\epsilon^{ip}_{\lambda j}\cos(|\lambda'_j|t + \gamma_j) + \frac{\lambda'_j}{|\lambda'_j|}\Delta\epsilon^{op}_{\lambda j}\sin(|\lambda'_j|t + \gamma_j), \tag{56}$$

$$\Delta\psi = \sum_{j=0}^{\infty} \frac{\lambda'_j}{|\lambda'_j|} \Delta\psi^{ip}_{\lambda j} \sin(|\lambda'_j|t + \gamma_j) + \Delta\psi^{op}_{\lambda j} \cos(|\lambda'_j|t + \gamma_j), \tag{57}$$

where $\gamma_j = \beta_j = -\phi_j$ if $\Delta\omega_j$ is positive (and λ'_j negative), and $\gamma_j = -\beta_j = \phi_j$ if $\Delta\omega_j$ is negative (and λ'_j positive). We can write, generally

$$\Delta\epsilon = \sum_{j=0}^{\infty} \Delta\epsilon^{ip}_j \cos(|\lambda'_j|t + \gamma_j) + \Delta\epsilon^{op}_j \sin(|\lambda'_j|t + \gamma_j) \tag{58}$$

$$= \sum_{j=0}^{\infty} \Delta\epsilon^{ip}_j \cos(|\Delta\omega_j|t + \gamma_j) + \Delta\epsilon^{op}_j \sin(|\Delta\omega_j|t + \gamma_j), \tag{59}$$

$$\Delta\psi = \sum_{j=0}^{\infty} \Delta\psi^{ip}_j \sin(|\lambda'_j|t + \gamma_j) + \Delta\psi^{op}_j \cos(|\lambda'_j|t + \gamma_j) \tag{60}$$

$$= \sum_{j=0}^{\infty} \Delta\psi^{ip}_j \sin(|\Delta\omega_j|t + \gamma_j) + \Delta\psi^{op}_j \cos(|\Delta\omega_j|t + \gamma_j), \tag{61}$$

with

$$\Delta\epsilon^{ip}_j = \Delta\epsilon^{ip}_{\lambda j} = \Delta\epsilon^{ip}_{\omega j}, \tag{62}$$

$$\Delta\epsilon^{op}_j = \frac{\lambda'_j}{|\lambda'_j|}\Delta\epsilon^{op}_{\lambda j} = \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\epsilon^{op}_{\omega j}, \tag{63}$$

$$\Delta\psi^{ip}_j = \frac{\lambda'_j}{|\lambda'_j|}\Delta\psi^{ip}_{\lambda j} = \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi^{ip}_{\omega j}, \tag{64}$$

$$\Delta\psi^{op}_j = \Delta\psi^{op}_{\lambda j} = \Delta\psi^{op}_{\omega j}. \tag{65}$$

These equations lead to

$$\Delta\epsilon^{ip}_{\lambda j} = -\left(A^{\text{retro}\,ip}_j + A^{\text{pro}\,ip}_j\right), \tag{66}$$

$$\Delta\epsilon^{op}_{\lambda j} = \frac{\lambda'_j}{|\lambda'_j|}\left(A^{\text{retro}\,op}_j - A^{\text{pro}\,op}_j\right), \tag{67}$$

$$\Delta\psi^{ip}_{\lambda j} \sin\epsilon_0 = \frac{-\lambda'_j}{|\lambda'_j|}\left(A^{\text{retro}\,ip}_j - A^{\text{pro}\,ip}_j\right), \tag{68}$$

$$\Delta\psi^{op}_{\lambda j} \sin\epsilon_0 = -\left(A^{\text{retro}\,op}_j + A^{\text{pro}\,op}_j\right), \tag{69}$$

and

$$A^{\text{pro}\,ip}_j = \frac{-1}{2}\left(\Delta\epsilon^{ip}_{\lambda j} - \frac{\lambda'_j}{|\lambda'_j|}\Delta\psi^{ip}_{\lambda j}\sin\epsilon_0\right), \tag{70}$$

$$A_j^{\mathrm{retro}\,ip} = \frac{-1}{2}\left(\Delta\epsilon_{\lambda j}^{ip} + \frac{\lambda_j'}{|\lambda_j'|}\Delta\psi_{\lambda j}^{ip}\sin\epsilon_0\right), \tag{71}$$

$$A_j^{\mathrm{pro}\,op} = \frac{-1}{2}\frac{\lambda_j'}{|\lambda_j'|}\left(\Delta\epsilon_{\lambda j}^{op} + \frac{\lambda_j'}{|\lambda_j'|}\Delta\psi_{\lambda j}^{op}\sin\epsilon_0\right), \tag{72}$$

$$A_j^{\mathrm{retro}\,op} = \frac{1}{2}\frac{\lambda_j'}{|\lambda_j'|}\left(\Delta\epsilon_{\lambda j}^{op} - \frac{\lambda_j'}{|\lambda_j'|}\Delta\psi_{\lambda j}^{op}\sin\epsilon_0\right), \tag{73}$$

or

$$\Delta\epsilon_{\omega j}^{ip} = -\left(A_j^{\mathrm{retro}\,ip} + A_j^{\mathrm{pro}\,ip}\right), \tag{74}$$

$$\Delta\epsilon_{\omega j}^{op} = \frac{\Delta\omega_j}{|\Delta\omega_j|}\left(A_j^{\mathrm{retro}\,op} - A_j^{\mathrm{pro}\,op}\right), \tag{75}$$

$$\Delta\psi_{\omega j}^{ip}\sin\epsilon_0 = -\frac{\Delta\omega_j}{|\Delta\omega_j|}\left(A_j^{\mathrm{retro}\,ip} - A_j^{\mathrm{pro}\,ip}\right), \tag{76}$$

$$\Delta\psi_{\omega j}^{op}\sin\epsilon_0 = -\left(A_j^{\mathrm{retro}\,op} + A_j^{\mathrm{pro}\,op}\right), \tag{77}$$

and

$$A_j^{\mathrm{pro}\,ip} = \frac{-1}{2}\left(\Delta\epsilon_{\omega j}^{ip} - \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{ip}\sin\epsilon_0\right), \tag{78}$$

$$A_j^{\mathrm{retro}\,ip} = \frac{-1}{2}\left(\Delta\epsilon_{\omega j}^{ip} + \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{ip}\sin\epsilon_0\right), \tag{79}$$

$$A_j^{\mathrm{pro}\,op} = \frac{-1}{2}\frac{\Delta\omega_j}{|\Delta\omega_j|}\left(\Delta\epsilon_{\omega j}^{op} + \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{op}\sin\epsilon_0\right), \tag{80}$$

$$A_j^{\mathrm{retro}\,op} = \frac{1}{2}\frac{\Delta\omega_j}{|\Delta\omega_j|}\left(\Delta\epsilon_{\omega j}^{op} - \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{op}\sin\epsilon_0\right). \tag{81}$$

Note that all these relations are given for the convention that

$$A_j^{\mathrm{pro}} = A_j^{\mathrm{pro}\,ip} + iA_j^{\mathrm{pro}\,op}, \tag{82}$$

$$A_j^{\mathrm{retro}} = A_j^{\mathrm{retro}\,ip} + iA_j^{\mathrm{retro}\,op}. \tag{83}$$

Herring et al. (1986), as well as Wahr and Sasao (1981), define the prograde and retrograde amplitudes by:

$$A^{\mathrm{pro}} = A^{\mathrm{pro}\,ip} - iA^{\mathrm{pro}\,op}, \tag{84}$$

$$A^{\mathrm{retro}} = A^{\mathrm{retro}\,ip} - iA^{\mathrm{retro}\,op}, \tag{85}$$

so that with their convention we have

$$\Delta\epsilon + i\sin\epsilon_0\Delta\psi = -\left(\sum_{j=0}^{\infty}\left(A^{\text{retro}\,ip} - iA^{\text{retro}\,op}\right) \mathrm{e}^{i(|\Delta\omega_j|t+\phi_j)} + \left(A^{\text{pro}\,ip} - iA^{\text{pro}\,op}\right) \mathrm{e}^{-i(|\Delta\omega_j|t+\phi_j)}\right) \tag{86}$$

which gives the link between the variations in longitude and obliquity and the amplitudes of the retrograde and prograde motions:

$$A_j^{\text{pro}\,ip} = \frac{-1}{2}\left(\Delta\epsilon_{\omega j}^{ip} - \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{ip}\sin\epsilon_0\right) = \frac{-1}{2}\left(\Delta\epsilon_{\lambda j}^{ip} - \frac{\lambda_j'}{|\lambda_j'|}\Delta\psi_{\lambda j}^{ip}\sin\epsilon_0\right), \tag{87}$$

$$A_j^{\text{retro}\,ip} = \frac{-1}{2}\left(\Delta\epsilon_{\omega j}^{ip} + \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{ip}\sin\epsilon_0\right) = \frac{-1}{2}\left(\Delta\epsilon_{\lambda j}^{ip} + \frac{\lambda_j'}{|\lambda_j'|}\Delta\psi_{\lambda j}^{ip}\sin\epsilon_0\right), \tag{88}$$

$$A_j^{\text{pro}\,op} = \frac{1}{2}\frac{\Delta\omega_j}{|\Delta\omega_j|}\left(\Delta\epsilon_{\omega j}^{op} + \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{op}\sin\epsilon_0\right) = \frac{1}{2}\frac{\lambda_j'}{|\lambda_j'|}\left(\Delta\epsilon_{\lambda j}^{op} + \frac{\lambda_j'}{|\lambda_j'|}\Delta\psi_{\lambda j}^{op}\sin\epsilon_0\right), \tag{89}$$

$$A_j^{\text{retro}\,op} = -\frac{1}{2}\frac{\Delta\omega_j}{|\Delta\omega_j|}\left(\Delta\epsilon_{\omega j}^{op} - \frac{\Delta\omega_j}{|\Delta\omega_j|}\Delta\psi_{\omega j}^{op}\sin\epsilon_0\right) = -\frac{1}{2}\frac{\lambda_j'}{|\lambda_j'|}\left(\Delta\epsilon_{\lambda j}^{op} - \frac{\lambda_j'}{|\lambda_j'|}\Delta\psi_{\lambda j}^{op}\sin\epsilon_0\right), \tag{90}$$

which is exactly what is presented in Mathews and Shapiro (1992).

Acknowledgements

We would like to thank P.M. Mathews and an anonymous reviewer who carefully reviewed our paper and whose remarks were very constructive. P. Defraigne is financially supported by the Belgian National Funds for Scientific Research (NFSR).

References

Capitaine, N.: 1982, Effets de la non-rigidité de la Terre sur son mouvement de rotation : étude théorique et utilisation d'observations, *Ph. D. Thesis*, Univ. Pierre et Marie Curie-Paris IV, 205.

Herring, T. A., Gwinn, C. R. and Shapiro, I. I.: 1986, 'Geodesy by radio interferometry: studies of forced nutations of the Earth. 1. Data Analysis', *J. Geophys. Res.*, **91** 4745–54.

Kinoshita H. and Souchay, J.: 1990, 'The theory of the nutation for a rigid Earth model at the second order', *Celest. Mech.*, **48**, 187–265.

Mathews, P. M. and Shapiro, I. I.: 1992, 'Nutations of the Earth', *Ann. Rev. of Earth and Planet. Sc.*, **20**, 469–500.

Melchior P. and Georis, B.: 1968, 'Earth tides, precession-nutation and secular retardation of Earth's rotation', *Phys. Earth and Planet. Inter.*, **1**, 267–287.

Roosbeek, F.: 1995, 'A tide generating potential precise to the nanogal level', submitted in *Geophys. J. Int.*

Smith, M. L.: 1974, 'The scalar equations of infinitesimal elastic-gravitational motion for a rotating', slightly elliptical Earth, *Geophys. J. R. Astron. Soc.*, **37**, 491–526.

Wahr, J.M.: 1979, 'The tidal motions of a rotating, elliptical, elastic and oceanless Earth', Ph.D.Thesis, University of Colorado, USA.

Wahr J. M.: 1981, 'The forced nutations of an elliptical, rotating, elastic and oceanless Earth', *Geophys. J. R. Astron. Soc.*, **64**, 705–727.

Wahr J.M. and Sasao, T.: 1981, 'A diurnal response in the ocean tide and the Earth's load response due to the resonant free "core nutation" ', *Geophys. J. R. Astron. Soc.*, **64**, 747–765.