COMPARISON BETWEEN DEPRIT AND DRAGT-FINN PERTURBATION METHODS

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Abstract. In this paper, the relationship between the Dragt-Finn transform and the classical Lie transform introduced by Deprit is discussed. The relative performance of the algorithms used for the computations of the transformed functions is compared, and the relation between their generators is given. These generators produce the same transform which insures the construction of the same invariants.

Key words: Lie methods, canonical transformations

1. Introduction

Our aim is to consider formal canonical transformations from the Lie algebraic point of view. We consider the two principal transformations that play a important role in the Lie perturbation theory. The Lie transform is the application of a single non-autonomous hamiltonian flow (Deprit transform) and we consider also the composition of successive applications of autonomous flows (Dragt-Finn transform). Both are families of near-identity transformations depending on a small parameter. Both have the advantage of giving explicit changes of variables which preserve the Hamilton form of the equations of motion. Both can be computed by iteration. In this paper, the Hamiltonian has been restricted to be autonomous, and a formal series on the small parameter but other cases may be treated.

The Lie transform is generally used in celestial mechanics and the factored product rather in optics [19], plasma physics or molecular vibrational dynamics [8].

Let us consider an autonomous hamiltonian system with n degrees of freedom near an equilibrium point. By a linear transformation, this equilibrium can be taken as the origin of the phase space $\mathbb{R}^{2n}$. Assume that in the coordinates $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n) = (z_1, \ldots, z_{2n})$, one can write

$$h(p, q) = h_0(p, q) + \textstyle\sum_{n\geq 1}\varepsilon^n h_n(p, q), \tag{1}$$

where ε denotes a small parameter, and the h_i are homogeneous polynomials of degree $i + 2$. We can suppose also that the linearized part is the Hamiltonian

$$h_0 = \frac{1}{2}\sum_{l=1}^{n}\omega_l\left(p_l^2 + q_l^2\right), \tag{2}$$

whose first integrals are the actions $I_l = \frac{1}{2}(p_l^2 + q_l^2)$, but other cases may also be considered (see [19]). A classical problem is to find formal integrals for h of the form

$$\xi_l = I_l + \textstyle\sum_{n\geq 1}\varepsilon^n \xi_{l,n}, l = 1, \dots, n, \tag{3}$$

where the $\xi_{l,i}(p,q)$ are homogeneous polynomials of degree $i+2$. In this paper we will see these first integrals as the result of the actions by a family of canonical formal transformations close to identity. A convenient transformation is picked in order to bring the Hamiltonian into a much simpler form (for instance formally completely integrable). If for example, the transformed Hamiltonian $K(p,q)$ does depend only on the actions $\frac{1}{2}(p_i^2 + q_i^2)$, then they will be n first integrals for this system and by applying the inverse transform, one obtains explicitly the integrals we are looking for.

If the linear part of the Hamiltonian h has some good properties, for example the harmonic frequencies $\omega_1, \dots, \omega_n$ are linearly independent over $\mathbb{Q}$, one can find such a normal form by using some Lie transform. The normal form generally does not converge, unless the system has one degree of freedom. The reader must bear in mind that we are working with formal functions, neglecting all convergence problems although in many cases truncated parts of these forms will provide useful approximations for the study of the stability of the solutions (see for example [18; 9; 11]).

After having given some definitions and elementary properties, we will describe the Deprit transform and the Dragt-Finn transform. In particular, we will give the relations between them and their generators. These two transformations induce methods to bring the Hamiltonian into a normal form and we will compare the complexity of the algorithms required to obtain the results.

2. Preliminaries

We briefly recall some definitions and properties that the reader can find for example in [14; 3; 9; 2; 16]. We denote by L_f or $L(f)$ the Lie operator

$$L_f g = \{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{4}$$

T is a canonical transformation if it preserves the Poisson bracket $\{,\}$:

$$\{z_i, z_j\} = \{Tz_i, Tz_j\}, 1 \leq i \leq n, 1 \leq j \leq n. \tag{5}$$

We identify a canonical transformation T with a transformation on the functions on the phase space according to $Tf(z) = f(T(z))$. In terms of Lie operators we get for a canonical transformation T,

$$TL_fT^{-1} = L_{Tf}. \tag{6}$$

From the state evolution of the hamiltonian system $\dot{z}_i = \{z_i, h\}$, we deduce that for any time-independent function $f(z)$ we have

$$\dot{f} = \frac{df}{dt} = \sum_{i=1}^{n}\left[\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right] = \sum_{i=1}^{n}\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i} = \{f, h\}. \tag{7}$$

For a time-independent canonical transformation T, we have $\dot{\widehat{Tz}} = \{Tz, T^{-1}h(Tz)\}$ so $\dot{Z}_i = \{Z_i, T^{-1}h(Z_i)\}$ for $Z_i = Tz_i$. The Hamiltonian h becomes $K = T^{-1}h$ in the new variables $Z_i = Tz_i$.

In this paper, E_i will denote the space of homogeneous polynomials of degree i in the variables (p, q). We recall that $\dim E_i = \binom{2n+i-1}{i}$. For f in E_{p+2}, L_f is a linear mapping from E_k to E_{k+p} for each k.

e^{L_f} will denote the exponential of the Lie operator L_f, that is to say, for g being a function on the phase space

$$e^{L_f} g = \textstyle\sum_{n\geq 0}\frac{L_f^n}{n!}g. \tag{8}$$

Both Lie method and Dragt-Finn method are based on finding canonical transforms

$$T = I + \textstyle\sum_{n\geq 1}\varepsilon^n T_n \quad \text{and} \quad T^{-1} = I + \sum_{n\geq 1}\varepsilon^n (T^{-1})_n \tag{9}$$

that bring the Hamiltonian h into a much simpler form

$$K = T^{-1}h = K_0 + \textstyle\sum_{n\geq 1}\varepsilon^n K_n. \tag{10}$$

This point of view has the advantage of giving the explicit change of variables and requires only the use of the inverse transform.

Let us remind two theorems [5; 16] that explain the natural use of the transformations we are looking for.

– Given a family of canonical C^∞ transformations T_θ, there exists a function $w(z, \theta)$ such that T_θ^{-1} is the application at time θ of the flow associa ted to the non autonomous hamiltonian vector field of the Hamiltonian w, that is to say, if $z^* = T_\theta z$, we get

$$\frac{\partial z^*}{\partial \theta} = \{z^*, w(z^*, \theta)\}. \tag{11}$$

– Given any formal canonical transformation M near the identity, such that

$$M z_i = z_i + \textstyle\sum_{|j|\geq 1} a_{j,i} z^j \tag{12}$$

for each i, then there exists a series $g = \sum_{n\geq 1} g_n \in \prod_{n\geq 1} E_{n+2}$ such that

$$M = e^{-L_{g_1}} \cdots e^{-L_{g_k}} \cdots. \tag{13}$$

Here $j = (j_1, \dots, j_{2n}) \in \mathbb{N}^{2n}$, $|j| = \sum_{k=1}^{2n} j_k$ and z^j denotes $z_1^{j_1} \cdots z_{2n}^{j_{2n}}$.

As $\frac{d}{dt}\left[e^{tL_{g_k}}\right] = L_{g_k} e^{tL_{g_k}}$, $e^{L_{g_k}}$ may be seen as the application at time θ of the flow associated to the autonomous Hamiltonian g_k and so M^{-1} is the result of these successive transformations.

Of course we do not attempt to obtain analytical transformations, but we can deduce that any formal canonical transformation M_θ such that

$$M_\theta z_i = z_i + \textstyle\sum_{|j|\geq 1} \theta^{|j|} a_{j,i} z^j \tag{14}$$

can be factored as

$$M_\theta = e^{-\theta L_{g_1}} \cdots e^{-\theta^n L_{g_n}} \cdots . \tag{15}$$

Each component appears to be the application at time θ^k of the flow associated to the autonomous Hamiltonian g_k or may be considered as the application at time 1 of the flow associated to the non autonomous Hamiltonian $k\theta^{k-1} g_k$.

2.1. THE DEPRIT TRANSFORM

Given a formal series $w = \sum_{n \geq 1} \varepsilon^n w_n$, we build up the so-called Deprit transform as the flow at time ε of a non-autonomous hamiltonian vector field given by $\frac{\partial w}{\partial \varepsilon}$. We use $\frac{\partial w}{\partial \varepsilon}$ instead of w in order to get the same transformation as the Dragt-Finn transformation generated by $g = w$ when $w = w_n$ is an homogeneous term. From (11) we deduce that

$$\frac{\partial T z}{\partial \varepsilon} = -\{\frac{\partial w}{\partial \varepsilon}(Tz), Tz\} = -\{T\frac{\partial w}{\partial \varepsilon}(z), Tz\} = -TL(\frac{\partial w}{\partial \varepsilon})z \tag{16}$$

so

$$\frac{\partial T}{\partial \varepsilon} = -TL(\frac{\partial w}{\partial \varepsilon}) \tag{17}$$

and from $TT^{-1} = I$ we have

$$\frac{\partial T^{-1}}{\partial \varepsilon} = L(\frac{\partial w}{\partial \varepsilon})T^{-1}. \tag{18}$$

We thus deduce the formal expansion of T and T^{-1} with respect to ε

$$T_0 = I, T_n = -\sum_{p=1}^{n} \frac{p}{n} T_{n-p} L_{w_p} \quad \text{and} \quad (T^{-1})_n = \sum_{p=1}^{n} \frac{p}{n} L_{w_p} (T^{-1})_{n-p}. \tag{19}$$

As each T_n is a sum of products of linear transforms, T is linear. Furthermore T preserves the product and the Poisson bracket (see [4]).

$f = \sum_{n \geq 0} \varepsilon^n f_n$ being a formal series in ε, we deduce (see [3; 14]) that $F = T^{-1} f$ is a formal series in ε and

$$T^{-1} f = \sum_{n\geq 0} \varepsilon^n F_n \quad \text{where} \quad F_n = \sum_{0 \leq p \leq n} (T^{-1})_p f_{n-p}. \tag{20}$$

One can see recursively on n that

$$(T^{-1})_n = \sum_{n>m_1>\cdots>m_r} \frac{n-m_1}{n}\frac{m_1-m_2}{m_1}\cdots\frac{m_r}{m_r}L_{w_{n-m_1}}\cdots L_{w_{m_r}}. \tag{21}$$

The number of terms in this sum is the number of subsets of $\{1,\ldots,n-1\}$, that is 2^{n-1}.

2.2. The Dragt-Finn transform

We consider a Lie generating function $g = \sum_{n\geq 1}\varepsilon^n g_n$ where $g_p \in E_p$. The Dragt-Finn transformation M_g so as its inverse M_g^{-1} are the product of single transformations

$$M_g = e^{-\varepsilon L_{g_1}}\cdots e^{-\varepsilon^n L_{g_n}}\cdots,\quad M_g^{-1} = \cdots e^{\varepsilon^n L_{g_n}}\cdots e^{\varepsilon L_{g_1}}, \tag{22}$$

where e^{L_g} is defined in (8). From

$$\frac{\partial}{\partial\varepsilon}\left[e^{-\varepsilon^k L_{g_k}}\right] = -k\varepsilon^{k-1}L_{g_k}e^{-\varepsilon^k L_{g_k}} = L\left(\frac{\partial}{\partial\varepsilon}\left[-\varepsilon^k L_{g_k}\right]\right)e^{-\varepsilon^k L_{g_k}}, \tag{23}$$

we deduce that M_g is a product of Deprit transformations. Thus M_g is canonical, linear and preserves products (see [19] a for direct proof). We can formally expand M and M^{-1} as

$$M = I + \textstyle\sum_{n\geq 1}\varepsilon^n M_n,\; M^{-1} = I + \sum_{n\geq 1}\varepsilon^n (M^{-1})_n, \tag{24}$$

where

$$(M^{-1})_n = \sum_{\wp(1,n;n)} \frac{L_{g_n}^{m_n}\cdots L_{g_1}^{m_1}}{m_n!\cdots m_1!}. \tag{25}$$

$\wp(p,q;n)$ denotes the set $\{m_p,\ldots,m_q \in \mathbb{N}; pm_p+\cdots+qm_q = n\}$. $(M^{-1})_n$ appears to be a sum of $p(n)$ products of Lie operators where $p(n)$ is the cardinal of $\wp(1,n;n)$, that is the number of partitions of $\{1,\ldots,n\}$. It is known (see [1]) that $p(n)\simeq_\infty \exp(\pi\sqrt{\frac{2n}{3}})/4n\sqrt{3} \leq 2^{n-1}$. As above with the Deprit transform (20), for f being a formal series in ε, we get the formal expansion

$$F = M^{-1}f = \textstyle\sum_{n\geq 0}\varepsilon^n F_n \quad \text{where} \quad F_n = \displaystyle\sum_{m=0}^{n}(M^{-1})_m f_{n-m}. \tag{26}$$

3. Relationship between the Transformations

Given a generating function $w = \sum_{n\geq 1}\varepsilon^n w_n$, and the corresponding Lie transform T_w, the theorem of Dragt and Finn [5; 19] insures that there exists a generating function $g = \sum_{n\geq 1}\varepsilon^n g_n$, such that T_w may be factored as the product

$$M_g = e^{-\varepsilon L_{g_1}}\cdots e^{-\varepsilon^n L_{g_n}}\cdots. \tag{27}$$

We will see in this section the direct relation and the one-to-one correspondence between the generators. This result had been found by Finn [7]. We will give here the general formula in terms of Lie polynomials. Considering a generating series g and differentiating the corresponding formal transformation M_g with respect to ε, we get

$$\frac{\partial M_g}{\partial \varepsilon} = \sum_{n\geq 1} \left[e^{-\varepsilon L_{g_1}} \cdots e^{-\varepsilon^{n-1} L_{g_{n-1}}}\right] \left[\frac{\partial}{\partial \varepsilon}\left[e^{-\varepsilon^n L_{g_n}}\right]\right] \left[e^{-\varepsilon^{n+1} L_{g_{n+1}}} \cdots\right] \tag{28}$$

$$= \sum_{n\geq 1} \left[e^{-\varepsilon L_{g_1}} \cdots e^{-\varepsilon^{n-1} L_{g_{n-1}}}\right] \left[-n\varepsilon^{n-1} L_{g_n} e^{-\varepsilon^n L_{g_n}}\right] \left[e^{-\varepsilon^{n+1} L_{g_{n+1}}} \cdots\right] \tag{29}$$

$$= M_g \sum_{n\geq 1} M_g^{-1} \left[e^{-\varepsilon L_{g_1}} \cdots e^{-\varepsilon^{n-1} L_{g_{n-1}}}\right] \left[-n\varepsilon^{n-1} L_{g_n}\right] \left[e^{-\varepsilon^n L_{g_n}} \cdots\right] \tag{30}$$

$$= M_g \sum_{n\geq 1} \left[\cdots e^{\varepsilon^n L_{g_n}}\right] \left[-n\varepsilon^{n-1} L_{g_n}\right] \left[e^{-\varepsilon^n L_{g_n}} \cdots\right] \tag{31}$$

$$= M_g L \left[\sum_{n\geq 1} -n\varepsilon^{n-1} \left[\cdots e^{\varepsilon^n L_{g_n}}\right] g_n\right]. \tag{32}$$

Thus we have

$$\frac{\partial M_g}{\partial \varepsilon} = -M_g L(S), \tag{33}$$

in which

$$S = \sum_{n\geq 1} n\varepsilon^{n-1} \left[\cdots e^{\varepsilon^{n-1} L_{g_{n-1}}}\right] g_n \tag{34}$$

$$= \sum_{n\geq 1} \varepsilon^{n-1} \left[\sum_{k=1}^{n} k \sum_{\wp(k+1,n-k;n-k)} \frac{L_{g_{n-k}}^{m_{n-k}} \cdots L_{g_{k+1}}^{m_{k+1}}}{m_{k+1}! \cdots m_{n-k}!} g_k\right]. \tag{35}$$

$\wp(p,q;n)$ denotes the set $\{m_p, \ldots, m_q \in \mathbb{N}; pm_p + \cdots + qm_q = n\}$. On the other hand we have (19)

$$\frac{\partial T_w}{\partial \varepsilon} = -T_w L(\frac{\partial w}{\partial \varepsilon}), \tag{36}$$

so from the initial conditions $T_0 = M_0 = I$, a necessary and sufficient condition to have $T_w = M_g$ is

$$S = \frac{\partial w}{\partial \varepsilon}, \tag{37}$$

that is, when we consider each term of the formal expansion

$$w_n = \sum_{1\leq k\leq n} \frac{k}{n} \sum_{\wp(k+1,n-k;n-k)} \frac{L_{g_{n-k}}^{m_{n-k}} \cdots L_{g_{k+1}}^{m_{k+1}}}{m_{k+1}! \cdots m_{n-k}!} g_k \tag{38}$$

$$= g_n + \sum_{1\le k\le n-1} \frac{k}{n} \sum_{\wp(k+1,n-k;n-k)} \frac{L_{g_{n-k}}^{m_{n-k}} \cdots L_{g_{k+1}}^{m_{k+1}}}{m_{k+1}! \cdots m_{n-k}!.} \tag{39}$$

These formula give directly w_n as a linear combination of Poisson brackets of the g_i's. Because the connection between the w_i's and the g_i's has a triangular form, we deduce the same property for the g_i's, that is, each g_i may be written as a linear combination of Poisson brackets of the w_j's. Some formulas will be found at the appendix.

4. Perturbation Method

With the Deprit method as with the Dragt-Finn method, we search for a generating function w (resp. g) that produces the inverse transformation T_w^{-1} (resp. M_g^{-1}) defined by

$$T_0^{-1} = I,\ (T^{-1})_n = \sum_{p=1}^{n} \frac{p}{n} L_{w_p} (T^{-1})_{n-p} \tag{40}$$

or

$$M_0^{-1} = I,\ (M^{-1})_n = \sum_{\wp(1,n;n)} \frac{L_{g_n}^{m_n} \cdots L_{g_1}^{m_1}}{m_n! \cdots m_1!}. \tag{41}$$

These two transformations provide a change of variables and two transformed Hamiltonians $K^w = T^{-1}h$ and $K^g = M^{-1}h$ that are respectively given by (20) or (26)

$$K_n^w = \sum_{m=0}^{n} (T^{-1})_m h_{n-m} \quad \text{and} \quad K_n^g = \sum_{m=0}^{n} (M^{-1})_m h_{n-m}. \tag{42}$$

For both methods, we try to construct simultaneously step by step the generating function w (resp. g) and the transformed Hamiltonian K^w (resp. K^g). As

$$K_n^w = (T^{-1})_n h_0 + \sum_{p=0}^{n-1} (T^{-1})_{n-p} h_p \tag{43}$$

$$= L_{w_n} h_0 + \sum_{p=1}^{n-1} \frac{p}{n} L_{w_p} (T_{n-p})^{-1} h_0 + \sum_{p=0}^{n-1} (T^{-1})_{n-p} h_p, \tag{44}$$

we have to solve the so-called Lie equation

$$L_{h_0} w_n + K_n^w = \sum_{p=1}^{n-1} \frac{p}{n} L_{w_p} (T^{-1})_{n-p} h_0 + \sum_{p=0}^{n-1} (T^{-1})_{n-p} h_p = R_n^w. \tag{45}$$

where R_n^w depends only on h_i and w_i for $1 \le i \le n-1$.

For the Dragt-Finn transform we have

$$K_n^g = L_{g_n} h_0 + \sum_{\wp(1,n-1;n)} \frac{L_{g_n}^{m_n} \cdots L_{g_1}^{m_1}}{m_n! \cdots m_1!} h_0 + \sum_{m=0}^{n-1} (M^{-1})_m h_{n-m} \tag{46}$$

so we must solve the Lie equation

$$L_{h_0} g_n + K_n^g = R_n^g. \tag{47}$$

One of the goal of the construction of normal forms is finding formal integrals for the system. We thus have to solve (45) or (47) with the extra condition : $\{f, K\} = 0$ for some function f that does not depend on the small parameter ε. That is to say we have to solve for each n

$$L_{h_0} w_n + K_n^w = R_n^w \quad \text{with} \quad \{f, K_n^w\} = 0 \tag{48}$$

or

$$L_{h_0} g_n + K_n^g = R_n^g \quad \text{with} \quad \{f, K_n^g\} = 0. \tag{49}$$

As the equation (48) or (49) are not necessary consistent, we will suppose until the end that the range and the kernel of L_{h_0} are in direct sum and take h_0 for the first integral of K. In this case the equations (45) or (47) are consistent and are the natural decomposition into two supplementary subspaces. In this case K is said to be in normal form up to order r if $K_0, \ldots, K_r$ belong to $\ker L_{h_0}$. The computing of R^w (resp. R^g) and the resolution of the equations (45) and (47) are the two fundamental steps for the perturbation method.

5. Resolution of the Equation of Perturbation

Let a, b be two eigenvectors of L_{h_0} with eigenvalues λ, μ. From

$$\{h_0, ab\} = \{h_0, a\}b + \{h_0, b\}a = (\lambda + \mu)ab \tag{50}$$

ab is an eigenvector of L_{h_0} related to the eigenvalue $(\lambda + \mu)$.

With $h_0 = \frac{1}{2} \sum_{i=1}^n \omega_i (p_i^2 + q_i^2)$ and no rational dependencies on the ω_i's, let us make the symplectic change of variables

$$x_l = \frac{1}{\sqrt{2}}(p_l + \mathrm{i}q_l), \; y_l = \frac{\mathrm{i}}{\sqrt{2}}(p_l - \mathrm{i}q_l), \; l = 1, \ldots, n, \tag{51}$$

that brings h_0 into

$$h_0 = -\mathrm{i} \sum_{l=1}^n \omega_l x_l y_l. \tag{52}$$

From $\{h_0, x_l\} = \mathrm{i}\omega_l x_l$ and $\{h_0, y_l\} = -\mathrm{i}\omega_l y_l$, the eigenvectors of L_{h_0} are the $x^\lambda y^\mu$, corresponding to the pure imaginary eigenvalues $\mathrm{i}\omega.(\lambda - \mu)$, so we deduce that L_{h_0} may be diagonalized.

Since the ω_l are rationally independents we deduce that an eigenvector $x^\lambda y^\mu$ belongs to the kernel if and only if $\lambda = \mu$ and thus the kernel is the subspace spanned by the products of the actions

$$(p_1^2 + q_1^2)^{k_1} \cdots (p_n^2 + q_n^2)^{k_n}. \tag{53}$$

If

$$R_n = \sum_{|k|+|l|=n} r_{k,l} x^k y^l \tag{54}$$

then we solve (47) by taking

$$K_n = \sum_{|k|+|l|=n, k=l} r_{k,l} x^k y^l \quad \text{and} \quad g_n = -\mathrm{i} \sum_{|k|+|l|=n, k\neq l} \frac{r_{k,l}}{(k-l).w} x^k y^l. \tag{55}$$

In practice, if L_{h_0} is not in diagonal form, we have to invert by any method the consistent linear system $L_{h_0} L_{h_0} g_n = L_{h_0} R_n$. It may be sometimes more difficult to diagonalize the system then to solve the preceding linear system.

6. Uniqueness of the Normal Form

We just suppose now the following properties on h_0.

- (i) $h_0 \in E_2$,
- (ii) the range and the kernel of L_{h_0} are in direct sum,
- (iii) for any $a, b \in \ker L(h_0)$, we have $\{a, b\} = 0$.

For $h_0 = \frac{1}{2}\sum_{i=1}^n \omega_i(p_i^2 + q_i^2)$ and no rational dependencies on the ω_i's, we have clearly the two first conditions. As the kernel of L_{h_0} is the subspace spanned by the product of the actions, from

$$\{I_1^{k_1} \cdots I_n^{k_n}, I_1^{l_1} \cdots I_n^{l_n}\} = I_1^{k_1+l_1} \cdots I_n^{k_n+l_n} \sum_{i=1}^n \sum_{j=1}^n \frac{k_i l_j}{I_i I_j} \{I_i, I_j\} = 0 \tag{56}$$

the third condition holds.

Under these conditions we will prove the uniqueness of the normal form obtained by the Deprit method, that is to say, if there is a normal form K and a generating function w such that $K = T_w^{-1} h$, then K is unique. Moreover w is not unique.

Let u and v be two generating functions such that $K^u = T_u^{-1} h$ and $K^v = T_v^{-1} h$ are in normal form up to a given order r. If T denotes $T_u^{-1} T_v$ we get $K^v = T^{-1} K^u$ and by derivation we obtain

$$\frac{\partial T^{-1}}{\partial \varepsilon} = \frac{\partial T_v^{-1}}{\partial \varepsilon} T_u + T_v^{-1} \frac{\partial T_u}{\partial \varepsilon} = L(\frac{\partial v}{\partial \varepsilon}) T_v^{-1} T_u - T_v^{-1} T_u L(\frac{\partial u}{\partial \varepsilon}) \tag{57}$$

so

$$\frac{\partial T^{-1}}{\partial \varepsilon} = \left[L(\frac{\partial v}{\partial \varepsilon}) - T^{-1} L(\frac{\partial u}{\partial \varepsilon}) T\right] T^{-1} = L\left[\frac{\partial v}{\partial \varepsilon} - T^{-1}\frac{\partial u}{\partial \varepsilon}\right] T^{-1} \tag{58}$$

and T appears as the Lie transform generated by w where

$$\frac{\partial w}{\partial \varepsilon} = \frac{\partial v}{\partial \varepsilon} - T^{-1}\frac{\partial u}{\partial \varepsilon}, \tag{59}$$

that brings the normal form K^u into the normal form K^v up to the same order r. By (45) and $h_0 = K_0^u = K_0^v$, we get for each n the Lie equation

$$\{h_0, w_n\} + K_n^v = R_n^u = K_n^u + \sum_{p=1}^{n-1}(T^{-1})_p K_{n-p}^u + \sum_{p=1}^{n-1}\frac{p}{n}L(w_p)(T^{-1})_{n-p}h_0. \tag{60}$$

We will prove that for each $0 \le n \le r$, we have for each $1 \le i, i+j \le n$

$$R_n^u = K_n^u = K_n^v, \{h_0, w_n\} = 0 \quad \text{and} \quad (T^{-1})_i K_j^u = 0. \tag{61}$$

Given any order r, (60) becomes for $n = 1$

$$\{h_0, w_1\} + K_1^v = K_1^u. \tag{62}$$

As the kernel and the range of L_{h_0} are in direct sum, and both K_1^v and K_1^u belong to the kernel, we have

$$\{h_0, w_1\} = 0 \quad \text{and} \quad K_1^u = K_1^v. \tag{63}$$

Suppose we have (61) for $1 \le i < n \le r$: then (60) becomes

$$\begin{aligned}
\{h_0, w_n\} + K_n^v &= K_n^u + \sum_{p=1}^{n-1}(T^{-1})_p K_{n-p}^u + \sum_{p=1}^{n-1}\frac{p}{n}L_{w_p}\underbrace{(T^{-1})_{n-p}h_0}_{=0(61)} && (64)\\
&= K_n^u + \sum_{p=1}^{n-1}\sum_{q=1}^{p-1}\frac{q}{p}L_{w_q}\underbrace{(T^{-1})_{p-q}K_{n-p}^u}_{=0(61)} + \sum_{p=1}^{n-1}\underbrace{L_{w_p}K_{n-p}^u}_{=0(\text{iii})} && (65)\\
&= K_n^u && (66)
\end{aligned}$$

so

$$K_n^v = K_n^u \quad \text{and} \quad \{h_0, w_n\} = 0 \tag{67}$$

and for $1 \le i, i+j = n$ we have by (19)

$$(T^{-1})_i K_j^u = \sum_{p=1}^{i-1}\frac{p}{i}\underbrace{(T^{-1})_{i-p}K_j^u}_{=0(61)} + \underbrace{L_{w_p}K_j^u}_{=0(\text{iii})} = 0. \tag{68}$$

We thus have for each $n \le r$

$$\{h_0, w_n\} = 0 \quad \text{and} \quad K_n^u = K_n^v. \tag{69}$$

We thus have $K^u = K^v$ which proves the uniqueness of the normal form. On the other hand the transformation T is not necessarily the identity map, as w has only to be in $\ker L_{h_0}$.

7. Equivalence of the Two Methods

Under the three conditions (i), (ii), (iii), let K^w and K^g be two Hamiltonians in normal form obtained by the Deprit and the Dragt methods respectively. Let M_g be the corresponding Dragt transform such that $K^g = M_g^{-1}h$. By (39), there exists w' such that $T_{w'}^{-1} = M_g^{-1}$, so $K^{w'} = T_{w'}^{-1}h = K^g$. By uniqueness of the normal form obtained by Lie transform, we have $K^w = K^g$. Thus any normal form obtained by the Dragt transform is equal to the unique normal form obtained by any Deprit transform and is therefore unique.

The two methods produce the same unique normal form but not necessary the same transformation.

8. Computing of the Inverse Transform

In the light of the preceding sections, the main step is computing of the inverse transform. For the Deprit transform, we will use the algorithm proposed by Henrard [12], that is issued from the relations between the T_p^{-1} and is often used for example by Giorgilli [9] for several evaluations. For the Dragt-Finn transform, as we do not know any closed formula between the $(M^{-1})_p$, we will use the fact that the transformation is the product of single homogeneous transformations.

Given $f = \sum_{n\geq 0} \varepsilon^n f_n$ a formal series in ε, we call F the inverse transformed function $T^{-1}f = \sum_{n\geq 0} \varepsilon^n F_n$. These algorithms may also be used for computing the transformation as we will see.

8.1. The Deprit inverse transform

From $F_n = \sum_{k=0}^{p} (T^{-1})_p f_{n-p}$ and the relation

$$(T^{-1})_p f_q = \sum_{k=1}^{p} \frac{k}{p} L_{w_k} (T^{-1})_{p-k} f_q, \tag{70}$$

we will construct successively $F_{p,q} = (T^{-1})_p f_q$ for $p+q \leq r$. We have to fill a two dimensional array

$$\begin{array}{ccccccccc}
f_0 & f_1 & f_2 & \cdots & f_i & \cdots & \cdots & f_n \\
 & F_{1,0} & F_{1,1} & F_{1,2} & & \ddots & \vdots & F_{1,n-1} \\
 & & \ddots & \ddots & & \ddots & & \\
 & & & \ddots & \ddots & & \ddots & \vdots \\
 & & & & F_{i,0} & F_{i,1} & \cdots & F_{i,n-i} \\
 & & & & & \ddots & & \vdots \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \cdots & F_{n,0} \\
F_0 & F_1 & F_2 & \cdots & F_i & \cdots & & F_n
\end{array} \tag{71}$$

column by column. Each term $F(j, i-j)$ is deduced from the preceding $F(k, i-j)$ in the same diagonal which is independent from the others. In particular if $f_i = F(i, 0)$ we get $F(k, i) = 0$ for each k. Each term F_i is obtained as the sum of the terms of the column $F(j, i-j)$. From the initial values $F_0 = f_0, \ldots, F_r = f_r$, we will use the loop

for $i := 1$ **to** r **do**
  for $j := 1$ **to** i **do**
    for $k := 1$ **to** j **do** $F_{j,i-j} := F_{j,i-j} + \frac{k}{j}\{w_k, F_{j-k,i-j}\}$;
    $F_i := F_i + F_{j,i-j}$.

We have to evaluate a Poisson bracket of a term of order ε^p against a term of order ε^q when $k = p$ and $j - k + i - j = q$ or equivalently when $k = p$ and $i = p + q$ in the preceding algorithm. The number of such Poisson brackets is the cardinal of $\{j; p \leq j \leq p + q\}$, that is $q + 1$.

The number of Poisson brackets required is a good indication but we have also to pay attention to the number of terms kept in stack. In the algorithm proposed for the Deprit method, it is necessary to keep in stack all the $F_{p,q}$ for $p + q \leq r$ in order to calculate $F_0, \ldots, F_r$. Furthermore it may be interesting to keep them in order to calculate F_{r+1} if necessary. We would rather say that the Deprit transform uses the evaluation of $q + 1$ Poisson brackets involving polynomials of order p and q and to keep in stack $k + 1$ terms of order k.

For the computing of K^w and w, we can use the same algorithm. Given $h = h_0 + \sum_{n \geq 1} \varepsilon^n h_n$, we construct the same array $K^w(p, q)$ but we solve at each loop

$$L_{h_0} w_n + K_n^w = R_n^w = \sum_{p=1}^{n-1} \frac{p}{n}\{w_p, K^w(n-p, 0)\} + \sum_{p=1}^{n-1} K^w(n-p, p). \quad (72)$$

We will use a loop like

for $i := 1$ **to** r **do**
  for $j := 1$ **to** $i - 1$ **do**
    for $k := 1$ **to** j **do** $K_{j,i-j} := K_{j,i-j} + \frac{k}{j}\{w_k, K_{j-k,i-j}\}$;
    $R_i := R_i + K_{j,i-j}$;
  for $k := 1$ **to** $i - 1$ **do**
    $R_i := R_i + \frac{k}{i}\{w_k, K_{i-k,0}\}$
    $K_{i,0} := K_{i,0} + \frac{k}{i}\{w_k, K_{i-k,0}\}$;
  solve $L_{h_0} w_i + K_i = R_i$
  $K_{i,0} := K_{i,0} + \{w_i, K_{0,i}\}$.

8.2. THE DRAGT-FINN INVERSE TRANSFORM

As we do not know any closed formula for M_n^{-1}, the computation of $\sum_{m=0}^{n} M_{n-m}^{-1} f_m$ would require $\sum_{i \le n} p(i)$ steps which is quite less than 2^n. The fact that M^{-1} is the product of single transformations allow us to use the following algorithm. Given a function $f = \sum_{n \ge 0} \varepsilon^n f_n$, we consider the auxiliary series of functions, $F^0 = f$, $F^k = e^{\varepsilon^k L_{g_k}} F^{k-1}$. We have

$$F_n^k = \sum_{jk+p=n} \frac{1}{j!} L_{g_k}^j F_p^{k-1} \tag{73}$$

so $F_n = F_n^n$. Given $F_{|r}^{k-1} = \sum_{p=0}^{r} \varepsilon^p F_p^{k-1}$, we construct successively $F_{|r}^{k,0} = F_{|r}^{k-1}$, $F_{|r}^{k,i} = \frac{1}{i} L_{g_k} F_{|r}^{k,i-1}$ while $i \le \left[\frac{r}{k}\right] = \max\{n \in \mathbb{N}; n \le \frac{r}{k}\}$. We obtain $F_{|r}^k$ as $\sum_{ki \le r} F_{|r}^{k,i}$. We will use the following construction

$$\begin{array}{ccccccccc}
f_0 & f_1 & f_2 & \cdots & f_i & \cdots & \cdots & f_r & \\
 & F_1^1 & F_2^1 & F_3^1 & & \ddots & \vdots & F_r^1 & \\
 & & \ddots & \ddots & & \ddots & & & \\
 & & & \ddots & \ddots & & \ddots & \vdots & \\
 & & & & F_i^i & F_{i+1}^i & \cdots & F_r^i & \\
 & & & & & \ddots & & \vdots & \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \cdots & F_r^r & \\
F_0 & F_1 & F_2 & \cdots & F_i & \cdots & & F_r &
\end{array} \tag{74}$$

with the following algorithm

for $i := 1$ **to** r **do**
  for $k := i$ **to** r **do**
    for $j := 1$ **to** $\left[\frac{k}{i}\right]$ **do** $F_{i+j-1,k} := \frac{1}{j}\{g_i, F_{i+j-2,k-i}\}$;
  for $k := i$ **to** r **do**
    for $j := 2$ **to** $\left[\frac{k}{i}\right]$ **do** $F_{i,k} := F_{i,k} + F_{i+j-1,k}$;
    $F_{i,k} = F_{i,k} + F_{i-1,k}$.

The array is filled line by line, and that is the main difference with the preceding algorithm. We stack temporarily the $F_k^{i,j}$ in the same array in the $(i+j-1)^{\text{th}}$ line. A Poisson bracket involving a term of order ε^p and a term of order ε^q is obtained when $i = p$ and $k - i = q$, so the number of such Poisson brackets is $\left[\frac{p+q}{p}\right]$.

On the other hand, this algorithm uses an array which is partially erased after each loop. This array is filled with the terms $F_p^{i,j}$ for $ij \le p \le r$. During the computing of the F^i, the array is filled first by the $F^{i'}$ for $0 \le i' \le i$ that gives

$\inf(i, k+1)$ terms of a given order k and by the $F^{i,j}$ that gives also $[\frac{k}{i}]$ terms of order k. We thus have simultaneously $\inf(i, k+1) + [\frac{k}{i}]$ terms of order k in the array. If $i\epsilon k$, we get $\inf(i, k+1) + [\frac{k}{i}] = k+1$, otherwise $\inf(i, k+1) + [\frac{k}{i}] = i + [\frac{k}{i}] \leq i + \frac{k}{i} \leq k+1$ so the array contains at most $k+1$ terms of order k.

For the computation of K^g and g, we will use the same algorithm but from

$$K_n = K_n^n = K_n^{n-1} + L_{g_n} h_0, \tag{75}$$

we will replace the computation of $K_n^{n,1}$ with the resolution of the equation (45). Thus the algorithm we will use is

for $i := 1$ **to** r **do**
  solve $L_{h_0} g_i + K_{i,i} = K_{i-1,i}$;
  for $k := i+1$ **to** r **do**
    for $j := 1$ **to** $[\frac{k}{i}]$ **do** $K_{i+j-1,k} := \frac{1}{j}\{g_i, K_{i+j-2,k-i}\}$;
  for $k := i$ **to** r **do**
    for $j := 2$ **to** $[\frac{k}{i}]$ **do** $K_{i,k} := K_{i,k} + K_{i+j-1,k}$
    $K_{i,k} := K_{i,k} + K_{i-1,k}$.

9. Number of Poisson Brackets Evaluations

From $[\frac{p+q}{p}] \leq \frac{p+q}{p} \leq q+1$, we deduce that for given orders p and q, the computation of the Dragt transform uses less Poisson brackets involving polynomials of order p and q than the Deprit transform. Furthermore the total number of Poisson brackets required to compute $F_0, \ldots, F_r$ with the Lie transform is

$$\begin{aligned} t'_r &= \sum_{1 \leq p+q \leq r} q+1 = \sum_{i=1}^{r} \sum_{q=0}^{i-1} q+1 \\ &= \sum_{i=1}^{r} \frac{i(i+1)}{2} = \frac{r(r+1)(r+2)}{6} = \frac{r^3}{6} + O(r^2). \end{aligned} \tag{76}$$

For the Dragt transform it is

$$\begin{aligned} m'_r &= \sum_{1 \leq p+q \leq r} [\tfrac{p+q}{p}] = \sum_{i=1}^{r} \sum_{p=1}^{i} (\frac{i}{p} + O(1)) \\ &= \sum_{i=1}^{r} (i \log i + O(i)) = \frac{1}{2} r^2 \log r + O(r^2). \end{aligned} \tag{77}$$

For given orders r of the perturbation, we give in table (I) the number of products of Lie operators involved in $\sum_{k \leq r} (T^{-1})_k$ (resp. $\sum_{k \leq r} (M^{-1})_k$) and the number

TABLE I

Transformations Cost

	Lie Operators		Evaluated Poisson brackets	
Order	$\sum_{k\leq r}(M^{-1})_k$	$\sum_{k\leq r}(T^{-1})_k$	$F_r = M^{-1}f$	$F_r = T^{-1}f$
1	1	1	1	1
2	3	3	4	4
3	6	7	9	10
4	11	15	17	20
5	18	31	27	35
6	29	63	41	56
7	44	127	57	84
8	66	255	77	120
9	96	511	100	165
10	138	1023	127	220
15	683	32767	314	680
20	2713	1048575	600	1540
30	28628	1073741823	1492	4960

of Poisson brackets evaluated during the computation by the preceding algorithms of $F_{|r} = T^{-1}f$ (resp. M^{-1}).

The Dragt-Finn transform requires clearly less Poisson brackets evaluations than the Deprit transform. On the other hand, the Deprit transform and the Dragt transform both require to keep in stack $k+1$ terms of order k (that can be polynomials of degree k or $k+2$) that are necessary either to succeed the computing at the order $r+1$ or to reduce the cost for the computation of next orders in the future.

10. Conclusion

As the Deprit method and the Dragt-Finn methods produce the same new Hamiltonians when the frequencies of h_0 have no linear dependencies over $\mathbb{Q}$, it seems that the Dragt-Finn one is more efficient, regards to the number of steps required for the computation. Nevertheless, the Deprit method can be more easily adapted for non-autonomous hamiltonian systems (see [3]) and is may be more practical to implement, while it does not need any use of auxiliary functions that are erased after each loop. This aspect could explain the apparent better behavior of the Deprit transform in [8], especially when the order of the perturbation is low.

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Appendix

A. Relations between the Lie Transformations

We give in this section the first relations between the generators of the two considered canonical transforms. We wrote these formulas down with respect to the Lyndon basis with the reverse lexicographic order. We just recall a few definitions (see [17; 20]).

Given an alphabet A (finite or endless), we consider A^* the set of the words built from A. This set is ordered with the lexicographic order. If $w = uv$ is a word we will say that vu is conjugate to w and the Lyndon words are the words not greater than all their conjugates. Each Lyndon word is the concatenation of two Lyndon words but the factorization is not unique. We will call it standard factorization when the right factor is as great as possible. There is a one-to-one mapping between the Lyndon words and the Lyndon brackets in the free Lie algebra over A. Let w be a Lyndon word. If w is a single letter than the associated Lie bracket $\Lambda(w)$ is w, otherwise we have $\Lambda(w) = [\Lambda(u), \Lambda(v)]$ where $w = uv$ is the standard factorization. The Lyndon brackets are a basis of the free Lie algebra over the alphabet A. If we take for A the infinite alphabet $g_1, \ldots, g_k, \ldots$ with the reverse lexicographic order then the formula (39) is the decomposition on the Lyndon basis.

A.1. The Deprit Generators

$$
\begin{aligned}
w_1 &= g_1 \\
w_2 &= g_2 \\
w_3 &= g_3 + \tfrac{1}{3}\{g_2, g_1\} \\
w_4 &= g_4 + \tfrac{1}{4}\{g_3, g_1\} \\
w_5 &= g_5 + \tfrac{2}{5}\{g_3, g_2\} + \tfrac{1}{5}\{g_4, g_1\} + \tfrac{1}{10}\{g_2, \{g_2, g_1\}\} \\
w_6 &= g_6 + \tfrac{1}{3}\{g_4, g_2\} + \tfrac{1}{6}\{g_5, g_1\} + \tfrac{1}{6}\{g_3, \{g_2, g_1\}\} \\
w_7 &= g_7 + \tfrac{3}{7}\{g_4, g_3\} + \tfrac{2}{7}\{g_5, g_2\} + \tfrac{1}{7}\{g_6, g_1\} + \tfrac{1}{14}\{g_3, \{g_3, g_1\}\} \\
&\quad + \tfrac{1}{7}\{g_4, \{g_2, g_1\}\} + \tfrac{1}{42}\{g_2, \{g_2, \{g_2, g_1\}\}\} \\
w_8 &= g_8 + \tfrac{3}{8}\{g_5, g_3\} + \tfrac{1}{4}\{g_6, g_2\} + \tfrac{1}{8}\{g_7, g_1\} + \tfrac{1}{8}\{g_3, \{g_3, g_2\}\} \\
&\quad + \tfrac{1}{8}\{g_4, \{g_3, g_1\}\} + \tfrac{1}{8}\{g_5, \{g_2, g_1\}\} + \tfrac{1}{16}\{g_3, \{g_2, \{g_2, g_1\}\}\} \\
w_9 &= g_9 + \tfrac{4}{9}\{g_5, g_4\} + \tfrac{1}{3}\{g_6, g_3\} + \tfrac{2}{9}\{g_7, g_2\} + \tfrac{1}{9}\{g_8, g_1\} + \tfrac{2}{9}\{g_4, \{g_3, g_2\}\} \\
&\quad + \tfrac{1}{18}\{g_4, \{g_4, g_1\}\} + \tfrac{1}{9}\{g_5, \{g_3, g_1\}\} + \tfrac{1}{9}\{g_6, \{g_2, g_1\}\} \\
&\quad + \tfrac{1}{18}\{g_3, \{g_3, \{g_2, g_1\}\}\} + \tfrac{1}{18}\{g_4, \{g_2, \{g_2, g_1\}\}\} \\
&\quad + \tfrac{1}{216}\{g_2, \{g_2, \{g_2, \{g_2, g_1\}\}\}\} \\
w_{10} &= g_{10} + \tfrac{2}{5}\{g_6, g_4\} + \tfrac{3}{10}\{g_7, g_3\} + \tfrac{1}{5}\{g_8, g_2\} + \tfrac{1}{10}\{g_9, g_1\}
\end{aligned}
$$

$$+\tfrac{1}{10}\{g_4,\{g_4,g_2\}\}+\tfrac{1}{5}\{g_5,\{g_3,g_2\}\}+\tfrac{1}{10}\{g_5,\{g_4,g_1\}\}+\tfrac{1}{10}\{g_6,\{g_3,g_1\}\}$$
$$+\tfrac{1}{10}\{g_7,\{g_2,g_1\}\}+\tfrac{1}{60}\{g_3,\{g_3,\{g_3,g_1\}\}\}+\tfrac{1}{10}\{g_4,\{g_3,\{g_2,g_1\}\}\}$$
$$+\tfrac{1}{20}\{g_5,\{g_2,\{g_2,g_1\}\}\}+\tfrac{1}{60}\{g_3,\{g_2,\{g_2,\{g_2,g_1\}\}\}\}$$

$$w_{11}=g_{11}+\tfrac{5}{11}\{g_6,g_5\}+\tfrac{4}{11}\{g_7,g_4\}+\tfrac{3}{11}\{g_8,g_3\}+\tfrac{2}{11}\{g_9,g_2\}$$
$$+\tfrac{1}{11}\{g_{10},g_1\}+\tfrac{3}{22}\{g_4,\{g_4,g_3\}\}+\tfrac{2}{11}\{g_5,\{g_4,g_2\}\}+\tfrac{1}{22}\{g_5,\{g_5,g_1\}\}$$
$$+\tfrac{2}{11}\{g_6,\{g_3,g_2\}\}+\tfrac{1}{11}\{g_6,\{g_4,g_1\}\}+\tfrac{1}{11}\{g_7,\{g_3,g_1\}\}$$
$$+\tfrac{1}{11}\{g_8,\{g_2,g_1\}\}+\tfrac{1}{33}\{g_3,\{g_3,\{g_3,g_2\}\}\}+\tfrac{1}{22}\{g_4,\{g_3,\{g_3,g_1\}\}\}$$
$$+\tfrac{1}{22}\{g_4,\{g_4,\{g_2,g_1\}\}\}+\tfrac{1}{11}\{g_5,\{g_3,\{g_2,g_1\}\}\}+\tfrac{1}{22}\{g_6,\{g_2,\{g_2,g_1\}\}\}$$
$$+\tfrac{1}{44}\{g_3,\{g_3,\{g_2,\{g_2,g_1\}\}\}\}+\tfrac{1}{66}\{g_4,\{g_2,\{g_2,\{g_2,g_1\}\}\}\}$$
$$+\tfrac{1}{1320}\{g_2,\{g_2,\{g_2,\{g_2,\{g_2,g_1\}\}\}\}\}$$

$$w_{12}=g_{12}+\tfrac{5}{12}\{g_7,g_5\}+\tfrac{1}{3}\{g_8,g_4\}+\tfrac{1}{4}\{g_9,g_3\}+\tfrac{1}{6}\{g_{10},g_2\}+\tfrac{1}{12}\{g_{11},g_1\}$$
$$+\tfrac{1}{4}\{g_5,\{g_4,g_3\}\}+\tfrac{1}{12}\{g_5,\{g_5,g_2\}\}+\tfrac{1}{6}\{g_6,\{g_4,g_2\}\}+\tfrac{1}{12}\{g_6,\{g_5,g_1\}\}$$
$$+\tfrac{1}{6}\{g_7,\{g_3,g_2\}\}+\tfrac{1}{12}\{g_7,\{g_4,g_1\}\}+\tfrac{1}{12}\{g_8,\{g_3,g_1\}\}+\tfrac{1}{12}\{g_9,\{g_2,g_1\}\}$$
$$+\tfrac{1}{12}\{g_4,\{g_3,\{g_3,g_2\}\}\}+\tfrac{1}{24}\{g_4,\{g_4,\{g_3,g_1\}\}\}+\tfrac{1}{24}\{g_5,\{g_3,\{g_3,g_1\}\}\}$$
$$+\tfrac{1}{12}\{g_5,\{g_4,\{g_2,g_1\}\}\}+\tfrac{1}{12}\{g_6,\{g_3,\{g_2,g_1\}\}\}+\tfrac{1}{24}\{g_7,\{g_2,\{g_2,g_1\}\}\}$$
$$+\tfrac{1}{72}\{g_3,\{g_3,\{g_3,\{g_2,g_1\}\}\}\}+\tfrac{1}{24}\{g_4,\{g_3,\{g_2,\{g_2,g_1\}\}\}\}$$
$$+\tfrac{1}{72}\{g_5,\{g_2,\{g_2,\{g_2,g_1\}\}\}\}+\tfrac{1}{288}\{g_3,\{g_2,\{g_2,\{g_2,\{g_2,g_1\}\}\}\}\}$$

A.2. THE DRAGT-FINN GENERATORS

We have just inverted the previous triangular system and rewritten it on the Lyndon basis untill the order 8

$$g_1=w_1$$
$$g_2=w_2$$
$$g_3=w_3-\tfrac{1}{3}\{w_2,w_1\}$$
$$g_4=w_4-\tfrac{1}{4}\{w_3,w_1\}+\tfrac{1}{12}\{\{w_2,w_1\},w_1\}$$
$$g_5=w_5-\tfrac{2}{5}\{w_3,w_2\}-\tfrac{1}{5}\{w_4,w_1\}-\tfrac{7}{30}\{w_2,\{w_2,w_1\}\}$$
$$+\tfrac{1}{20}\{\{w_3,w_1\},w_1\}-\tfrac{1}{60}\{\{\{w_2,w_1\},w_1\},w_1\}$$
$$g_6=w_6-\tfrac{1}{3}\{w_4,w_2\}-\tfrac{1}{6}\{w_5,w_1\}-\tfrac{1}{10}\{w_3,\{w_2,w_1\}\}+\tfrac{3}{20}\{\{w_3,w_1\},w_2\}$$
$$+\tfrac{1}{30}\{\{w_4,w_1\},w_1\}+\tfrac{1}{15}\{w_2,\{\{w_2,w_1\},w_1\}\}-$$
$$\tfrac{1}{120}\{\{\{w_3,w_1\},w_1\},w_1\}+\tfrac{1}{360}\{\{\{\{w_2,w_1\},w_1\},w_1\},w_1\}$$
$$g_7=w_7-\tfrac{3}{7}\{w_4,w_3\}-\tfrac{2}{7}\{w_5,w_2\}-\tfrac{1}{7}\{w_6,w_1\}-\tfrac{5}{28}\{w_3,\{w_3,w_1\}\}$$

$$+ \tfrac{1}{21} \{w_4, \{w_2, w_1\}\} + \tfrac{4}{35} \{\{w_3, w_2\}, w_2\} + \tfrac{11}{105} \{\{w_4, w_1\}, w_2\}$$
$$+ \tfrac{1}{42} \{\{w_5, w_1\}, w_1\} - \tfrac{19}{210} \{w_2, \{w_2, \{w_2, w_1\}\}\}$$
$$+ \tfrac{31}{420} \{w_3, \{\{w_2, w_1\}, w_1\}\} - \tfrac{13}{420} \{\{w_3, w_1\}, \{w_2, w_1\}\} -$$
$$\tfrac{1}{28} \{\{\{w_3, w_1\}, w_1\}, w_2\} - \tfrac{1}{210} \{\{\{w_4, w_1\}, w_1\}, w_1\} -$$
$$\tfrac{1}{70} \{w_2, \{\{\{w_2, w_1\}, w_1\}, w_1\}\} - \tfrac{11}{630} \{\{w_2, w_1\}, \{\{w_2, w_1\}, w_1\}\}$$
$$+ \tfrac{1}{840} \{\{\{\{w_3, w_1\}, w_1\}, w_1\}, w_1\} - \tfrac{1}{2520} \{\{\{\{\{w_2, w_1\}, w_1\}, w_1\}, w_1\}, w_1\}$$
$$g_8 = w_8 - \tfrac{3}{8} \{w_5, w_3\} - \tfrac{1}{4} \{w_6, w_2\} - \tfrac{1}{8} \{w_7, w_1\} - \tfrac{11}{40} \{w_3, \{w_3, w_2\}\} -$$
$$\tfrac{1}{14} \{w_4, \{w_3, w_1\}\} + \tfrac{1}{28} \{w_5, \{w_2, w_1\}\} + \tfrac{9}{70} \{\{w_4, w_1\}, w_3\}$$
$$+ \tfrac{1}{12} \{\{w_4, w_2\}, w_2\} + \tfrac{13}{168} \{\{w_5, w_1\}, w_2\} + \tfrac{1}{56}\{\{w_6, w_1\}, w_1\} -$$
$$\tfrac{26}{105} \{w_3, \{w_2, \{w_2, w_1\}\}\} + \tfrac{23}{560}\{w_3, \{\{w_3, w_1\}, w_1\}\}$$
$$+ \tfrac{1}{28} \{w_4, \{\{w_2, w_1\}, w_1\}\} - \tfrac{2}{105} \{\{w_4, w_1\}, \{w_2, w_1\}\} -$$
$$\tfrac{19}{420} \{\{w_3, \{w_2, w_1\}\}, w_2\} - \tfrac{29}{560} \{\{\{w_3, w_1\}, w_2\}, w_2\} -$$
$$\tfrac{3}{140} \{\{\{w_4, w_1\}, w_1\}, w_2\} - \tfrac{1}{336} \{\{\{w_5, w_1\}, w_1\}, w_1\}$$
$$+ \tfrac{47}{1680} \{w_2, \{w_2, \{\{w_2, w_1\}, w_1\}\}\} - \tfrac{13}{840} \{w_3, \{\{\{w_2, w_1\}, w_1\}, w_1\}\} -$$
$$\tfrac{3}{560} \{\{w_3, w_1\}, \{\{w_2, w_1\}, w_1\}\} - \tfrac{29}{630} \{\{w_2, \{w_2, w_1\}\}, \{w_2, w_1\}\}$$
$$+ \tfrac{1}{120}\{\{\{w_3, w_1\}, w_1\}, \{w_2, w_1\}\} + \tfrac{11}{1680} \{\{\{\{w_3, w_1\}, w_1\}, w_1\}, w_2\}$$
$$+ \tfrac{1}{1680} \{\{\{\{w_4, w_1\}, w_1\}, w_1\}, w_1\}$$
$$+ \tfrac{5}{2016} \{w_2, \{\{\{\{w_2, w_1\}, w_1\}, w_1\}, w_1\}\}$$
$$+ \tfrac{1}{252} \{\{w_2, w_1\}, \{\{\{w_2, w_1\}, w_1\}, w_1\}\} -$$
$$\tfrac{1}{6720} \{\{\{\{\{w_3, w_1\}, w_1\}, w_1\}, w_1\}, w_1\}$$
$$+ \tfrac{1}{20160} \{\{\{\{\{\{w_2, w_1\}, w_1\}, w_1\}, w_1\}, w_1\}, w_1\}$$

B. Normal Forms

B.1. THE PENDULUM

We give the normal form up to order 30 of the Hamiltonian of a pendulum near the origin which is an equilibrium point. We give also the generator associated to the corresponding Dragt transform.

Hamiltonian (up to order 30)

$$
\begin{aligned}
h = \tfrac{1}{2}p^2 + (1-\cos q) = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 - \tfrac{1}{24}q^4 + \tfrac{1}{720}q^6 - \tfrac{1}{40320}q^8 + \tfrac{1}{3628800}q^{10} \\
- \tfrac{1}{479001600}q^{12} + \tfrac{1}{87178291200}q^{14} - \tfrac{1}{20922789888000}q^{16} + \tfrac{1}{6402373705728000}q^{18} \\
- \tfrac{1}{2432902008176640000}q^{20} + \tfrac{1}{1124000727777607680000}q^{22} \\
- \tfrac{1}{620448401733239439360000}q^{24} + \tfrac{1}{403291461126605635584000000}q^{26} \\
- \tfrac{1}{304888344611713860501504000000}q^{28} + \tfrac{1}{265252859812191058636308480000000}q^{30}
\end{aligned}
$$

New Hamiltonian (up to order 30)

$$
\begin{aligned}
K = \tfrac{1}{2}\left(q^2+p^2\right) - \tfrac{1}{64}\left(q^2+p^2\right)^2 - \tfrac{1}{2048}\left(q^2+p^2\right)^3 - \tfrac{5}{131072}\left(q^2+p^2\right)^4 \\
- \tfrac{33}{8388608}\left(q^2+p^2\right)^5 - \tfrac{63}{134217728}\left(q^2+p^2\right)^6 - \tfrac{527}{8589934592}\left(q^2+p^2\right)^7 \\
- \tfrac{9387}{1099511627776}\left(q^2+p^2\right)^8 - \tfrac{175045}{140737488355328}\left(q^2+p^2\right)^9 \\
- \tfrac{422565}{2251799813685248}\left(q^2+p^2\right)^{10} - \tfrac{4194753}{144115188075855872}\left(q^2+p^2\right)^{11} \\
- \tfrac{1330745}{288230376151711744}\left(q^2+p^2\right)^{12} - \tfrac{4403374207}{590295810358705651712}\left(q^2+p^2\right)^{13} \\
- \tfrac{578183175}{4722366482869645213696}\left(q^2+p^2\right)^{14} - \tfrac{12308013927}{604462909807314587353088}\left(q^2+p^2\right)^{15}
\end{aligned}
$$

Generating function (up to order 9)

$$
\begin{aligned}
g = -\tfrac{1}{64}qp^3 - \tfrac{5}{192}q^3p - \tfrac{1}{1024}qp^5 - \tfrac{1}{384}q^3p^3 - \tfrac{17}{15360}q^5p - \tfrac{11}{131072}qp^7 - \tfrac{121}{393216}q^3p^5 \\
- \tfrac{1513}{5898240}q^5p^3 - \tfrac{2657}{24772608}q^7p
\end{aligned}
$$

Position variable with respect to the new variables (up to order 9)

$$
\begin{aligned}
Q = q - \tfrac{3}{64}qp^2 - \tfrac{5}{192}q^3 - \tfrac{37}{8192}qp^4 - \tfrac{37}{4096}q^3p^2 - \tfrac{11}{122880}q^5 - \tfrac{39}{524288}qp^6 \\
- \tfrac{731}{1572864}q^3p^4 - \tfrac{727}{1572864}q^5p^2 + \tfrac{1415}{14155776}q^7 + \tfrac{83}{134217728}qp^8 - \tfrac{463}{100663296}q^3p^6 \\
- \tfrac{1687}{67108864}q^5p^4 - \tfrac{51281}{1509949440}q^7p^2 - \tfrac{438161}{54358179840}q^9
\end{aligned}
$$