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Energy Transfer and Phase Slip by Quantum Vortex Motion in Superfluid ^{4}He

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The purpose of the present article is to emphasize the usefulness of the ideas of E. R. Huggins in thinking about vortex motion and phase slip in superfluid ^{4}He, and is primarily pedagogical. Several explicit illustrations of vortex motion and phase-slip processes are considered. In addition, it is shown that Huggins's results lead to a generalization and a more complete understanding of the familiar expression $E + \mathbf{v}_s \cdot \mathbf{p}$ for the energy in the rest system of an excitation in the flowing superfluid, as applied to vortex excitations. Here, E is the energy and **p** *is the momentum of the excitation in the moving system, and $\mathbf{v}_s$ is the superfluid velocity.*

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1. INTRODUCTION

Some years ago, E. R. Huggins derived expressions for energy transfer in a classical ideal incompressible fluid in which singular vortex lines are present.[1] This work continued a line of development pursued by P. W. Anderson and others in previous years to describe phase slip in superfluid ^{4}He by the motion of quantum vortices,[2] and was followed by the work of K. W. Schwarz and others.[3] In recent years, the observation of discrete phase-slip events in the flow of superfluid ^{4}He through tiny apertures[4-6] and the observation in these apertures of critical velocities which decreased with temperature in a consistent way[5-8] have stimulated new interest in this area.[9,10]

The purpose of the present article is to emphasize the usefulness of Huggins's results in thinking about vortex motion and phase slip, and is primarily pedagogical. Several explicit illustrations of vortex motion and phase-slip processes are considered. In addition, it is shown that Huggins's results lead to a generalization and a more complete understanding of the familiar expression $E + \mathbf{v}_s \cdot \mathbf{p}$ for the energy in the rest system of an excitation in the flowing superfluid, as applied to vortex excitations. Here, E is the energy and **p** is the momentum of the excitation in the moving system, and $\mathbf{v}_s$ is the

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superfluid velocity. A somewhat different approach to energy transfers in vortex motion and phase slip in superfluid ^4He is contained in the article by K. W. Schwarz to appear in this same journal issue.

In this article, we shall assume that the results for a classical ideal incompressible fluid in the limit of an infinitesimal vortex core radius can be applied directly to the superfluid component of ^4He. At a minimum, this assumption limits velocities to much less than the speed of sound (and second sound in the two-fluid region) and lengths to much larger than the interatomic distance in liquid ^4He. In making this application we shall assume that with an appropriate choice of core radius parameter the superfluid ^4He vortex energy can be treated as if it were all kinetic.[11]

2. ENERGY TRANSFER

The results derived by Huggins on which this paper is based are the following. Imagine the flow of an ideal nonviscous incompressible fluid in a channel with fixed walls completely filled by the fluid. For definiteness and simplicity, let us imagine the channel to be simply-connected and to be closed at both ends by pistons which move with the fluid, as shown in Fig. 2.1. The velocities of the pistons may vary with time. Further, assume that in addition to potential flow between the pistons, vortices with filamentary cores may be present. These vortex lines either close upon themselves or terminate at the walls. For the application of this formalism to superfluid ^4He, we assume that all of the vortices have the same circulation κ. Further, let us assume that the ends of the flow region at the pistons are far away from a central region in which vortices are present. As suggested by the figure, we shall be interested primarily, but not exclusively, in cases in which the channel contains a

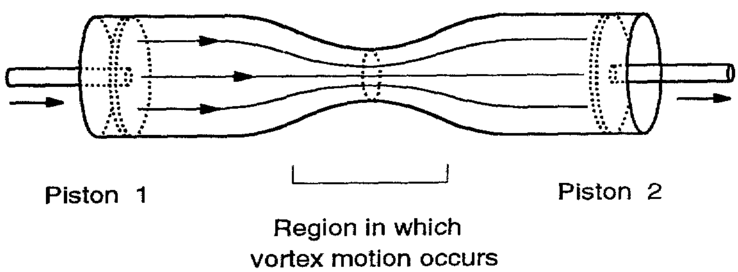

Fig. 2.1. Schematic view of the flow tube in which the fluid is contained between two pistons that are located far from the region in which vortex motion takes place.

constriction, in particular a small aperture in a partition.

For a given velocity of one of the pistons and a given configuration of vortex lines, the velocity field $\mathbf{v}(\mathbf{r})$ is uniquely determined and may uniquely be decomposed into a sum of a potential flow field $\mathbf{v}_p(\mathbf{r})$ and a vortex flow field $\mathbf{v}_v(\mathbf{r})$. Furthermore, as Huggins shows, the total kinetic energy of the fluid T may be written as the sum of a kinetic energy of potential flow T_p and a kinetic energy of vortex flow T_v, without any interaction term, assuming that $\mathbf{v}_v = 0$ at the pistons, so that the normal component of $\mathbf{v}_v$ at the boundaries is everywhere zero.

Following Huggins, we assume that in addition to the force on the fluid exerted by the walls due to the pressure of the fluid, there are at most a conservative body force per unit mass $\mathbf{g}_\Omega(\mathbf{r})$ acting on the fluid derivable from a potential $\Omega(\mathbf{r})$ ($\mathbf{g}_\Omega = -\,\mathrm{grad}\,\Omega$) and a local force per unit mass $\mathbf{g}_e(\mathbf{r})$ that cannot be derived from a potential and that acts on the fluid only near the vortex cores. Forces of type $\mathbf{g}_e$ might arise from the action of an electric field on an ion trapped on a vortex core or from the scattering of elementary excitations in the fluid or in the walls by the cores.

It is then possible to show that the time derivative of T_p and T_v are given by the expressions

$$dT_p/dt = I\,(\chi_1 - \chi_2) + \rho \int \mathbf{v}_p \cdot (\mathbf{v} \times \omega + \mathbf{g}_e)\, dV, \tag{2.1}$$

$$dT_v/dt = -\rho \int \mathbf{v}_p \cdot (\mathbf{v} \times \omega + \mathbf{g}_e)\, dV + \rho \int \mathbf{v} \cdot \mathbf{g}_e\, dV. \tag{2.2}$$

Here $I = I_p$ is the total mass current flowing in the channel from piston 1 to piston 2. The potential χ is given by the expression

$$\chi = \Omega + P/\rho, \tag{2.3}$$

where P is the pressure and ρ is the density, and χ_1 and χ_2 are the values of χ at pistons 1 and 2, respectively, at each of which χ is assumed to be uniform. The integrals run over the entire volume of the fluid, and $\omega = \mathrm{curl}\mathbf{v} = \mathrm{curl}\mathbf{v}_v$. The derivation of Eq. (2.2) depends on the normal component of $\mathbf{v}_v$ being zero at the boundaries, an additional reason for assuming that any vorticity is located far from the pistons. (These expressions differ slightly from the corresponding ones given by Huggins, in that here the time derivatives are total derivatives for a region with moving boundaries (at the pistons) rather than derivatives for a region with boundaries at the pistons fixed at the instantaneous positions of the pistons.)

The first term on the right-hand side of Eq. (2.1) represents the rate at which work is being done on the fluid by the pistons and the force $\mathbf{g}_\Omega$, whereas the second term on the right-hand side of Eq. (2.2) is the rate at which work is being done on the fluid by the force $\mathbf{g}_e$. Hence when Eqs. (2.1) and (2.2) are added together, we have an overall kinetic-energy-work equation for the fluid. Furthermore, we may interpret the equations individually as if the pistons and force $\mathbf{g}_\Omega$ act only to change the energy of potential flow and the force $\mathbf{g}_e$ acts only to change the energy of vortex flow, if at the same time the expression

$$- \rho \int \mathbf{v}_p \cdot (\mathbf{v} \times \omega + \mathbf{g}_e)\, dV \tag{2.4}$$

is taken to represent the rate dE/dt (p → v) at which energy is transferred from potential flow to vortex flow. We may then rewrite Eqs. (2.1) and (2.2) in the shortened forms

$$dT_p/dt = dW_p/dt - dE/dt\,(p \rightarrow v), \tag{2.5}$$

$$dT_v/dt = dE/dt\,(p \rightarrow v) + dW_e/dt, \tag{2.6}$$

where the terms correspond to those in Eqs. (2.1) and (2.2).

In the absence of work being done on the fluid by the pistons, by $\mathbf{g}_\Omega$, and by $\mathbf{g}_e$, the interpretation of expression (2.4) as dE/dt (p → v) seems clearly established. However, in the presence of work being done by any of these means, the interpretations above do not seem to have any justification beyond Eqs. (2.1) and (2.2). Nevertheless, they provide a consistent and useful bookkeeping scheme for energy transfers involving the fluid.

As Huggins has shown, the expression (2.4) for dE/dt (p → v) has a simple geometrical form when as in the present case, the vorticity ω is concentrated in filamentary vortex cores. First, we may transform the volume integral in (2.4) into a line integral along the vortex cores, obtaining

$$dE/dt\,(p \rightarrow v) = - \rho \int \mathbf{v}_p \cdot (\mathbf{v}_{av} \times \kappa + \mathbf{h}_e)\, dl, \tag{2.7}$$

where $\mathbf{v}_{av}$ is a vorticity-weighted average velocity at the core given by

$$\mathbf{v}_{av} \times \kappa = \int \mathbf{v} \times \omega\, dS \tag{2.8}$$

integrated over a cross section of the core perpendicular to the axis of the core. In these formulas, κ is a vector whose magnitude equals the circulation of the vortex and whose direction is parallel to the core in the direction of ω, $\rho\mathbf{h}_e$ is the force per unit length of vortex due to $\mathbf{g}_e$ acting on the fluid, and dl is an element of length along the vortex core.

The transverse velocity of the core $\mathbf{v}_c$ satisfies the relationship[1]

$$\mathbf{v}_c \times \kappa = \int (\mathbf{v} \times \omega + \mathbf{g}_{e\,perp})\, dS = \mathbf{v}_{av} \times \kappa + \mathbf{h}_{e\,perp}, \tag{2.9}$$

where the subscript perp denotes the component perpendicular to the core. Hence if we assume that in their entirety $\mathbf{g}_e$ and $\mathbf{h}_e$ act locally perpendicular to the core,[12] we then have

$$dE/dt\,(p \rightarrow v) = - \rho\, \kappa \int \mathbf{v}_p \cdot \mathbf{v}_c \times \mathbf{dl}, \tag{2.10}$$

where we have replaced κdl by κ**dl**.

The expression $- \mathbf{v}_c \times \mathbf{dl}$ represents the rate at which the element of core dl is sweeping out area, as illustrated in Fig. 2.2. (The sign is chosen here to produce a result in agreement with Huggins's convention.) Thus, the dot product of $\rho \mathbf{v}_p$ with $-\mathbf{v}_c \times d\mathbf{l}$ represents the rate at which the mass current associated with $\mathbf{v}_p$ is crossed by dl. The right-hand side of Eq. (2.10) then represents the product of κ and the net rate at which potential flow mass current is being crossed by vortex cores, the sense, for a vortex loop, being such that when the net potential flow mass current through the loop decreases due to vortex motion, $dE/dt\ (p \to v)$ is positive. Here we assume that the positive sense of the area spanning the loop is defined in relation to the sense of κ by the usual right-hand rule. We may write Eq. (2.10) as

$$dE/dt\,(p \to v) = \kappa\,(dI_p/dt)_{\text{crossing}}. \tag{2.11}$$

We may now describe the dissipation of superfluid flow by vortex motion in the following way. If the vortex moves so as to sweep across streamlines of potential flow in the proper sense, energy is transferred from potential flow to vortex flow. Vortex flow energy may in some circumstances be expended concurrently against external forces, while in others it may be transported by vortex motion to some other part of the system to be expended later against external forces.

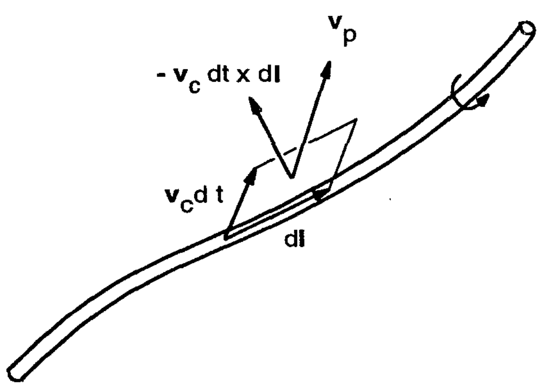

Fig. 2.2. Perspective view of a section of vortex core showing the area swept out by an element dl in time dt, an area through which potential flow with velocity $\mathbf{v}_p$ is taking place.

3. EXAMPLES

Let us now consider several examples of processes in which energy transfers into and out of vortex motion occur.

3.1. Example A

First consider the process envisaged by Iordanskii and Langer and Fisher as occurring above threshold in their homogeneous nucleation theories of superfluid critical velocity.[13,14] As shown in Fig. 3.1, a circular vortex ring with self-induced velocity directed to the left is present in the superfluid, which undergoes potential flow with uniform speed v_p to the right. Assume that the radius R of the ring is large enough so that the speed v_r of the ring relative to the fluid is less than v_p and that the ring is transported to the right with speed $v_p - v_r$. In the absence of any resistance, simple translation is all that will occur, the vortex ring remaining unchanged in size. No energy transfer will take place between potential and vortex flow.

In the presence of a stationary normal-fluid component, a resistive force will act on the ring to the left, causing the ring to expand. In so doing, the vortex core cuts streamlines of potential flow in such a direction as to transfer energy from potential flow to vortex flow. At the same time, the resistive force does negative work on the vortex, but this loss in vortex energy is over-balanced by the transfer of energy to the vortex from potential flow, and the net change in vortex energy is positive. Here we have an illustration of an energy transfer between potential and vortex flows that requires the concurrent action of a local force. (It is interesting to note that if we view this process from the frame of the superfluid, the work done by the resistive force on the vortex is positive, and the increase in vortex energy is simply equal to this work.)

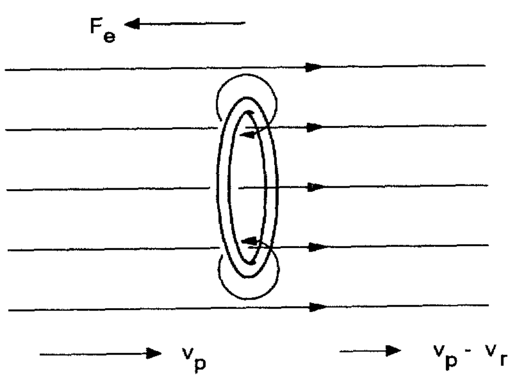

Fig. 3.1. Example A: A circular vortex ring with self-induced velocity v_r to the left is being swept to the right in a uniform potential flow field v_p, with $v_p > v_r$. A resistive force F_e acts on the vortex core to the left, causing the radius of the ring to grow.

3.2. Example B

Next consider a circular vortex ring placed symmetrically on the axis of a circular aperture through which potential flow is taking place, as shown in Fig. 3.2. In this example, discussed by Huggins,[1] let both the potential flow and the self-induced velocity of the vortex ring be directed to the right. In the absence of any local forces, the elements of the vortex core will tend to be swept to the right and outward along the streamlines of potential flow, at the same time moving in addition to the right with the self-induced velocity of the vortex. The dotted curves in Fig. 3.2 show a calculated representative trajectory of the core, neglecting any influence of the image of the vortex in the wall containing the aperture. In so moving, the vortex core cuts streamlines of potential flow in such a direction that $dE/dt\ (p \rightarrow v) > 0$, and the increase in vortex energy is manifest in the increase in ring radius. Here we have an illustration of a process in which transfer of energy from potential flow to vortex flow occurs in the presence of diverging potential flow without the action of any local forces.

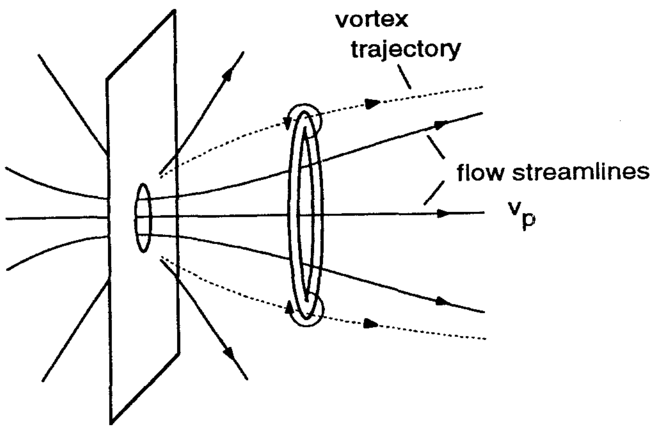

Fig. 3.2. Example B: A circular vortex ring with self-induced velocity to the right moves in a potential flow field directed to the right and diverging from an orifice in a plane wall. The dotted curves show a calculated representative trajectory for the core of the ring.

3.3. Example C

Third, consider the process illustrated in Fig. 3.3, in which half of a circular vortex ring moves in a potential flow field diverging in a half-space from a point, an idealization of the situation that we would have for potential

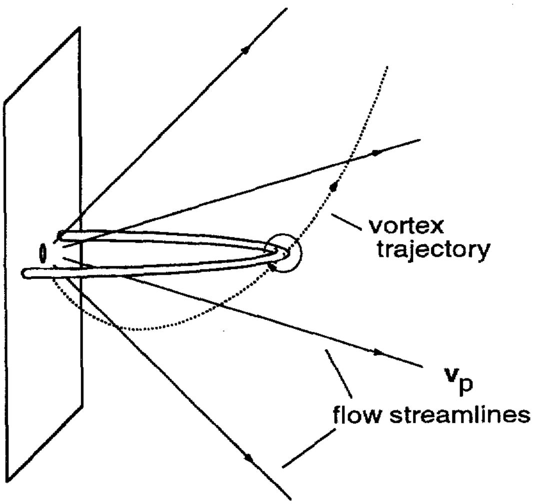

Fig. 3.3. Example C: A vortex loop with self-induced velocity upward moves in a potential flow field directed to the right and diverging from a small orifice in a plane wall. The dotted curve shows a calculated representative trajectory for the outermost portion of the core of the loop.

flow through a circular orifice in a plane wall if the ring radius were much larger than the radius of the orifice.[15] Here we assume that the axis of the half-ring lies in the half-plane and passes through the source. We also assume that the ends of the half-ring move freely over the walls without pinning. If as in Fig. 3.3 the self-induced motion of the half-ring is upward, a small initial half-ring below the origin in a sufficiently strong potential flow field will first be swept downward and outward from the half-plane. However, as the ring grows in radius, its self-induced motion will eventually dominate, and the ring will eventually move upward and outward. The dotted curve shows a calculated representative trajectory for the outermost portion of vortex core. Symmetry allows the half-ring to retain its plane circular shape throughout the motion, and its axis remains fixed.

In the absence of any local forces the motion of the core is a simple combination of radial outward convection in the potential flow field and self-induced motion perpendicular to the plane of the half-ring. In so moving, the core cuts streamlines of potential flow in such a direction as to transfer energy from potential flow to vortex flow as the half-ring grows in size. Here, as in Example B, we have a case in which energy transfer occurs without the action of local forces.

In this case, the trajectory shown in Fig. 3.3 is simply the result of the vector addition of a radial flow $v_p = (A/2\pi)r/r^3$, where A is the volume rate of flow from the point source into the right-hand half-space and **r** is a position vector with origin at the source, evaluated at the vortex core, and the upward self-induced velocity of the half-ring of magnitude $(\kappa/4\pi R)\ln(8R/ea)$, where R is the radius of the half-ring, e is the base of the natural logarithms, and a is the core radius parameter.[11]

3.4. Example D

Last, consider uniform flow with velocity v_p past a stationary rod ("wing") of finite length aligned perpendicular to the flow.[16] Suppose that there is a circulation κ around the rod and that attached to the ends of the rod is a vortex loop of circulation κ, as shown in Fig. 3.4. Assume that at the instant under consideration the vortex core lies in the plane of v_p and the rod. The right-hand end of the the vortex is being carried to the right with velocity v_p, and the upper and lower legs of the vortex are being lengthened. At the same time, the vortex experiences a nonuniform motion downward, induced by circulation around the rod and around the vortex itself. As a result, the right-hand end of the vortex cuts lines of potential flow in such a direction as to transfer energy from potential to vortex flow. Of course, the simple plane configuration considered here is a momentary one and will evolve rapidly into some more complicated nonplanar one.

We may calculate the instantaneous rate of energy transfer as follows. If the length of the right-hand end segment of the vortex is d, the vertical component of the velocity of an element of this segment at position x along the segment, in the limit that the segment is far from the rod, is due to the two (~semi-infinite) segments lying parallel to the flow and is given by

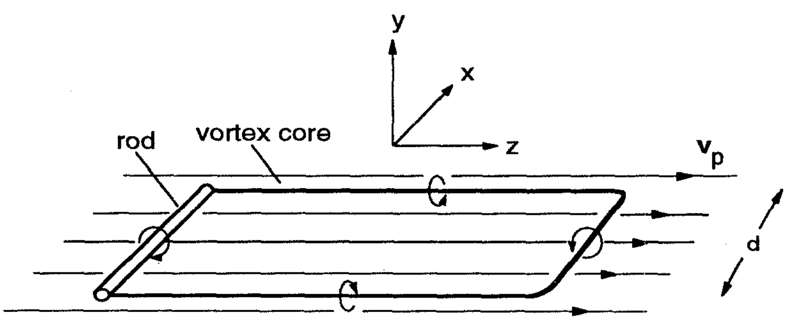

Fig. 3.4. Example D: A rectangular vortex loop, attached to a stationary rod and having a downward self-induced velocity, is being extended by uniform potential flow to the right.

$$v_{cy}(x) = -\kappa/[4\pi((d/2)-x)] - \kappa/[4\pi((d/2)+x)] \quad (3.1)$$

in terms of the axes shown in the figure. The origin is chosen to lie at the midpoint of the right-hand end segment. The rate $dE/dt\ (p \to v)$ may then be evaluated from Eq. (3.1) by integrating along the right-hand end segment. The result is

$$dE/dt\ (p \to v) = (\rho v_p \kappa^2/2\pi)\ln(d/\varepsilon), \quad (3.2)$$

if the integration is taken from $-(d/2)+\varepsilon$ to $(d/2)-\varepsilon$, ε being a cutoff that may be chosen to be the core radius parameter a. Note that this result is then equal to the product of v_p and the energy per unit length

$$E = (\rho\kappa^2/2\pi)\ln(d/a) \quad (3.3)$$

of a rectilinear pair of vortices with opposite circulations separated by a distance d.[11] This product is just the rate of increase of kinetic energy in the vortex system. Here again we have an example of the transfer of energy from potential flow to vortex flow without the action of local forces.

4. PHASE-SLIP EXPERIMENTS

Recent experiments have shown evidence for the occurrence of discrete energy losses in the flow of superfluid ^{4}He through small apertures equal to κI, where I is the (average) mass current during the loss event.[4-6] These losses are thus consistent with vortex motion events that involve the transfer of energy from potential flow to vortex flow in which the vortex cuts across the entire potential flow. In so far as such motion can be regarded as bringing about a slip of 2π in the difference between the phases of the superfluid order parameter at some upstream point and some downstream point, they are often referred to as 2π phase slips.[17]

Examples A, B, and C all provide candidates for such events, at least as far as topology goes. In example A, such an event might involve the vortex ring originating at the center of the flow channel, growing radially outward, and eventually being annihilated at the wall, in so doing cutting all of the streamlines of potential flow. In example B, we might imagine a ring initially having the radius of the orifice originating at the orifice and moving out into the flow, asymptotically cutting all of the streamlines of potential flow as it moves far to the right. In example C, we may imagine a small ring originating below the orifice close to the wall and ending by moving far above, again asymptotically cutting all of the potential flow streamlines.

In all of these examples, the vortex motion serves as an energy receiver to which energy of potential flow is transferred and in which the energy can be convected to another location. In example A, local forces act concurrently to dissipate the vortex energy. In examples B and C, local forces presumably act eventually to dissipate the vortex energy, perhaps near distant walls, but it is not necessary for them to act before the 2π phase slip is essentially complete. Because the nucleation of the initial vortex seems unlikely, Example B should not be taken as a realistic suggestion for a phase-slip process in ^4He. Nevertheless, it illustrates in principle how energy transfers might take place in such events. On the other hand, Example C and another model topologically equivalent to Example C have recently been proposed as realistic possibilities for phase slip at an aperture.[15,10]

It is interesting to compare the situation in three dimensions to that in two dimensions, i.e., in films thin enough so that vortices are limited to those lying perpendicular to the plane of the film. In this case, the length of a vortex line has no opportunity to grow continuously as in three dimensions, and the ability for a vortex to act as an energy receiver and transporter is much weaker.

Where does the energy come from which is transferred from potential flow to vortex flow during phase slip? If the piston velocities remain constant, so does T_p, and the energy is provided by work done on the fluid by the pistons (in the presence of a pressure difference between the pistons) or by work done on the fluid by a g_Ω force. On the other hand, if no work is done by such forces, the energy transferred must come from T_p, as the potential flow velocities decrease.

5. IMPULSE

Under the assumptions of the development described in this article, the vortex velocity field involves no net momentum. However, there can be defined a momentum-like variable, the impulse, which is often made use of in discussing vortices in superfluid ^4He, and which we will refer to in Section 6.[18-20] If we restrict our attention to the case in which all of the vorticity is concentrated in the cores of filamentary vortex lines of circulation κ, we may define the impulse **I** as

$$\mathbf{I} = \rho \kappa \int d\mathbf{S}, \tag{5.1}$$

where the integral is taken over a surface or surfaces spanning the vortex configuration. If we have a single vortex loop in an unbounded fluid, the integral is defined uniquely, if the surface does not run off to infinity, even though the surface itself is not unique. However, with boundaries present, **I** defined by Eq. (5.1) is not unique. Consider the situation shown in Fig. 5.1. In addition to a surface spanning the vortex ring without contact with the wall, a surface may be constructed which also terminates at the wall in an arbitrary closed curve. In general, **I** will differ for these two cases, and in the latter case **I** will depend upon the curve in the wall. If a vortex is present whose core terminates at the wall at its two ends, a surface spanning it will necessarily also terminate at some closed curve in the wall connecting the end points of the core, and, here too, **I** will depend on that curve.

Nevertheless, the time rate of change of impulse can be given a unique

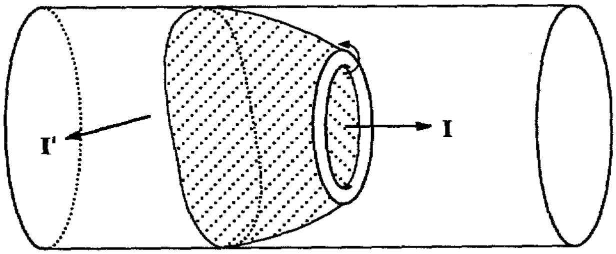

Fig. 5.1. Two possible surfaces for defining the impulse of a vortex ring in the presence of a cylindrical wall.

meaning as follows. We may write

$$dI/dt = \rho\,\kappa\,(d/dt)\int dS = \rho\,\kappa \int v_c \times dl. \qquad (5.2)$$

In calculating this time rate of change of impulse we observe the following restrictions. If the surface spanning a vortex loop also terminates at the boundary, the closed curve of termination in the boundary is assumed to remain at rest. If we have a vortex line terminating at two points on the boundary, the curve in the boundary connecting these two points and serving as a termination for the surface spanning the vortex is also assumed to remain at rest. If the ends of the vortex are not pinned at the boundary, an exception must be made to this latter assumption to allow alterations in the curve consisting of portions of the paths followed by the ends of the vortex as they move along the boundary, as shown in Fig. 5.2. As a result, the latter integral involves only the vortex cores.

With the help of Eq. (2.9), the latter integral can be changed into the form

$$dI/dt = \rho\,\kappa \int v_{av} \times dl + \rho \int h_e\, dl. \qquad (5.3)$$

Thus dI/dt has two sources. The first is convective, that is, a source due to fluid flow alone, and may contribute whether or not any g_e acts. The second is due to the action of g_e. If g_e acts impulsively, the net change in I over the infinitesimal period of time during which g_e acts is dominated by the second integral (since v_{av} remains finite) and equals the net impulse supplied by g_e. Herein lies the justification for the use of the term "impulse" for I. We could imagine creating a vortex system from a state with (almost) no vortices by the impulsive action of g_e, assuming that appropriate vortex "nuclei" exist. (The nonuniqueness of I for a given vortex configuration reflects the different ways in which the configuration might be created.)

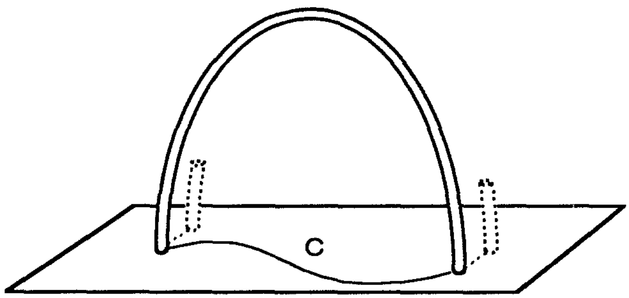

Fig. 5.2. A vortex loop moving along a boundary wall. The heavier dotted lines show the new position of the vortex core, and the lighter dotted lines show the additions to the contour C in the wall connecting the ends of the core, as in the discusssion following Eq. (5.2).

It is interesting to realize that because of the convective term, the impulse of a vortex system is not in general a conserved quantity in the absence of g_e, even with the restrictions following Eq. (5.2). We see illustrations of nonconservation of impulse in such a situation in Examples B, C, and D of Section 3. Although Examples B and C involve diverging flows, Example D shows that a diverging flow is not necessary for nonconservation of I.

Another example of nonconservation of impulse in the absence of g_e is the case of a plane vortex ring encountering head-on a plane wall parallel to its own plane.[20-22] The ring expands under the influence of its image, whose flow field must be regarded as part of the v_v field in the formalism of this article. Here we have another illustration of nonconservation of impulse in the absence not only of g_e but also of a diverging v_p field. As Fetter has shown in detail, such an encounter, with change of vortex impulse, involves no net delivery of an ordinary impulse to the fluid by the wall.[22]

Since no momentum is involved in the vortex flow field, it is perhaps not surprising that the impulse is not in general a conserved quantity in the absence of g_e. But if some g_e force is present, it is natural to ask where the momentum is that should be created by it. The answer, of course, is that under the assumptions of this article, if the motion of the pistons remains unchanged as g_e acts, so that the momentum asssociated with v_p remains unchanged, an additional net force opposite and equal to that due to g_e must act on the fluid at the same time. This counter force might be exerted by some combination of pressure at the boundaries and the action of some g_Ω.

6. FREE ENERGY OF A VORTEX SYSTEM

Use is often made of the relationship

$$F = E + \mathbf{v}_s \cdot \mathbf{p} \tag{6.1}$$

to express the energy in the laboratory frame of an elementary excitation in flowing superfluid ^{4}He, where E and $\mathbf{p}$ are the energy and momentum of the excitation (in the frame of the moving superfluid), and $\mathbf{v}_s$ is the superfluid velocity in the laboratory frame. For example, use of this relationship is made in the derivation of the Landau critical velocity.[23] Use is also made of this relation in connection with vortex configurations, such as in the ILF theory of homogeneous nucleation and growth, where $\mathbf{p}$ is taken to be the impulse of the vortex system.[13,14] But is this application justified in view of the fact that the vortex system possesses no actual momentum? Further, how is this relation to be applied when the superfluid velocity varies over an extended vortex system?

A simple answer to these questions is provided by the earlier discussion of this article. The result is an appropriate generalization of Eq. (6.1), at least in differential form. From Eqs. (2.6) and (2.10) we have

$$dW_e/dt = dT_V/dt + \rho\,\kappa \int \mathbf{v}_p \cdot \mathbf{v}_c \times d\mathbf{l}. \tag{6.2}$$

From Eq. (5.2), $\rho\kappa\mathbf{v}_c \times d\mathbf{l}$ is just the increment in the time rate of change of the impulse coming from element $d\mathbf{l}$ of vortex core as it moves. Thus we have

$$dW_e = dT_V + \int \mathbf{v}_p \cdot dd\mathbf{I}, \tag{6.3}$$

where the double d before I denotes the increment both with respect to time and path length along the core, and the integral runs along all of the vortex cores in the system.

Equation (6.3) is the generalization of Eq. (6.1) in differential form that we seek. We identify the element of work done by an external force dW_e as the increment in effective energy of the vortex system in the laboratory frame and dT_V as the increment in energy of the excitation. The integral is the generalization of $\mathbf{v}_s \cdot \mathbf{p}$. It reduces to $\mathbf{v}_p \cdot d\mathbf{I}$, where $d\mathbf{I}$ is the total increment in impulse, when $\mathbf{v}_p$ is uniform over the vortex system. In so far as we confirm the applicability of Eq. (6.1) to vortices in the uniform flow case, we support the conclusion reached by Schwarz by a somewhat different derivation.[3]

Equations (6.1) and (6.3) suggest an analogy between the vortex system and a thermodynamic system, in which the potential flow system acts as the analog of a thermodynamic reservoir. In this analogy, the increment of F in Eq. (6.1) or dW_e in Eq. (6.3) is the counterpart of an increment in free energy.

7. CONCLUDING REMARKS

In this article we have looked at certain general properties of energy transfer in the flowing superfluid component of liquid ^{4}He when quantum vortices are present. We have been particularly interested in the application of these ideas to phase slip by vortex motion. It must be emphasized, however, that in order to understand phase slip in detail, one must go far beyond these general considerations. There are two aspects of the problem, both of which are currently under active study. One is the nucleation of quantum vortices by thermal and quantum fluctuations to the point where they can undergo hydrodynamic growth.[24,25] The second is the detailed hydrodynamic evolution of vortex configurations once they are nucleated, using the same classical ideas underlying this article.[10,26] The importance of understanding the inverse processes leading to vortex annihilation should also be kept in mind.

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vorticity already present in the core.[1] In a core of finite radius, the net effect might be the expansion and twisting of the vorticity in the core. In the superfluid component of ^{4}He, such a process seems to be ruled out, unless the force is intense enough to create vortex rings encircling the original core.

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