GRAVITATIONAL POTENTIAL HARMONICS FROM THE SHAPE OF AN HOMOGENEOUS BODY

GEORGES BALMINO
Department of Terrestrial and Planetary Geodesy, Centre National d'Etudes Spatiales, 18 Av. Ed. Belin, F–31055 Toulouse Cedex, France

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Abstract. The spherical harmonic coefficients of the gravitational potential of an homogeneous body are analytically derived from the harmonics describing its shape. General formulas are given as well as detailed expressions up to the fifth order of the topography harmonics. The volume, surface and inertia tensor of the body are obtained as by-products. The case of a triaxial ellipsoid is given as example and used for numerical checking. Another numerical scheme for verification is provided. The application to Phobos is made and the convergence of the expressions for the harmonics is numerically established.

Résumé. Les harmoniques du champ de gravitation d'un corps homogène de forme donnée sont calculés analytiquement à partir des harmoniques du développement en série du rayon vecteur exprimant la forme de la surface du corps. Outre la formule générale, des expressions détaillées, au cinquième ordre des harmoniques du rayon vecteur, sont données sous une forme bien adaptée à la programmation. Le volume, la surface et le tenseur d'inertie du corps sont calculés analytiquement à partir des formules générales. Le cas de l'ellipsoide triaxial est pris comme test des formules établies. Un autre test numérique est fourni dans le cas le plus général. Ceci est appliqué à Phobos, et la convergence des expressions fournissant les harmoniques est numériquement démontrée.

Key words: Gravitational potential, topography, spherical harmonics, Phobos

1. Introduction

The problem of computing the spherical harmonics of the gravity field expansion of an homogeneous body of known shape and density may look very theoretical since, in many cases, the first order theory is sufficient for geodetic and geophysical investigations. However, in the case of bodies such as the small natural satellites of the solar system, or the asteroids, refined expressions are needed to represent the gravitational potential with enough accuracy for application to navigation of spacecraft flying by or orbiting these bodies. Planetary missions such as VIKING at Mars (where the orbiter came very close to Phobos) and, soon to come, NEAR with a rendez-vous with Asteroid 433 Eros planned for december 1998, are cases which need a fine gravity field representation of the body under investigation. Even in the case of the Earth, the linear theory may be too crude, for instance to compute the global isostatic potential coefficients (Rummel *et al.*, 1988).

We start from the integral expression of the complex gravity harmonics, we then expand the radius vector of the integrand as given by a spherical harmonics expansion, we use the product-sum conversion formulas for spherical functions and we perform the integration term by term. The obtained expressions, which become more complicated as the order of the expansion increases, must then be carefully checked, which is done numerically by two methods: (I) the triaxial ellipsoid case is considered, for which we derive ad hoc formulas, not found in the literature, for the radius vector and gravity coefficients in their spherical harmonic expansions; (II) an alternative form of the gravitational harmonics, best suited for direct numerical evaluation, is given. We finally study the case of Phobos for which the difference between the analytical and the numerical techniques is investigated; the convergence of the formula giving the gravitational potential coefficients is also shown numerically.

2. Complex Forms of the Spherical Harmonic Expansion

The gravitational potential, U, of the body in a reference frame $\{X\}$ attached to it is taken in the form (Balmino and Borderies, 1978)

$$U(r,\varphi,\lambda) = \frac{GM}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l \sum_{m=-l}^{l} K_{lm}Y_{lm}(\varphi,\lambda) \tag{1}$$

with

r, φ, λ	=	spherical coordinates in $\{X\}$, respectively the radius vector, latitude and longitude
G	=	gravitational constant
M	=	mass of body
R	=	reference length
K_{lm}	=	complex dimensionless coefficient of degree l and order m
Y_{lm}	=	complex surface harmonic function of degree l and order m.

The Y_{lm} functions are related to the usual Legendre polynomials ($m = 0$) and associated functions ($m \neq 0$) by

$$Y_{lm}(\varphi,\lambda) = P_{lm}(\sin\varphi)\, e^{im\lambda} \tag{2}$$

where, for $m \geq 0$:

$$P_{lm}(u) = \frac{(1-u^2)^{m/2}}{2^l l!}\frac{\mathrm{d}^{l+m}}{\mathrm{d}u^{l+m}}[(u^2-1)^l]$$

$$P_{l,-m}(u) = (-1)^m\frac{(l-m)!}{(l+m)!}P_{lm}(u).$$

With these definitions, the K_{lm} coefficients are expressed as

$$K_{lm} = \frac{(-1)^m}{MR^l}\iiint_V r'^l Y_{l,-m}(\varphi',\lambda')\,\mathrm{d}M' \tag{3}$$

the integral being extended to the volume V of the body; r', φ', λ' are the spherical coordinates of the current point in V and $\mathrm{d}M'$ is the mass element.

The series (1) is uniformly convergent for $r > R^*$ where R^* is the radius of the sphere centered at the origin of coordinates and including all masses. The question of its convergence inside this sphere and especially down to the surface of the body in the general case is quite delicate. An account of the modern developments and results on this matter can be found in Moritz (1980).

The complex harmonics are related to the usual real coefficients, C_{lm} and S_{lm}, by

$$\begin{aligned}(2-\delta_{0m})K_{lm} &= C_{lm} - iS_{lm} \\ (2-\delta_{0m})K_{l,-m} &= (C_{lm} + iS_{lm})\frac{(l+m)!}{(l-m)!}(-1)^m\end{aligned} \tag{4}$$

for $m \geq 0$ and with δ being the Kronecker symbol.

It is commonplace in geodesy to use the following normalized Legendre functions and harmonics

$$\begin{aligned}\bar{Y}_{lm} &= Y_{lm}N_{lm} \\ \bar{K}_{lm} &= K_{lm}/N_{lm}\end{aligned} \tag{5}$$

where (Heiskanen and Moritz, 1967)

$$N_{lm} = [(2-\delta_{0m})(2l+1)(l-m)!/(l+m)!]^{1/2} \tag{6}$$

is defined for any m, $-l \leq m \leq +l$.

Of course, we have, for $m \geq 0$:

$$(2-\delta_{0m})\bar{K}_{lm} = \bar{C}_{lm} - i\bar{S}_{lm} \tag{7}$$

where $\bar{C}_{lm}$, $\bar{S}_{lm}$ are the usual, real and normalized coefficients, and

$$\bar{K}_{l,-m} = (-1)^m \bar{K}^*_{lm} \tag{8}$$

where $(\ldots)^*$ is the complex conjugate of $(\ldots)$. Also

$$\bar{Y}_{l,-m} = (-1)^m \bar{Y}^*_{lm}. \tag{9}$$

It must be noted that, over the unit sphere σ_1

$$\frac{1}{4\pi}\iint_{\sigma_1} \bar{Y}_{lm}\bar{Y}^*_{l'm'}\,\mathrm{d}\sigma = (2-\delta_{0m})\delta_{ll'}\delta_{mm'}. \tag{10}$$

Because of the factors 4π and $(2-\delta_{0m})$ in this scalar product, it is most advantageous, for further computation, to define another set of normalized spherical functions by

$$Y_l^m = \frac{(-1)^m}{\sqrt{4\pi}}\left[\frac{(2l+1)(l-m)!}{(l+m)!}\right]^{1/2} Y_{lm} \tag{11}$$

or

$$Y_l^m = \frac{(-1)^m}{\sqrt{4\pi}} \frac{1}{\sqrt{2-\delta_{0m}}} \bar{Y}_{lm}. \tag{12}$$

These are of commun use in quantum physics. The interesting properties for our purpose are

$$Y_l^{-m} = (-1)^m Y_l^{m*} \tag{13}$$

and

$$\iint_{\sigma_1} Y_l^m Y_{l'}^{m'*} \, \mathrm{d}\sigma = \delta_{ll'} \delta_{mm'}. \tag{14}$$

Also, the Legendre addition formula writes

$$P_l(\cos\Psi) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_l^m(\varphi,\lambda) Y_l^{m*}(\varphi',\lambda') \tag{15}$$

(Ψ is the angular distance between the points (φ, λ) and (φ', λ') on σ_1).

The harmonic coefficients which go with the Y_l^m functions are naturally

$$K_l^m = (-1)^m \sqrt{4\pi}\, \sqrt{2-\delta_{0m}}\, \bar{K}_{lm} \tag{16}$$

which satisfy

$$K_l^{-m} = (-1)^m K_l^{m*}. \tag{17}$$

Combining (7) and (16) we find, for $m \geq 0$:

$$\begin{aligned} \bar{C}_{lm} &= (-1)^m [(2-\delta_{0m}/4\pi]^{1/2} \, \mathrm{Re}\, K_l^m \\ \bar{S}_{lm} &= (-1)^m [(2-\delta_{0m}/4\pi]^{1/2} \, \mathrm{Im}\, K_l^m \end{aligned} \tag{18}$$

with

$$K_l^m = \frac{4\pi}{2l+1} \frac{1}{MR^l} \iiint_V r^l Y_l^{m*}(\varphi,\lambda)\, \mathrm{d}M \tag{19}$$

(the primes have been dropped since there is no ambiguity).

3. The Description of the Body Shape

We assume that the surface limiting the body can be described by the following spherical harmonic expansion

$$r(\varphi,\lambda) = R_0(l+S)$$

$$S = \sum_{j\geq 1} \sum_{k=0}^{j} (\bar{A}_{jk} \cos k\lambda + \bar{B}_{jk} \sin k\lambda) \bar{P}_{jk}(\sin\varphi) \tag{20}$$

where R_0 is a given reference length (may be different from R). Such an expansion, infinite in principle, may be truncated in practice depending on the degree of knowledge of the body surface. Recent models for the Earth, Venus and Mars shapes, complete to degree 1080, 720, and 720 respectively, have been recently determined (Balmino, 1993); a model of degree 12 exists for the Moon (Bills and Ferrari, 1977) and a model of degree 8 has been established for Phobos by Duxbury (1991). Equation (20) implies that a single determination of r is found for any (φ, λ), and that R_0 is the mean of r over the unit sphere.

We first convert the $\bar{A}$'s and $\bar{B}$'s coefficients into complex ones according to (18) for $q \geq 0$:

$$T_j^q = (-1)^q[4\pi/(2-\delta_{0q})]^{1/2}(\bar{A}_{jq} + i\bar{B}_{jq}) \tag{21}$$

and then we use (17) to define T_j^{-q}. To simplify the analytic expression to come, we then order the coefficients T_j^q and Legendre functions Y_j^q by increasing j from zero to, in practice, a maximum value J, and for each j by increasing order q, from $-j$ to $+j$. Renaming the coefficient T of rank n as D_n and the function $Y_j^q(\varphi, \lambda)$ of same rank as $Z_n(\varphi, \lambda)$, we simply have

$$S = \sum_{n=1}^{N} D_n Z_n(\varphi, \lambda) = \sum_{n=1}^{N} T_{j_n}^{q_n} Y_{j_n}^{q_n}(\varphi, \lambda) \tag{22}$$

with $N = (J+1)^2 - 1$ and where, for a given n, we have

$$\begin{aligned} j_n &= \text{int}\{\sqrt{n}\} \\ q_n &= n - j_n(j_n + 1). \end{aligned} \tag{23}$$

4. Derivation of the K_l^m Harmonics from the T_j^q's

Writing the mass element $\mathrm{d}M$ as $\rho r^2\,\mathrm{d}r\,\mathrm{d}\sigma$ with ρ constant, integrating with respect to r from 0 to its surface value $r(\varphi, \lambda)$ as given by (20) and (22), we find

$$K_l^m = \frac{1}{(2l+1)(l+3)} \frac{4\pi\rho R_0^3}{M} \left(\frac{R_0}{R}\right)^l \iint_{\sigma_1} (1+S)^{l+3} Y_l^{m*}\,\mathrm{d}\sigma. \tag{24}$$

We now expand $(1+S)^{l+3}$:

$$(1+S)^{l+3} = \sum_{H=0}^{l+3} \binom{l+3}{H} S^H = 1 + (l+3)S + \frac{(l+3)(l+2)}{2} S^2 + \cdots \tag{25}$$

which tells us that a term of order H in S (assumed to have a norm smaller than 1) does not contribute to gravity harmonics of degree smaller than $H-3$. Also, it is clear that we may have non vanishing K_l^m coefficients for any degree l even if the series S is finite. The contribution of the terms of order H to K_l^m will be denoted $K_l^m(H)$.

In order to show how the general term of order H in S, that is of order H in any T_j^q, can be obtained, we will proceed by step. Formulas up to the fifth order in any T_j^q will be given explicitly, since we found that this was sufficient in the case of the celestial bodies afore mentioned. The same approach was used by Chao and Rubincam (1989), was limited to third order and applied to Phobos. More recently, this procedure was also applied by Martinec (1991) but not published in a very explicit way suited for verification. We feel that the presentation below is easier to understand and to program, especially when using our recursive formulas on the product-sum coefficients for spherical harmonics (annex A).

$H = 0$

Since we have the integral of $Y_l^{m^*}$ over σ_1, the only non zero term is obtained for $l = m = 0$ (the integral is $\sqrt{4\pi}$). So

$$\begin{aligned} K_0^0(0) &= \frac{4}{3}\,\pi R_0^3 \frac{\rho}{M}\,\sqrt{4\pi} \\ K_l^m(0) &= 0 \quad \text{if} \quad l \neq 0 \text{ or } m \neq 0. \end{aligned} \tag{26}$$

$H = 1$

We write S in its usual form with coefficients T_j^q and we use (14). We then find the classical linear relationship

$$K_l^m(1) = \frac{4\pi\rho R_0^3}{(2l+1)M}\left(\frac{R_0}{R}\right)^l T_l^m \tag{27}$$

which, in the case of a body close to a sphere of mean density ρ_0 with superimposed topography T_l^m reduces to the usual form

$$K_l^m(1) = \frac{3}{2l+1}\,\frac{\rho}{\rho_0}\,T_l^m \tag{28}$$

$H = 2$

We need S^2 which may be written as

$$\sum_{j=1}^{J}\sum_{q=-j}^{j}(T_j^q)^2(Y_j^q)^2 + 2\sum_{h=1}^{N-1}\sum_{k=h+1}^{N} T_{j_h}^{q_h}T_{j_k}^{q_k}Y_{j_h}^{q_h}Y_{j_k}^{q_k}$$

and we have to integrate products of three surface harmonics, that is

$$\int\!\!\int_{\sigma_1} Y_j^q Y_{j'}^{q'} Y_l^{m^*}\,\mathrm{d}\sigma = (-1)^m \int\!\!\int_{\sigma_1} Y_j^q Y_{j'}^{q'} Y_l^{-m}\,\mathrm{d}\sigma.$$

Such integrals will be denoted $I_{jj'l}^{qq'm}$ being understood that the last function, of degree l and order m, comes in as its complex conjugate. Their evaluation requires to transform the product $Y_j^q Y_{j'}^{q'}$ into a summation, which involves the Clebsch–Gordan coefficients. This and formulas for the product of any number of spherical functions and their corresponding integrals are given in Annex A. Therefore we find

$$K_l^m(2) = \frac{4\pi\rho R_0^3}{M(2l+1)}\left(\frac{R_0}{R}\right)^l \frac{l+2}{2} \times \left\{\sum_{j=1}^{J}\sum_{q=-j}^{j}(T_j^q)^2 I_{jjl}^{qqm} + 2\sum_{h=1}^{N-1}\sum_{k=h+1}^{N} T_{j_h}^{q_h} T_{j_k}^{q_k} I_{j_h j_k l}^{q_h q_k m}\right\}. \tag{29}$$

$H = 3$

There are three groups of terms

$$K_l^m(3) = \frac{4\pi\rho R_0^3}{M(2l+1)}\left(\frac{R_0}{R}\right)^l \frac{(l+1)(l+2)}{6} \times \left\{\sum_{j=1}^{J}\sum_{q=-j}^{j}(T_j^q)^3 I_{jjjl}^{qqqm} + 3\sum_{i=1}^{N}\sum_{\substack{h=1\\h\neq i}}^{N}(T_{j_i}^{q_i})^2 T_{j_h}^{q_h} I_{j_i j_i j_h l}^{q_i q_i q_h m} + 6\sum_{i=1}^{N-2}\sum_{h=i+1}^{N-1}\sum_{k=h+1}^{N} T_{j_i}^{q_i} T_{j_h}^{q_h} T_{j_k}^{q_k} I_{j_i j_h j_k l}^{q_i q_h q_k m}\right\}. \tag{30}$$

$H = 4$

The expansion of S^4 using multinomial coefficients produces 5 groups of terms

$$K_l^m(4) = \frac{4\pi\rho R_0^3}{M(2l+1)}\left(\frac{R_0}{R}\right)^l \frac{l(l+1)(l+2)}{24} \times \left\{\sum_{j=1}^{J}\sum_{q=-j}^{j}(T_j^q)^4 I_{jjjjl}^{qqqqm} + 4\sum_{i=1}^{N}\sum_{\substack{h=1\\h\neq i}}^{N}(T_{j_i}^{q_i})^3 T_{j_h}^{q_h} I_{j_i j_i j_i j_h l}^{q_i q_i q_i q_h m} + 6\sum_{i=1}^{N-1}\sum_{h=i+1}^{N}(T_{j_i}^{q_i})^2 (T_{j_h}^{q_h})^2 I_{j_i j_i j_h j_h l}^{q_i q_i q_h q_h m} + 12\sum_{i=1}^{N}\sum_{h=1}^{N-1-d}\sum_{k=h+1}^{N-d}(T_{j_i}^{q_i})^2 T_{j_h}^{q_h} T_{j_k}^{q_k} I_{j_i j_i j_h j_k l}^{q_i q_i q_h q_k m} + 24\sum_{i=1}^{N-3}\sum_{h=i+1}^{N-2}\sum_{k=h+1}^{N-1}\sum_{n=k+1}^{N} T_{j_i}^{q_i} T_{j_h}^{q_h} T_{j_k}^{q_k} T_{j_n}^{q_n} I_{j_i j_h j_k j_n l}^{q_i q_h q_k q_n m}\right\}. \tag{31}$$

In this formula, $d = \delta_{iN}$.

$H = 5$

There are seven groups of terms

$$K_l^m(5) = \frac{4\pi\rho R_0^3}{M(2l+1)} \left(\frac{R_0}{R}\right)^l \frac{l(l^2-l)(l+2)}{120}$$

$$\times \left\{ \sum_{j=1}^{J} \sum_{q=-j}^{j} (T_j^q)^5 I_{jjjjjl}^{qqqqqm} + 5 \sum_{i=1}^{N} \sum_{\substack{h=1 \\ h\neq i}}^{N} (T_{ji}^{q_i})^4 T_{jh}^{q_h} I_{j_ij_ij_ij_ij_hl}^{q_iq_iq_iq_iq_hm} \right.$$

$$+ 10 \sum_{i=1}^{N} \sum_{\substack{h=1 \\ h\neq i}}^{N} (T_{ji}^{q_i})^3 (T_{jh}^{q_h})^2 I_{j_ij_ij_ij_hj_hl}^{q_iq_iq_iq_hq_hm}$$

$$+ 20 \sum_{i=1}^{N} \sum_{\substack{h=1 \\ h\neq i}}^{N-1-d} \sum_{\substack{k=h+1 \\ k\neq i}}^{N-d} (T_{ji}^{q_i})^3 T_{jh}^{q_h} T_{jk}^{q_k} I_{j_ij_ij_ij_hj_kl}^{q_iq_iq_iq_hq_km}$$

$$+ 30 \sum_{i=1}^{N} \sum_{\substack{h=1 \\ h\neq i}}^{N-1-d} \sum_{\substack{k=h+1 \\ k\neq i}}^{N-d} T_{ji}^{q_i} (T_{jh}^{q_h})^2 (T_{jk}^{q_k})^2 I_{j_ij_hj_hj_kj_kl}^{q_iq_hq_hq_kq_km}$$

$$+ 60 \sum_{i=1}^{N} \sum_{\substack{h=1 \\ h\neq i}}^{N-2-d} \sum_{\substack{k=h+1 \\ k\neq i}}^{N-1-d} \sum_{\substack{n=k+1 \\ n\neq i}}^{N-d} (T_{ji}^{q_i})^2 T_{jh}^{q_h} T_{jk}^{q_k} T_{jn}^{q_n} I_{j_ij_ij_hj_kj_nl}^{q_iq_iq_hq_kq_nm}$$

$$\left. + 120 \sum_{i=1}^{N-4} \sum_{h=i+i}^{N-3} \sum_{k=h+1}^{N-2} \sum_{n=k+1}^{N-1} \sum_{s=n+1}^{N} T_{ji}^{q_i} T_{jh}^{q_h} T_{jk}^{q_k} T_{jn}^{q_n} T_{js}^{q_s} I_{j_ij_hj_kj_nj_sl}^{q_iq_hq_kq_nq_sm} \right\}. \quad (32)$$

As in (31), d is defined as δ_{iN}.

GENERAL FORM

For any H, we expand the single indexed form of S by the multinomial theorem, that is

$$S^H = \sum_{\alpha_1+\alpha_2+\cdots+\alpha_N=H} \frac{H!}{\alpha_1!\alpha_2!\ldots\alpha_N!}$$
$$\times\ D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_N^{\alpha_N} Z_1^{\alpha_1}(\varphi,\lambda) Z_2^{\alpha_2}(\varphi,\lambda) \ldots Z_N^{\alpha_N}(\varphi,\lambda).$$

The general expression for $K_l^m(H)$ follows

$$K_l^m(H) = \frac{4}{3}\pi\rho R_0^3 \frac{3}{(l+3)(2l+1)} \left(\frac{R_0}{R}\right)^l \binom{l+3}{H}$$

$$\times \sum_{\Sigma_i \alpha_i = H} \frac{H!}{\prod_i(\alpha_i!)} \prod_{i=1}^{N} (T_{j_i}^{q_i})^{\alpha_i} I_{j_1^{\alpha_1} j_2^{\alpha_2} \ldots j_N^{\alpha_N} l}^{q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_N^{\alpha_N} m} \tag{33}$$

where

$$I_{j_1^{\alpha_1} j_2^{\alpha_2} \ldots j_N^{\alpha_N} l}^{q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_N^{\alpha_N} m} = \iint_{\sigma_1} (Y_{j_1}^{q_1})^{\alpha_1} (Y_{j_2}^{q_2})^{\alpha_2} \ldots (Y_{j_N}^{q_N})^{\alpha_N} Y_l^{m^*} \, \mathrm{d}\sigma. \tag{34}$$

There are $\binom{N+H-1}{H}$ such terms.

Formulas (29) to (32) are peculiar cases of (33), in which the summations have been arranged in an efficient way for programming.

5. Some Applications of the General Formulas

VOLUME AND AREA OF THE BODY

The volume computation is straightforward since it is

$$V = \frac{R_0^3}{3} \iint_{\sigma_1} (1+S)^3 \, \mathrm{d}\sigma \tag{35}$$

that is

$$V = \frac{K_0^0}{\sqrt{4\pi}} \frac{M}{\rho}. \tag{36}$$

From Equations (24) and (25) and from the remark which followed, we exactly have

$$V = \frac{M}{\rho\sqrt{4\pi}} [K_0^0(0) + K_0^0(1) + K_0^0(2) + K_0^0(3)].$$

Using (26) and the fact that $K_0^0(1) \equiv 0$, we find

$$V = \frac{4}{3}\pi R_0^3 + \frac{M}{\rho\sqrt{4\pi}} [K_0^0(2) + K_0^0(3)]. \tag{37}$$

The area A is of course

$$A = R_0^2 \iint_{\sigma_1} (1 + 2S + S^2) \, \mathrm{d}\sigma.$$

Noting that the integrals of all terms in S are zero and that only terms in S^2 with the same degree and the same order do not vanish, we find

$$A = 4\pi R_0^2 \left[1 + \frac{1}{4\pi} \sum_{j=1}^{J} \sum_{q=-j}^{j} (-1)^q T_j^q T_j^{-q} \right]. \tag{38}$$

INERTIA TENSOR

Let $\{I_{jk}\}$ be the inertia integrals of the body in the reference frame $\{X\}$ defined as

$$I_{jk} = \iiint_v x_j x_k \rho \, dv \tag{39}$$

where

$$\begin{aligned} x_1 + ix_2 &= rY_{11}(\varphi, \lambda) \\ x_3 &= rY_{10}(\varphi, \lambda) \end{aligned}$$

We compute these integrals from the following relations

$$\begin{aligned} &I_{11} + I_{22} = \rho \iiint_v r^2 Y_{11} Y_{11}^* \, dv \\ &I_{11} - I_{22} + 2iI_{12} = \rho \iiint_v r^2 Y_{11}^2 \, dv \\ &I_{33} = \rho \iiint_v r^2 Y_{10}^2 \, dv \\ &I_{13} + iI_{23} = \frac{1}{3} \iiint_v r^2 Y_{21} \, dv. \end{aligned} \tag{40}$$

We proceed as before, we expand r^2 as $R_0^2(1+2S+S^2)$ and transform the products of Legendre functions into summations. For example

$$I_{11} - I_{22} + 2iI_{12} = \frac{8\pi}{15} \rho R_0^5 \iint_{\sigma_1} (1+S)^5 (Y_1^1)^2 \, d\sigma.$$

We write $(Y_1^1)^2 = \sqrt{3/10\pi}\, Y_2^2$ (from formula A2, Annex 2), and finally

$$I_{11} - I_{22} + 2iI_{12} = \sqrt{\frac{32\pi}{375}} \rho R_0^5 \sum_{H=0}^{5} \binom{5}{H} \iint_{\sigma_1} S^H Y_2^{-2^*} \, d\sigma.$$

The other relations are treated in a similar way. For many applications in planetology, it suffices to have the inertia tensor as a function of the shape coefficients up to degree and order 2. In real form, using the unnormalized harmonics A_{jk} and B_{jk}, we find, up to the square of the A's and B's:

$$\begin{aligned} \mathbf{A} &= I_{22} + I_{33} = \mu \Big[1 + \frac{A_{20}}{2} - 3A_{22} + 4(A_{10}^2 + B_{11}^2) + 2A_{11}^2 + \frac{16}{7} A_{20}^2 \\ &\quad + \frac{36}{7} A_{21}^2 + \frac{54}{7} B_{21}^2 + \frac{144}{7} (A_{22}^2 + B_{22}^2) + \frac{24}{7} A_{22} A_{20} \Big] \\ \mathbf{B} &= I_{11} + I_{33} = \mu \Big[1 + \frac{A_{20}}{2} + 3A_{22} + 4(A_{10}^2 + A_{11}^2) + 2B_{11}^2 + \frac{16}{7} A_{20}^2 \end{aligned}$$

$$+ \frac{54}{7} A_{21}^2 + \frac{36}{7} B_{21}^2 + \frac{144}{7} (A_{22}^2 + B_{22}^2) - \frac{24}{7} A_{22}A_{20}\Big]$$

$$\mathbf{C} = I_{11} + I_{22} = \mu \Big[1 - \frac{A_{20}}{2} + 2A_{10}^2 + 4(A_{11}^2 + B_{11}^2) + \frac{10}{7} A_{20}^2$$

$$+ \frac{36}{7} (A_{21}^2 + B_{21}^2) + \frac{216}{7} (A_{22}^2 + B_{22}^2)\Big]$$

$$\mathbf{D} = -I_{23} = \mu \left[\frac{3}{2} B_{21} + 2A_{10}B_{11} - \frac{36}{7} (A_{22}B_{21} - A_{21}B_{22}) + \frac{6}{7} A_{20}B_{21}\right]$$

$$\mathbf{E} = -I_{31} = \mu \left[\frac{3}{2} A_{21} + 2A_{10}A_{11} + \frac{36}{7} (A_{22}A_{21} + B_{22}B_{21}) + \frac{6}{7} A_{20}A_{21}\right]$$

$$\mathbf{F} = -I_{12} = \mu \left[3B_{22} + 2A_{11}B_{11} + \frac{18}{7} A_{21}B_{21} + \frac{24}{7} A_{20}B_{22}\right]. \qquad (41)$$

The inertia tensor **I** has been written

$$\mathbf{I} = \begin{bmatrix} \mathbf{A} & -\mathbf{F} & -\mathbf{E} \\ -\mathbf{F} & \mathbf{B} & -\mathbf{D} \\ -\mathbf{E} & -\mathbf{D} & \mathbf{C} \end{bmatrix} \quad \text{and} \quad \mu = \frac{8\pi}{15} \rho R_0^5.$$

GRAVITY HARMONICS OF A MULTILAYERED BODY

Let us assume that the body is now composed of one part of density $\rho = \rho_0$ still limited by the surface of Equation (20), plus another part, inside the first one, of density ρ_1 limited by another surface of equation

$$r(\varphi, \lambda) = R_1(1 + S_1) \qquad (42)$$

For instance, this surface could be a triaxial ellipsoid (see Section 6).

We will denote the S-series of the outer surface by S_0, for sake of homogeneity. Equation (24) now becomes

$$K_l^m = \frac{1}{(2l+1)(l+3)} \frac{4\pi\rho}{MR^l} \Big[\rho_0 R_0^{l+3} \iint_{\sigma_1} (1 + S_0)^{l+3} Y_l^{m*} \,\mathrm{d}\sigma$$

$$- \rho_0 R_1^{l+3} \iint_{\sigma_1} (1 + S_1)^{l+3} Y_l^{m*} \,\mathrm{d}\sigma + \rho_1 R_1^{l+3} \iint_{\sigma_1} (1 + S_1)^{l+3} Y_l^{m*} \,\mathrm{d}\sigma\Big]$$

that is

$$K_l^m = \frac{1}{(2l+1)(l+3)} \frac{4\pi R_0^3}{M} \left(\frac{R_0}{R}\right)^l [\rho_0 k_l^m\{0\} + (\rho_1 - \rho_0)\xi_1^{l+3} k_l^m\{1\}] \quad (43)$$

where we have introduced ($\nu = 0, 1$)

$$k_l^m(\nu) = \iint_{\sigma_1} (1 + S_\nu)^{l+3} Y_l^{m*} \,\mathrm{d}\sigma \quad \text{and} \quad \xi_1 = R_1/R_0 \quad (< 1).$$

Introducing further

$$\begin{aligned} \Delta\rho_\nu &= \rho_\nu - \rho_{\nu-1} \quad (\text{and } \rho_{-1} = 0) \\ \xi_0 &= 1 \end{aligned} \tag{44}$$

this expression can be generalized to any number of layers of constant density ρ_ν

$$K_l^m = \frac{1}{(2l+1)(l+3)} \frac{4\pi R_0^3}{M} \left(\frac{R_0}{R}\right)^l \sum_\nu \Delta\rho_\nu \xi_\nu^{l+3} k_l^m\{\nu\}. \tag{45}$$

6. Example and Check of Our Formulas: The Triaxial Ellipsoid

To check all our formulas, we chose a figure of reference for which the shape could be easily expanded in spherical harmonics and which gravity coefficients of the corresponding volume could be derived exactly. The triaxial ellipsoid is an interesting figure for it is often used as an approximate model of the real shape of asteroids and satellites. We give below an elementary derivation of the two classes of harmonics.

Harmonic Expansion of the Radius Vector

Let us start from the equation of the ellipsoid (E), here written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{with} \quad a \geq b \geq c. \tag{46}$$

The meridian plane of longitude λ intersects (E) along an ellipse (E_λ) of semi-major axis $a(\lambda)$ and semi-minor axis c, with

$$\begin{aligned} a(\lambda) &= a(1 + e'^2 \sin^2\lambda)^{-1/2} \\ &= a \sum_{k=0}^{\infty} \varepsilon_{2k} e_\lambda'^{2k} \sin^{2k}\lambda \end{aligned} \tag{47}$$

where

e' = second eccentricity of the equuatorial ellipse = $(a^2/b^2 - 1)^{1/2}$. If $a < b\sqrt{2}$, we have $e' < 1$ which ensures the uniform convergence of the series.

$\varepsilon_0 = 1$

$\varepsilon_{2k} = -\varepsilon_{2k-2}(2k-1)/2k, \quad k \geq 1.$

Then, the radius vector at latitude φ on (E_λ) is

$$r(\varphi, \lambda) = a(\lambda) \sum_{n=0}^{\infty} \varepsilon_{2n} E'^{2n} \sin^{2n}\varphi \tag{48}$$

with $E'^2 = a(\lambda)^2/c^2 - 1$. This series is uniformly convergent provided that $E'^2 < 1$ for any λ that is $\max\{a(\lambda)\}^2/c^2 - 1 < 1$ or, since $\max\{a(\lambda)\} = a$

$$a < c\sqrt{2} \tag{49}$$

which implies $a < b\sqrt{2}$, the condition for the convergence of (47). We write E'^{2n} as

$$E'^{2n} = \left(\frac{1-\alpha}{\alpha}\right)^n (1 - \gamma \sin^2 \lambda)^n (1 + e'^2 \sin^2 \lambda)^{-n}$$

with

$$\begin{aligned} \alpha &= c^2/a^2 \\ \beta &= c^2/b^2 \\ \gamma &= \frac{\beta - \alpha}{1 - \alpha} \end{aligned}$$

and one has

$$\begin{aligned} 0 &\le \frac{1-\alpha}{\alpha} < 1 \\ \frac{1}{2} &< \alpha \le \beta < 1. \end{aligned}$$

Expanding the two last factors of E'^{2n} we have

$$\begin{aligned} a(\lambda)E'^{2n} &= a\left(\frac{1-\alpha}{\alpha}\right)^n \sum_{k=0}^{\infty} \varepsilon_{2k}\, e'^{2k} sin^{2k} \lambda \\ &\times \sum_{j=0}^{n} (-1)^j \gamma^j \binom{n}{j} \sin^{2j} \lambda \sum_{t=0}^{\infty} e'^{2t} \binom{-n}{t} \sin^{2t} \lambda. \end{aligned}$$

Keeping the indices k, j and putting $p = k + j + t$, we find

$$a(\lambda)E'^{2n} = a\left(\frac{1-\alpha}{\alpha}\right)^n \sum_{p=0}^{\infty} R_p^n \sin^{2p} \lambda \tag{50}$$

where

$$R_p^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \gamma^j\, e'^{2(p-j)} \sum_{k=0}^{p-j} \varepsilon_{2k} \binom{-n}{p-k-j}. \tag{51}$$

We now transform $\sin^{2p} \lambda$ as

$$\sin^{2p} \lambda = \sum_{\mu=0}^{p} S_\mu^p \cos(2p - 2\mu)\lambda \tag{52}$$

with

$$S_\mu^p = (2-\delta_{\mu p})\frac{(-1)^{p+\mu}}{2^{2p}}\binom{2p}{\mu} \tag{53}$$

and we obtain

$$r(\varphi,\lambda) = a\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\sigma_k^n\cos 2k\lambda\right)\sin^{2n}\varphi \tag{54}$$

where the coefficients σ_k^n are given by

$$\sigma_k^n = \varepsilon_{2n}\left(\frac{1-\alpha}{\alpha}\right)^n\sum_{p=k}^{\infty}R_p^n S_{p-k}^p \tag{55}$$

and verify (conditions on the equator for $\lambda = 0$, $\pi/2$ and condition at the poles)

$$\sum_{k=0}^{\infty}\sigma_k^0 = 1$$

$$\sum_{k=0}^{\infty}(-1)^k\sigma_k^0 = b/a$$

$$\sum_{k=0}^{\infty}\sigma_k^n = \delta_{0k}c/a.$$

We can now write the a priori expansion (for reasons of symmetry)

$$r(\varphi,\lambda) = a\sum_l\sum_m A_{2l,2m}\cos 2m\lambda P_{2l,2m}(\sin\varphi) \tag{56}$$

where

$$A_{2l,2m} = (2-\delta_{0m})\frac{\nu_{lm}}{4\pi}\iint_{\sigma_1}\frac{r}{a}P_{2l,2m}(\sin\varphi)\cos 2m\lambda\,\mathrm{d}\sigma. \tag{57}$$

Taking (54) into account easily yields

$$A_{2l,2m} = \nu_{lm}\sum_{n=0}^{\infty}\sigma_m^n\int_0^1 u^{2n}P_{2l,2m}(u)\,\mathrm{d}u. \tag{58}$$

In (57) and (58), we have $\nu_{lm} = (4l+1)(2l-2m)!/(2l+2m)!$

When $m = 0$, it is known that the integral vanishes if $n < l$. We then find

$$A_{0,0} = \sum_{n=0}^{\infty}\frac{\sigma_0^n}{2n+1}\,. \tag{59}$$

For the zonal terms, we use the decomposition formula

$$u^N = \sum_{j=0}^{[N/2]} Z_{N-2j} P_{N-2j}(u) \tag{60}$$

with

$$Z_{N-2j} = 2^{N-2j}(2N-4j+1)\frac{N!(N-j)!}{j!(2N-2j+1)!}\,.$$

Taking $N = 2n$, then $n - j = l$ and integrating, we have

$$A_{2l,0} = 2^{2l}(4l+1)\sum_{n=l}^{\infty} \sigma_0^n \frac{(2n)!(n+l)!}{(n-l)!(2n+2l+1)!}\,. \tag{61}$$

When $m > 0$, we have to use the following expression of $P_{2l,2m}$

$$\begin{aligned} P_{2l,2m}(u) &= (1-u^2)^m \frac{1}{2^{2l}} \sum_{\eta=0}^{l-m} \frac{(-1)^\eta}{\eta!} \\ &\times \frac{(4l-2\eta)!}{(2l-\eta)!(2l-2m-2\eta)!} u^{2l-2m-2\eta}. \end{aligned} \tag{62}$$

Expanding $(1-u^2)^m$, multiplying by u^{2n} and integrating we find

$$\begin{aligned} A_{2l,2m} &= \frac{\nu_{lm}}{2^{2l}} \sum_{n=0}^{\infty} \sigma_m^n \sum_{j=0}^{m} \binom{m}{j} (-1)^j \sum_{\eta=0}^{l-m} \\ &\times \left[\frac{(-1)^\eta}{\eta!} \frac{(4l-2\eta)!}{(2l-\eta)!(2l-2m-2\eta)!} \{2(l+n+j-m-\eta)+1\}^{-1}\right]. \end{aligned} \tag{63}$$

These formulas, for which the summations on n in (59) and (61) and on p in (55) have in practice to be truncated, have been checked numerically against the exact value of the radius vector. It is interesting to note that there exist simple approximate formulas which may be sufficient in practical cases; they are given in Annex B. When condition (49) is not fulfilled (or if a is smaller but close to $c\sqrt{2}$), it is necessary (or more advantageous) to compute the coefficients $A_{2l,2m}$ by harmonic analysis – as shown in Section 7.

HARMONIC EXPANSION OF THE GRAVITATIONAL POTENTIAL

Due to symmetries, the expansion of the potential has a form analogous to (56) and can be a priori written as

$$U = \frac{GM}{r}\left[1 + \sum_{l=1}^{\infty}\left(\frac{R}{r}\right)^{2l} \sum_{m=1}^{l} C_{2l,2m} \cos 2m\lambda P_{2l,2m}(\sin\varphi)\right] \tag{64}$$

where

$$C_{2l,2m} = \mu_{lm} \iiint r^{2l} P_{2l,2m} \cos 2m\lambda \, \mathrm{d}v \tag{65}$$

with

$$\mu_{lm} = \frac{\rho}{MR^{2l}} (2 - \delta_{0m}) \frac{(2l-2m)!}{(2l+2m)!} . \tag{66}$$

R may here be taken equal to R_0 as defined by (B2) in annex B.

We start from the expression (62) of $P_{2l,2m}$. We multiply it by $r^{2l}\ e^{2im\lambda}$; $(x+iy)^{2m}$ gets formed, which we expand. Taking the real part, and introducing

$$T_{lm}^{ks} = (-1)^{k+s}(2m)!(4l-2k)!/ \\ [(2s)!(2m-2s)!(2l-k)!(2l-2m-2k)!]/2^{2l},$$

we find

$$r^{2l} P_{2l,2m} \cos 2m\lambda = \sum_{k=0}^{l-m} \sum_{s=0}^{m} T_{lm}^{ks} x^{2m-2s} y^{2s} z^{2(l-m-k)} (x^2+y^2+z^2)^k . \tag{67}$$

We now expand the last factor

$$(x^2+y^2+z^2)^k = \sum_{p=0}^{k} \sum_{q=0}^{p} \binom{k}{p} \binom{p}{q} x^{2q} y^{2p-2q} z^{2k-2p} .$$

To compute the integral, we make the transformation $x/a = X, y/b = Y, z/c = Z$; so $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = abc\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$. We now replace the new coordinates by the spherical coordinates r^*, φ^*, λ^* such that $X = r^* \cos\varphi^* \cos\lambda^*$, $Y = r^* \cos\varphi^* \sin\lambda^*$, $Z = r^* \sin\varphi^*$ and integrate with $\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = r^{*2}\,\mathrm{d}r^* \cos\varphi^*\,\mathrm{d}\lambda^*$. We find

$$C_{2l,2m} = \mu_{lm} I_{2l+2} \sum_{k,s,p,q} J_{2C+1}^{2S} K_{2Q}^{2P} \tag{68}$$

where

$$I_{2l+2} = \int_0^1 r^{*2l+2}\,\mathrm{d}r^* = \frac{1}{2l+3}$$

$$J_{2C+1}^{2S} = \int_{-\pi/2}^{+\pi/2} \sin^{2S}\varphi^* \cos^{2C+1}\varphi^*\,\mathrm{d}\varphi^* = 2\frac{(2C)!!(2S-1)!!}{(2S+2C+1)!!}$$

$$K_{2Q}^{2P} = \int_0^{2\pi} \sin^{2P}\lambda^* \cos^{2Q}\lambda^*\,\mathrm{d}\lambda^* = 2\pi\,\frac{(2P-1)!!(2Q-1)!!}{(2P+2Q)!!}$$

and with $S = l-m-p, C = m+p, P = s+p-q, Q = m-s+q$. We have used the double factorial symbol, with $(2v)!! = 2.4.6\ldots 2v, 2(v-1)!! = 1.3.5\ldots(2v-1)$, and $(-1)!! = 0!! = 1$.

Putting everything together and introducing the mass $M = \frac{4}{3}\pi abc\rho$, we arrive at

$$\begin{aligned} C_{2l,2m} &= \frac{3}{R^{2l}} \frac{l!(2m)!}{2^{2l}(2l+3)(2l+1)!} (2-\delta_{0m}) \frac{(2l-2m)!}{(2l+2m)!} \\ &\times \sum_{k=0}^{l-m} \frac{(-1)^k (4l-2k)!}{(2l-k)!(2l-2m-2k)!} \\ &\times \sum_{s=0}^{m} \frac{(-1)^s}{(2s)!(2m-2s)!} \sum_{p=0}^{k} \frac{(2l-2m-2p)!}{(l-m-p)!(k-p)!} \\ &\times \sum_{q=0}^{p} \frac{(2s+2p-2q)!(2m-2s+2q)!}{q!(p-q)!(s+p-q)!(m-s+q)!} \\ &\times \ a^{2(m-s+q)} b^{2(s+p-q)} c^{2(l-m-p)}. \end{aligned} \tag{69}$$

For example, we find

$$\begin{aligned} C_{20} &= \frac{1}{5R^2}\left(c^2 - \frac{a^2+b^2}{2}\right) \\ C_{22} &= \frac{1}{20R^2}(a^2-b^2) \\ C_{40} &= \frac{15}{7}(C_{20}^2 + 2C_{22}^2) \\ C_{42} &= \frac{5}{7} C_{20}C_{22} \\ C_{44} &= \frac{5}{28} C_{22}^2. \end{aligned} \tag{70}$$

Relationships (70) were quoted by Borderies and Yoder (1989) with some errors for C_{40}, C_{42}, C_{44}. In particular C_{40} is not proportional to C_{20}^2 if $a \neq b$. For an ellipsoid of revolution, we indeed find (Levallois, 1970)

$$C_{2l,0} = \frac{3}{R^{2l}} (a^2 - c^2)^l \frac{(-1)^l}{(2l+1)(2l+3)} \,. \tag{71}$$

and, for instance: $C_{40} = \frac{15}{7} C_{20}^2$.

Finally, let us note that the convergence of series (64) with the harmonics given by (69) is uniformly convergent for $a < c\sqrt{2}$ (Pick *et al.*, 1973) and down to the surface of the ellipsoid, as a consequence of a theorem by Laplace. This is the same condition found for the uniform convergence of the radius vector series.

NUMERICAL TESTS

With this set of results for the triaxial ellipsoid, we performed the numerical verification of formulas (26) to (32). We first transformed $A_{2l,2m}$ into A_{2l}^{2m}, computed the $K_l^m(H)$ quantities, summed up the results and went back to the $C_{2l,2m}(A)$ coefficients which we compared with those directly derived from formula (69). The agreement was found to be at the 10^{-13} level (in relative values) for flattenings $(a-b)/a$ and $(a-c)/a$ of the order of 0.001, and at the 10^{-6} level for flattenings of the order of 0.1. This check is however limited because of symmetry.

7. General Verification by Harmonic Analysis

The most convincing verification of our set of formulas must be made with a general kind of body, that is of S series (provided that they converge). One scheme is to take a sphere with superimposed topography $h(\varphi, \lambda)$, so that

$$h(\varphi, \lambda) = R_0 S(\varphi, \lambda) \tag{72}$$

as this is available in the case of the terrestrial planets, and to evaluate numerically the integrals defining $K_l^m(H)$ for $H \geq 2$. We write:

$$K_l^m(H) = \frac{4}{3}\pi \frac{\rho R_0^3}{M} \left(\frac{R_0}{R}\right)^l \frac{3}{(2l+1)(l+3)} \binom{l+3}{H} k_l^m(H) \tag{73}$$

where

$$k_l^m(H) = \iint_{\sigma_1} S^H Y_l^{m*} \, \mathrm{d}\sigma. \tag{74}$$

The computation of these integrals is done by a technique of fast harmonic analysis on the sphere which we have implemented as follows.

We first generate a grid of mean values of S^H, say $f_{kj} = \langle S^H(\varphi_k, \lambda_j)\rangle$ over bins of regular size $\Delta\varphi$. $\Delta\lambda$, where $\Delta\varphi = \pi/L$, L being the maximum degree of the harmonic expansion of $h(\varphi, \lambda)$. Then either $\Delta\lambda = \Delta\varphi$ as it is done in the algorithm here described, or $\Delta\lambda$ is made a function of φ if a pseudo-equal area decomposition of the sphere is adopted (the formulas given hereafter remain the same). The generation of those grids is efficiently made by a synthesis scheme, dual of the analysis one (see annex C). We used the usual normalized form, and then:

$$\bar{k}_{lm}(H) = \frac{\theta_m}{q_l} \sum_{k=1}^{K} \bar{I}_{lm}(k) \sum_{j=1}^{J} \alpha_{kj}^m \tag{75}$$

is an approximation of the integral (74) using mean values.

In this formula, evaluated up to degree and order L, we have
$K = \pi/\Delta\varphi (= L)$, $J = 2\pi/\Delta\lambda$ (may be a function of φ, that is of k).

$\theta_m = (2/m)\sin(m\Delta\lambda/2)$ if $m > 0$, $\theta_m = \Delta\lambda$ if $m = 0$.

q_l is a de-smoothing coefficient to account for the smoothing of the $\langle S^H \rangle$ functions, at wave-number l, with respect to S^H. According to Colombo (1981) we take: $q_l = \beta_l^2$ if $0 \leq l \leq L/3$, $q_l = \beta_l$ if $L/3 < l \leq L$, where the β_l's are coefficients (Meissl, 1971) defined by

$$\begin{aligned} \beta_0 &= 1 \\ \beta_l &= [P_{l-1}(\cos\Psi) - P_{l+1}(\cos\Psi)]/(2l+1)/(1-\cos\Psi) \end{aligned} \tag{76}$$

(the P_l's are the usual Legendre polynomials and $\cos\Psi = 1 - \Delta\varphi \cdot \Delta\lambda/2\pi$).

$\bar{I}_{lm}(k)$ = integral of $\bar{P}_{lm}(\sin\varphi)$ between the limiting latitudes $\varphi_1 = \varphi_k - \Delta\varphi/2$ and $\varphi_2 = \varphi_k + \Delta\varphi/2$ where $\varphi_k = \pi/2 - (k - 1/2)\Delta\varphi$. Stable formulas for the $\bar{P}_{lm}$'s and $\bar{I}_{lm}$'s, carefully tested up to degree and order 1200, are given in annex D.

α_{kj}^m = array computed from the mean values f_{kj} by recursion. We start from

$$\begin{aligned} \alpha_{kj}^0 &= f_{kj} \\ \alpha_{kj}^1 &= f_{kj}\, e^{-i\lambda_j} \end{aligned} \tag{77}$$

where $\lambda_j = \lambda_0 + (j - 1/2)\Delta\lambda$, $\lambda_0 = 0$ or π (origin of the grid in longitude); then

$$\alpha_{kj}^m = 2\cos\lambda_j \alpha_{kj}^{m-1} - \alpha_{kj}^{m-2}. \tag{78}$$

Numerical tests were carried up to $H = 5$ and up to degree and order 100 with the Martian topography. Comparisons with the analytic expressions (27) to (32) showed agreement in the individual harmonics to the fourth to seventh decimal places.

8. Application to Phobos

Bodies that are close to spherical provide a limited verification because of the lack of r dependence. A body with a complex shape, such as Phobos, provides a much better test.

We started from the spherical harmonics model of the topography of this body given by Duxbury (*ibid*), which is complete to degree and order 8. The coefficients were first normalized by (5), R_0 was taken equal to the initial A_{00} coefficient (= 11 040.045 m), and all the other harmonics were divided by R_0 to obtain the S series as in (20).

Although the individual uncertainties are not quoted by the author, we believe that they may be at the 1% level since an r.m.s. residual of 150 m is reported when using the expansion.

We computed the gravity harmonics up to degree and order 16 in order to have an idea of the contribution of the topographic harmonics to those coefficients above degree 8 although the numerical values may be far from reality due to lack of representation of the topography between the degrees 8 and 16. The computations were carried out analytically by formulas (33) and (34), yielding values K_l^m (analyt.), and by numerical integration as explained in the previous paragraph, yielding values K_l^m (numer.).

To evaluate the results we introduce the degree variances τ_j, γ_l of the topography ($1 \leq j \leq 8$) and of the gravitational potential ($1 \leq l \leq 16$) respectively, by

$$4\pi(2j+1)\tau_j^2 = \sum_{q=-j}^{+j} T_j^q T_j^{q^*} \tag{79}$$

$$4\pi(2l+1)\gamma_l^2 = \sum_{m=-l}^{+l} K_l^m K_l^{m^*}. \tag{80}$$

Similarly, we have

$$4\pi(2l+1)\delta_l^2 = \sum_{m=-l}^{+l} \delta K_l^m \cdot \delta K_l^{m^*} \tag{81}$$

with

$$\delta K_l^m = K_l^m(\text{analyt.}) - K_l^m(\text{numer.}) \tag{82}$$

by which we define the relative accuracy $\in_l$ of our formulas

$$\in_l = \delta_l/\gamma_l. \tag{83}$$

The quantities introduced are intrinsic in the sense that they are invariant by rotation (Moritz, *ibid*). Figure 1 shows the relative agreement of the analytical and of the numerical techniques for such a complex body, which is at the 10^{-5} level. This is satisfactory for the applications considering the uncertainties on the initial coefficients T_j^q – to which the uncertainties on the $K_l^m(1)$ are directly proportional according to (27), also in view of the total power of the disturbing potential (central part omitted)

$$\Gamma = \sum_{l\geq 1}(2l+1)\gamma_l^2 \tag{84}$$

from which the norm $\Gamma^{1/2}$ is found to be 3.7×10^{-2} (at this resolution).

Also of interest is the speed of convergence of the $K_l^m(H)$, with H, which depends on the S^H series. On Figure 2, we see that the convergence, measured by the total power $\Gamma(H)$ of the $K_l^m(H)$ coefficients, is fairly fast and reaches a satisfactory level of 10^{-5} (compatible with the level of accuracy of our formulas as demonstrated above) for $H = 9$.

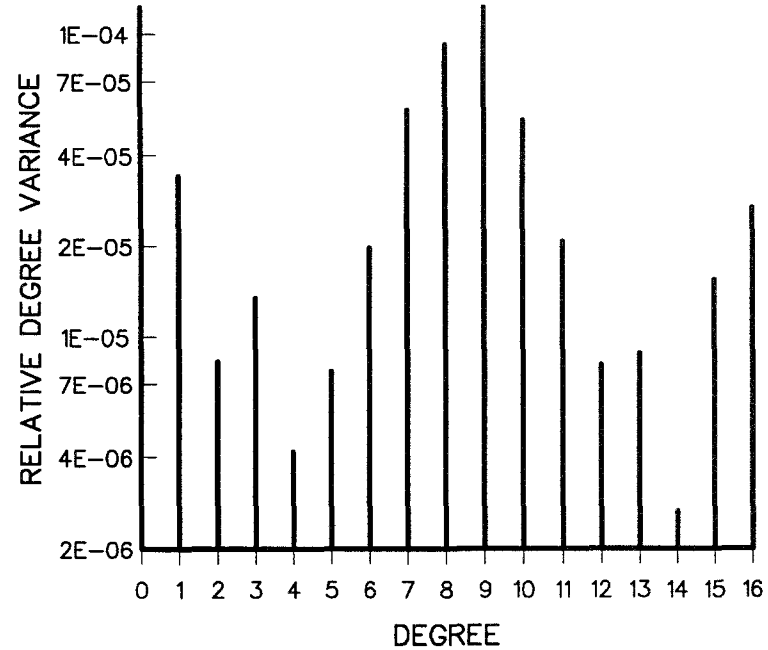

Fig. 1. Relative degree variances ϵ_l showing the agreement between the analytical and the numerical techniques.

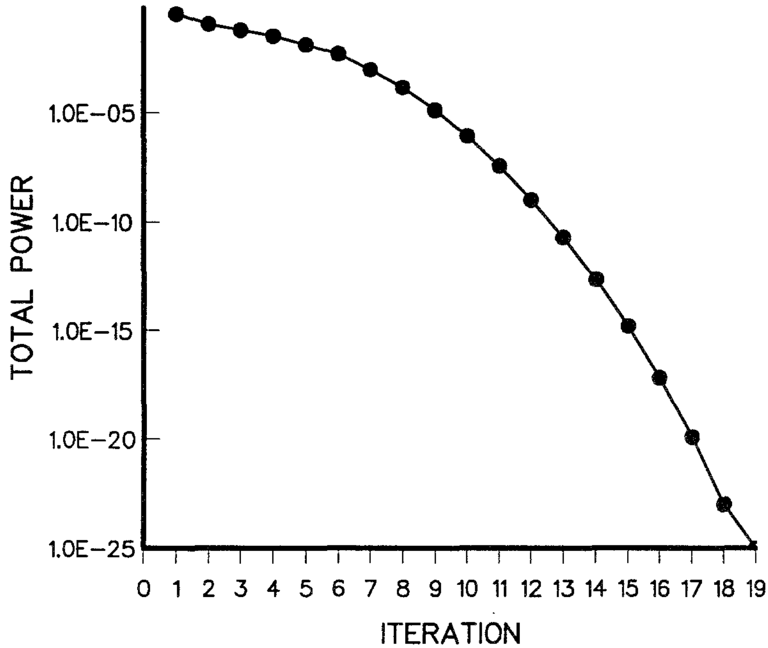

Fig. 2. Speed of convergence of formulas (33) and (73), measured by the total power $\Gamma(H)$ of the coefficients computed from S^H.

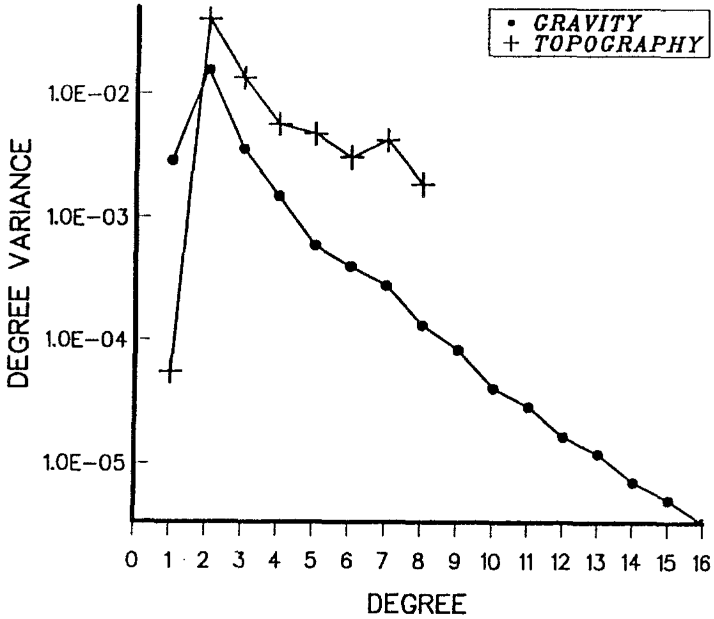

Fig. 3. Degree variances τ_l, γ_l of the topography and of the gravitational potential, respectively, as a function of the degree l of the harmonics.

Finally, Figure 3 gives the behaviour of the degree variances of the topography and gravitational harmonics. Ignoring the terms of degree one in the gravitational potential which reflect an offset between the center of mass and the center of figure (which may be real or due to uncertainties in the body shape), we may fit very well the gravitational variances by the empirical law

$$\gamma_l \approx \gamma_0 \, e^{-\alpha l} \tag{85}$$

with $\gamma_0 = 3.32 \times 10^{-2}$ and $\alpha = 0.6$. A fit may also be obtained with more conventional power laws, such as those experienced with the terrestrial planets (Balmino, 1993); we then find

$$\begin{aligned} \tau_l &\approx 0.16/l^2 \\ \gamma_l &\approx 0.40/l^4 \end{aligned} \tag{86}$$

for l in the range 2 to 8 (or 16 for the gravitational potential). It would be interested to test the validity of these laws with an improved model of Phobos as it may be obtained in future missions, and over a larger range for l.

With such laws, we may try to answer questions about the approximation due to truncation in the topographic and in the gravitational harmonics series. The lack of representation in the topographic series may be measured by

$$\Delta_n^T = [T_{(\infty)} - T_{(n)}]^{1/2} = \left[\sum_{l=n+1}^{\infty} (2l+1)\tau_l^2 \right]^{1/2} \tag{87}$$

or, since $(2l+1)\tau_l^2$ is here a positive decreasing function, it may be bounded as:

$$\Delta_n^T < \tau_0 \left\{ \int_n^{+\infty} \frac{(2x+1)}{x^4} \, dx \right\}^{1/2} .$$

That is

$$\Delta_n^T < \frac{\tau_0}{n} \left(1 + \frac{1}{3n} \right)^{1/2} . \tag{88}$$

Using $\tau_0 = 0.16$ and $n = 8$, we find $\Delta_8^T < 2.04 \times 10^{-2}$ to be compared with the total norm (0.185 using the law) or to the norm of the (8, 8) model equal to 0.173.

Similarly for the gravitational potential we may write

$$\Delta_n^K < \left[\int_n^{+\infty} (2x+1)\gamma_x \, dx \right]^{1/2} .$$

Using the exponential law (85) yields:

$$\Delta_n^K < \frac{\gamma_0}{\sqrt{\alpha}} \left(n + \frac{1}{2} + \frac{1}{2\alpha} \right)^{1/2} e^{-\alpha n} . \tag{89}$$

With the adopted constants we find $\Delta_8^K < 1.07 \times 10^{-3}$, and $\Delta_{16}^K < 1.2 \times 10^{-5}$.

If we adopt the power law instead, we have:

$$\Delta_n^K < \frac{\gamma_0}{n^{\alpha-1}} \left[\frac{1}{\alpha - 1} + \frac{1}{(2\alpha - 1)n} \right]^{1/2} . \tag{90}$$

That is $\Delta_8^K < 4.6 \times 10^{-4}$ and $\Delta_{16}^K < 5.7 \times 10^{-5}$, to be compared for instance to the found norm of the disturbing potential (3.7×10^{-2}). Therefore we conclude that, the topography harmonics of Phobos being given up to degree 8 (this, apart from errors in those harmonics, has a lack of representation of $\sim 12\%$ if we believe in a simple power law to represent the topography harmonics behaviour), adopting gravitational harmonics derived from them and up to the same degree results in an approximation at the 3% level – taking the worst case with the exponential law. This error of representation is small enough, considering the other errors, to accept and use the derived (8, 8) gravitational model in subsequent applications. Of course, this mathematical error can be reduced below 1% if the degree of the gravitational series is increased to 16, but the gravitational harmonics beyond degree 8 are not physically significant as said before.

9. Conclusion

We have derived exact expressions of the spherical harmonic coefficients of the gravity potential of an homogeneous body which shape is also given as a series of spherical harmonics. These formulas have been checked by means of the coefficients of the shape and potential of a triaxial ellipsoid, independently derived, and by direct numerical harmonic analysis especially in the case of Phobos. One practical importance of having these precise expressions for the gravitional harmonics is in the navigation of space vehicles visiting small bodies of the solar system such as small satellites or asteroids. It also gives the capability of testing whether such bodies are of constant density or not, which may give clues to the process of their formation.

Annex A. Product-Sum Conversion for Spherical Harmonics and Related Integrals

Because of the orthogonality of the surface spherical harmonic functions $Y_l^m(\varphi, \lambda)$, any product of such functions can be transformed into a linear form of the Y's. The question has been treated by many authors and notations and conventions vary widely. A very useful compendium of results has been compiled by Rotenberg *et al.* (1959) for the Y_l^m type of functions, where the authors make use of the Clebsch–Gordan coefficients with the $3-j$ notation of Wigner. A simpler form for the product of two non-normalized functions Y_{lm} was derived by Balmino (1978) as

$$Y_{lm} Y_{jq} = \sum_k Q_{lmjq}^k Y_{k,m+q} \tag{A1}$$

where k runs from $\max(|m+q|, |l-j|)$ to $l+j$ and where $l+j+k$ is always even. From this work, it was possible to compute the Q's in rational form for products of functions up to degree and order 20 (using 128 bit words). Above this degree, a more traditional programming yield the coefficients in floating form with sufficient accuracy up to about degree 40 for the normalized functions $\bar{Y}_{lm}$.

We decided to adopt the Rotenberg form of the decomposition formula, after satisfactory numerical experiments. It writes

$$\begin{aligned} Y_l^m Y_j^q &= \sum_k \left[\frac{(2l+1)(2j+1)(2k+1)}{4\pi}\right]^{1/2} \\ &\times \begin{pmatrix} l & j & k \\ m & q & -m-q \end{pmatrix} \begin{pmatrix} l & j & k \\ 0 & 0 & 0 \end{pmatrix} (-1)^{m+q} Y_k^{m+q} \end{aligned} \tag{A2}$$

where k runs as in formula (A1) and with

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= (-1)^{j_1-j_2-m_3} \\ \left[\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!\,(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}\right]^{1/2} &\sum_s \frac{(-1)^s}{s!} \\ \frac{[(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!]^{1/2}}{(j_1+j_2-j_3-s)!(j_1-m_1-s)!(j_2+m_2-s)!(j_3-j_2+m_1+s)!(j_3-j_1-m_2+s)!} & \end{aligned} \tag{A3}$$

This coefficient is the Wigner $3-j$ symbol (Wigner, 1958). In (A3), the index s runs from $\max(0, j_2-j_3-m_1, m_2+j_1-j_3)$ to $\min(j_1+j_2-j_3, j_1-m_1, j_2+m_2)$. The bracketted term in the summation does not depend on s but is put inside for numerical reasons when evaluating the coefficients. To compute them accurately, we computed the logarithm of each term (to avoid numerical overflows) and properly re-ordered all the terms before summation to reduce the numerical errors. A lot of properties exist for the $3-j$ coefficients. The most interesting one is that they vanish if one of the following conditions is not fulfilled

$j_1 + j_2 - j_3 \geq 0$
$j_1 - j_2 + j_3 \geq 0$
$-j_1 + j_2 + j_3 \geq 0$
$m_1 + m_2 + m_3 = 0$
$j_1 + j_2 + j_3$ even if $m_1 = m_2 = m_3 = 0$.

The following relation holds between our Q's and the $3-j$ coefficients

$$\begin{pmatrix} l & j & k \\ m & q & -m-q \end{pmatrix} \begin{pmatrix} l & j & k \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{m+q}}{2k+1} \left[\frac{(l-m)!(j-q)!(k+m+q)!}{(l+m)!(j+q)!(k-m-q)!} \right]^{1/2} Q^k_{lmjq}. \tag{A4}$$

The coefficients of formula (A3) were checked with the identity (A2) up to degree 100 and the relative accuracy was always better than 10^{-25} (in quadruple precision). We now introduce the notation

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & * \end{bmatrix} = (-1)^{m_1+m_2}[(2l_1+1)(2l_2+1)(2l_3+1)/4\pi]^{1/2} \times \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A5}$$

Then, using (A2) and the orthogonality of the Y's we find

$$\iint_{\sigma_1} Y_{l_1}^{m_1} Y_{l_2}^{m_2} Y_{l_3}^{m_3} \, d\sigma = (-1)^{m_3} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & * \end{bmatrix} \delta(m_1+m_2+m_3, 0).$$

In the case of four functions, we first write

$$\begin{aligned} Y_{l_1}^{m_1} Y_{l_2}^{m_2} Y_{l_3}^{m_3} &= \sum_k \begin{bmatrix} l_1 & l_2 & k \\ m_1 & m_2 & * \end{bmatrix} Y_k^{m_1+m_2} Y_{l_3}^{m_3} \\ &= \sum_k \sum_s \begin{bmatrix} l_1 & l_2 & k \\ m_1 & m_2 & * \end{bmatrix} \begin{bmatrix} k & l_3 & s \\ m_1+m_2 & m_3 & * \end{bmatrix} Y_s^{m_1+m_2+m_3} \\ &= \sum_s \begin{bmatrix} l_1 & l_2 & l_3 & s \\ m_1 & m_2 & m_3 & * \end{bmatrix} Y_s^{m_1+m_2+m_3}. \end{aligned} \tag{A6}$$

with

$$\begin{bmatrix} l_1 & l_2 & l_3 & s \\ m_1 & m_2 & m_3 & * \end{bmatrix} = \sum_k \begin{bmatrix} l_1 & l_2 & k \\ m_1 & m_2 & * \end{bmatrix} \begin{bmatrix} k & l_3 & s \\ m_1 + m_2 & m_3 & * \end{bmatrix}. \tag{A7}$$

In (A6), s runs from $|m_1 + m_2 + m_3|$ to $l_1 + l_2 + l_3$, $l_1 + l_2 + l_3 + s$ must be even and triangular inequalities on (l_1, l_2, k) and (k, l_3, s) must be satisfied.

In (A7), k runs from $\max(|m_1+m_2|, |l_1-l_2|)$ to l_1+l_2, l_1+l_2+k and l_3+k+s must be even. In consequence we have

$$\iint_{\sigma_1} Y_{l_1}^{m_1} Y_{l_2}^{m_2} Y_{l_3}^{m_3} Y_{l_4}^{m_4}\, \mathrm{d}\sigma$$
$$= (-1)^{m_4} \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & * \end{bmatrix} \delta(m_1 + m_2 + m_3 + m_4, 0). \tag{A8}$$

The general product-sum conversion formula is

$$\prod_1^N Y_{l_i}^{m_i} = Y_{l_1}^{m_1} Y_{l_2}^{m_2} \dots Y_{l_N}^{m_N} = \sum_k \begin{bmatrix} l_1 & l_2 & \dots & l_N & k \\ m_1 & m_2 & \dots & m_N & * \end{bmatrix} Y_k^{M_N} \tag{A9}$$

where $M_N = m_1+m_2+\ldots+m_N$, and k varies between $|M|$ and $l_1+l_2+\ldots+l_N = L$ with $L + k$ being even. The Nth order symbol is defined as

$$\left[\begin{pmatrix} l_i \\ m_i \end{pmatrix}^{(N)} \begin{matrix} k \\ * \end{matrix}\right] = \begin{bmatrix} l_1 & l_2 & \dots & l_{N-1} & l_N & k \\ m_1 & m_2 & \dots & m_{N-1} & m_N & * \end{bmatrix}$$
$$= \sum_j \begin{bmatrix} l_1 & l_2 & \dots & l_{N-1} & j \\ m_1 & m_2 & \dots & m_{N-1} & * \end{bmatrix} \begin{bmatrix} j & l_N & k \\ M_{N-1} & m_N & * \end{bmatrix} \tag{A10}$$

with $M_{N-1} = m_1 + m_2 + \ldots + m_{N-1}$.

The corresponding integral is

$$\iint_{\sigma_1} \prod_{i=1}^N Y_{l_i}^{m_i}\, \mathrm{d}\sigma = (-1)^{m_N} \left[\begin{pmatrix} l_i \\ m_i \end{pmatrix}^{(N-1)} \begin{matrix} l_N \\ * \end{matrix}\right] \delta(M_N, 0). \tag{A11}$$

As a consequence, the integrals needed in our work are

$$I_{l_1,l_2,\ldots,l_N l}^{m_1,m_2,\ldots,m_N m} = \iint_{\sigma_1} \left(\prod_{i=1}^N Y_{l_i}^{m_i}\right) Y_l^{m^*}\, \mathrm{d}\sigma = \left[\begin{pmatrix} l_i \\ m_i \end{pmatrix}^{(N)} \begin{matrix} l \\ * \end{matrix}\right] \delta(M_N, m)$$

Annex B. Approximate Expression in Spherical Harmonics of the Radius Vector of a Triaxial Ellipsoid

We start from the general equation of the surface of a triaxial ellipsoid, taken as

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fyx = 1 \tag{B1}$$

where there are no first degree terms in assuming that the center of the ellipsoid has been previously determined and taken as the origin of the coordinates.

It is more general than the basic form used in the core of this paper, but it has the advantage of leading to interesting similarities with the relationships between the gravity harmonics and the inertia tensor components of a body.

We use the following identities

$$\begin{aligned}
\frac{x^2}{r^2} &= \frac{1}{2}\cos^2\varphi(1+\cos 2\lambda) = \frac{1}{3}(1-P_{20}) + \frac{1}{6}P_{22}\cos 2\lambda \\
\frac{y^2}{r^2} &= \frac{1}{2}\cos^2\varphi(1-\cos 2\lambda) = \frac{1}{3}(1-P_{20}) - \frac{1}{6}P_{22}\sin 2\lambda \\
\frac{z^2}{r^2} &= \sin^2\varphi = \frac{2}{3}\left(\frac{1}{2}+P_{20}\right) \\
\frac{xy}{r^2} &= \frac{1}{6}P_{22}\sin 2\lambda \\
\frac{yz}{r^2} &= \frac{1}{3}P_{21}\sin\lambda \\
\frac{xz}{r^2} &= \frac{1}{3}P_{21}\cos\lambda.
\end{aligned}$$

Inserting these in (B1) we find

$$\begin{aligned}
\frac{1}{r^2} &= \frac{A+B+C}{3} + \frac{2C-A-B}{3}P_{20} + \frac{A-B}{6}P_{22}\cos 2\lambda \\
&\quad - \frac{2}{3}DP_{21}\sin\lambda - \frac{2}{3}EP_{21}\cos\lambda - \frac{1}{3}FP_{22}\sin 2\lambda.
\end{aligned}$$

Let

$$\begin{aligned}
R_0 &= \left(\frac{3}{A+B+C}\right)^{1/2} \\
\alpha &= \frac{A+B-2C}{A+B+C} \\
\beta &= \frac{1}{2}\frac{B-A}{A+B+C} \\
\gamma &= \frac{F}{A+B+C} \\
\delta &= \frac{2D}{A+B+C} \\
\varepsilon &= \frac{2E}{A+B+C}.
\end{aligned} \qquad \text{(B2)}$$

Then

$$r = R_0(1 - \mathbf{W})^{-1/2} = R_0\left(1 + \frac{1}{2}\,\mathbf{W} + \frac{3}{8}\,\mathbf{W}^2 + \ldots\right)$$

with

$$\mathbf{W} = \alpha P_{20} + \beta P_{22}\cos 2\lambda + \gamma P_{22}\sin 2\lambda + \delta P_{21}\sin\lambda + \varepsilon P_{21}\cos\lambda.$$

Consequently, we have to the second order in α, β, γ, δ, ε

$$r = R_0 \sum_{l=0}^{4}\sum_{m=0}^{l}(A_{lm}\cos m\lambda + B_{lm}\sin m\lambda)P_{lm}(\sin\varphi). \tag{B3}$$

The only contribution of order zero terms is to A_{00} and obviously

$$A_{00}(0) = 1.$$

The non zero first order terms, as directly coming from $\mathbf{W}$, are

$$\begin{aligned}
A_{20}(1) &= \frac{\alpha}{2} = R_0^2\left(\frac{A+B}{2} - C\right)\\
A_{22}(1) &= \frac{\beta}{2} = \frac{R_0^2}{3}\,\frac{B-A}{4}\\
B_{22}(1) &= \frac{\gamma}{2} = \frac{R_0^2}{3}\,\frac{F}{2}\\
A_{21}(1) &= \frac{\varepsilon}{2} = \frac{R_0^2}{3}\,E\\
B_{21}(1) &= \frac{\delta}{2} = \frac{R_0^2}{3}\,D.
\end{aligned} \tag{B4}$$

These relations are identical, except the common multiplying factor $R_0^2/3$, to those between the gravity harmonics and the tensor of inertia components of a general body.

The second order terms are obtained by expanding $\mathbf{W}^2$ and transforming the products of Legendre functions into sums. We used here our earlier work (Balmino, 1978, *ibid.*).

Useful relations are

$$P_{21}^2 = \frac{3}{7}\,P_{22} + \frac{6}{35}\,P_{42} = \frac{6}{5} + \frac{6}{7}\,P_{20} - \frac{72}{35}\,P_{40}.$$

The first form is to be associated with $\pm(\cos 2\lambda)/2$ and the second one to 1/2 in the decomposition of $\cos^2\lambda$ or $\sin^2\lambda$ into $(1 \pm \cos 2\lambda)/2$. Also

$$P_{22}^2 = \frac{3}{35}\,P_{44} = \frac{24}{5} - \frac{48}{7}\,P_{20} + \frac{72}{35}\,P_{40}$$

$$
\begin{aligned}
P_{20}P_{21} &= \frac{1}{7}\,P_{21} + \frac{9}{35}\,P_{41} \\
P_{20}P_{22} &= -\frac{2}{7}\,P_{22} + \frac{3}{35}\,P_{42} \\
P_{20}^2 &= \frac{1}{5} + \frac{2}{7}\,P_{20} + \frac{18}{35}\,P_{40}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
A_{00}(2) &= \frac{3}{40}\,[\alpha^2 + 12(\beta^2+\gamma^2) + 3(\delta^2+\varepsilon^2)] \\
A_{20}(2) &= \frac{3}{56}\,[2\alpha^2 - 24(\beta^2+\gamma^2) + 3(\delta^2+\varepsilon^2)] \\
A_{21}(2) &= \frac{3}{28}\,[(\alpha-\beta)\varepsilon + 6\delta\gamma] \\
B_{21}(2) &= \frac{3}{28}\,[(\alpha-\beta)\delta + 6\varepsilon\gamma] \\
A_{22}(2) &= -\frac{3}{112}\,[8\alpha\beta + 3(\delta^2-\varepsilon^2)] \\
B_{22}(2) &= \frac{3}{56}\,[-4\alpha\gamma + 3\delta\varepsilon] \\
A_{40}(2) &= \frac{27}{140}\,[\alpha^2 + 2\beta^2 - 2(\delta^2+\varepsilon^2-\gamma^2)] \\
A_{41}(2) &= \frac{27}{140}\,[(\alpha+\beta)\varepsilon - \delta\gamma] \\
B_{41}(2) &= \frac{27}{140}\,[(\alpha+\beta)\delta - \varepsilon\gamma] \\
A_{42}(2) &= \frac{9}{280}\,[2\alpha\beta - (\delta^2-\varepsilon^2)] \\
B_{42}(2) &= \frac{9}{140}\,[\alpha\gamma + \delta\varepsilon] \\
A_{43}(2) &= \frac{9}{280}\,[\beta\varepsilon - \delta\gamma] \\
B_{43}(2) &= \frac{9}{280}\,[\beta\delta + \varepsilon\gamma] \\
A_{44}(2) &= \frac{9}{560}\,[\beta^2 - \gamma^2] \\
B_{44}(2) &= \frac{9}{280}\,\beta\gamma. \qquad \text{(B5)}
\end{aligned}
$$

The other terms are zero up to second order in the small parameters.

Annex C. Fast Harmonic Synthesis over the Sphere

We want to compute values of a function known by a spherical harmonics approximation of high degree and order (typically with 10^5 to 10^6 terms) at the nodes (φ_i, λ_j) of a regular grid over a sphere (or part of it), or mean values over the cells defined by these nodes.

We have $\varphi_{i+1} - \varphi_i = \Delta\varphi$, $\lambda_{j+1} - \lambda_j = \Delta\lambda$ (possibly a function of latitude).

The algorithm given below is in some way equivalent to an FFT in longitude over the sphere. It is here written for point values. In the case of mean values, one has simply to replace the quantities $\cos m\lambda_j$ or $\sin m\lambda_j$ by $\theta_m \cos m\bar{\lambda}_j$ or $\theta_m \sin m\bar{\lambda}_j$ where

$$\begin{aligned} \theta_0 &= 1 \\ \theta_m &= \frac{\sin m\Delta\lambda/2}{m\Delta\lambda/2} \\ \bar{\lambda}_j &= (\lambda_j + \lambda_{j+1})/2 \\ \Delta\lambda &= \lambda_{j+1} - \lambda_j . \end{aligned} \tag{C1}$$

Also the Legendre functions at φ_i must be replaced by their integrals taken between φ_i and φ_{i+1}. So let us assume that we have to determine a set of grid values of a function expanded with real normalized harmonics, as

$$f(r, \varphi, \lambda) = f_0 \sum_{l=0}^{L} g_l(r, \varphi) \sum_{m=0}^{l} (\bar{A}_{lm} \cos m\lambda + \bar{B}_{lm} \sin m\lambda) \bar{P}_{lm}(\sin\varphi)$$

where the points (r, φ, λ) are the nodes (φ_i, λ_j) of the grid and with $r = r(\varphi)$ (for instance when some function is evaluated on an ellipsoid of revolution).

We first make the following transformation

$$f(r, \varphi, \lambda) = f_0 \sum_{m=0}^{L} f_m \tag{C2}$$

with

$$f_m = \alpha_m \cos m\lambda + \beta_m \sin m\lambda \tag{C3}$$

and where

$$\begin{bmatrix} \alpha_m \\ \beta_m \end{bmatrix} = \sum_{l=m}^{L} g_l[r(\varphi), \varphi] \bar{P}_{lm}(\sin\varphi) \begin{bmatrix} \bar{A}_{lm} \\ \bar{B}_{lm} \end{bmatrix} . \tag{C4}$$

Introducing $\lambda_j = \lambda_0 + j\Delta\lambda$, denoting $f_{m,j} = f_m(\lambda_j)$, and using the identities

$$\begin{aligned}\cos mj\Delta\lambda &= 2\cos m\Delta\lambda \cos m(j-1)\Delta\lambda - \cos m(j-2)\Delta\lambda \\ \sin mj\Delta\lambda &= 2\cos m\Delta\lambda \sin m(j-1)\Delta\lambda - \sin m(j-2)\Delta\lambda\end{aligned}$$

we find

$$\begin{aligned}f_{m,j} &= \alpha_m \cos m\lambda_j + \beta_m \sin m\lambda_j \\ &= 2\cos m\Delta\lambda \cdot f_{m,j-1} - f_{m,j-2}.\end{aligned} \tag{C5}$$

This recursive relation is initialized with

$$\begin{aligned}f_{m,0} &= \alpha_m \cos m\lambda_0 + \beta_m \sin m\lambda_0 \\ f_{m,1} &= f_{m,0}\cos m\Delta\lambda + \sin m\Delta\lambda(\beta_m \cos m\lambda_0 - \alpha_m \sin m\lambda_0).\end{aligned} \tag{C6}$$

Annex D. Computation of Legendre Functions and Integrals

The Legendre polynomials $\bar{P}_{l0}$ of degree l and associated functions $\bar{P}_{lm}$ of degree l and order m are fully normalized, so that

$$\frac{1}{4\pi}\iint_{\sigma_1} \bar{P}^2_{lm}(\sin\varphi)\begin{bmatrix}\cos^2 m\lambda \\ \sin^2 m\lambda\end{bmatrix}\cos\varphi \,\mathrm{d}\varphi\,\mathrm{d}\lambda = 1 \tag{D1}$$

(over the unit sphere σ_1).

Their definite integrals $\bar{I}_{lm}$ are computed efficiently by stable recursive formulas, for instance according to Gerstl (1980). The set listed below is a variant of Gerstl's work, adapted to much higher degree and order than originally studied by this author.

We assume that all computations are to be carried out to a maximum degree L and maximum order L, too. The definite integrals are evaluated between two latitudes φ_1, φ_2 ($\varphi_1 < \varphi_2$), that is

$$\bar{I}_{lm} = \int_{\varphi_1}^{\varphi_2} \bar{P}_{lm}(\sin\varphi)\cos\varphi\,\mathrm{d}\varphi \qquad (0 \le m \le l \le L). \tag{D2}$$

We first define $x = \sin\varphi$, $y = \cos\varphi$, $\bar{\varphi} = (\varphi_1 + \varphi_2)/2$, $x_i = \sin\varphi_i$, $y_i = \cos\varphi_i$, (i = 1, 2), and the following coefficients

$$\begin{aligned}\tau_l &= [(2l-1)(2l+1)]^{1/2} \\ W_1^1 &= \sqrt{3} \\ W_m^m &= [1 + 1/2m]^{1/2} \quad \text{for} \quad 2 \le m \le L \\ W_l^m &= \tau_l/[(l-m)(l+m)]^{1/2} \quad \text{for} \quad 1 \le m < l \le L.\end{aligned} \tag{D3}$$

The integrals of the Legendre polynomials are computed as follows

$$\begin{aligned} \bar{I}_{00} &= x_2 - x_1 \\ \bar{I}_{10} &= \sqrt{3}\,\bar{I}_{00}\bar{\varphi} \\ \bar{I}_{l0} &= \{\bar{P}_{l+1,0}/\tau_{l+1} - \bar{P}_{l-1,0}/\tau_l\}_1^2, \quad (2 \le l \le L) \end{aligned} \tag{D4}$$

where $\{\ldots\}_1^2$ is the difference between the values of the expression at φ_2 and φ_1. For these, we need the Legendre polynomials, computed by the sequel

$$\begin{aligned} \bar{P}_{10} &= \sqrt{3}\,x \\ \bar{P}_{20} &= \sqrt{5}\,(3x^2 - 1)/2 \\ \bar{P}_{l+1,0} &= \tau_{l+1}(x\bar{P}_{l0} - l\bar{P}_{l-1,0}/\tau_l)/(l+1), \quad (2 \le l \le L). \end{aligned} \tag{D5}$$

For the integrals of the associated functions, we need these functions themselves, $\bar{P}_{lm}$, in φ_1 and φ_2. They are evaluated by the recursive formulas

$$\begin{aligned} \bar{P}_{11} &= \sqrt{3}\,y \\ \bar{P}_{mm} &= yW_m^m\bar{P}_{m-1,m-1}, \quad (2 \le m \le L) \\ \bar{P}_{m+1,m} &= xW_{m+1}^m\bar{P}_{m,m}, \quad (1 \le m \le L) \\ \bar{P}_{lm} &= W_l^m(x\bar{P}_{l-1,m} - \bar{P}_{l-2,m}/W_{l-1}^m), \quad (m+2 \le l \le L). \end{aligned} \tag{D6}$$

Then, according to $\bar{\varphi}$, we initialize the sectorial integrals with the greatest possible accuracy. The critical latitude, φ_0, which sets the type of formulas to be used, has been empirically determined for L in the range 1 to 1200. We adopted $\varphi_0 = 15°$.

$$\begin{aligned} &-\text{if}|\bar{\varphi}| \le \varphi_0 : \bar{I}_{11} = \{\varphi + xy\}_1^2\sqrt{3}/2 \\ &\bar{I}_{mm} = W_m^m[mW_{m-1}^{m-1}\bar{I}_{m-2,m-2} + \{xP_{mm}\}_1^2]/(m+1) \\ &(2 \le m \le L) \end{aligned} \tag{D7}$$

$-\text{if}|\bar{\varphi}| > \varphi_0$: we take $M = \text{int}\,\{100(100 - \bar{\varphi})\}$ and

$$\begin{aligned} \bar{I}_{mm} &= -\text{sgn}\,(\bar{\varphi}) \cdot \Bigg\{\bar{P}_{mm}y^2\left[\frac{1}{m+2} + \frac{1}{2}\,y^2\left[\frac{1}{m+4} + \frac{3}{4}\,y^2\left(\frac{1}{m+6} + \frac{5}{6}\,y^2\right.\right.\right. \\ &\times \left.\left.\left.\left(\frac{1}{m+8} + \frac{7}{8}\,y^2 + \cdots + \frac{2M-1}{2M}\,y^2\left(\frac{1}{m+2M+2}\right)\cdots\right)\right)\right]\right]\Bigg\}_1^2 \\ &(m = L \text{ and } L-1) \end{aligned} \tag{D8}$$

$$\begin{aligned} &\bar{I}_{m-2,m-2} = [(m+1)\bar{I}_{mm} - \{x\bar{P}_{mm}\}_1^2]/(mW_m^mW_{m-1}^{m-1}) \\ &(m = L \text{ to } 4) \\ &\bar{I}_{11} = \{\varphi + xy\}_1^2\sqrt{3}/2. \end{aligned} \tag{D9}$$

Finally, the tesseral integrals ($l \neq m$) are computed via the recursive formulas:

$$\bar{I}_{m+1,m} = -W^m_{m+1}/(m+2)\{y^2\bar{P}_{mm}\}^2_1, \quad (1 \leq m \leq L-1)$$
$$\bar{I}_{l,m} = W^m_l[(l-2)\bar{I}_{l-2,m}/W^m_{l-1} - \{y^2\bar{P}_{l-1,m}\}^2_1]/(l+1)$$
$$(m+2 \leq l \leq L). \quad \text{(D10)}$$

This set of formulas has been programmed on a CDC Cyber 2000 V computer in single precision ($\approx$ 14 accurate digits in elementary operations). The precision was tested: (I) by comparisons with values obtained with a double precision ($\approx$ 28 significant figures) version of the program; (II) by evaluating the definite integrals numerically, by means of a 16, 32 or 64 knots Gaussian quadrature formula. These tests were carried out for $L = 1$ to 1200, $\bar{\varphi} = -90°$ to $90°$, $\varphi_2 - \varphi_1 = 0°.1$ to $5°$.

Largest relative 'errors' never exceeded 5×10^{-11} in this range. The programming itself was optimized for the Cyber vectorized compiler which resulted in very short computer times.

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