AN INTEGRABLE CASE OF A ROTATIONAL MOTION ANALOGOUS TO THAT OF LAGRANGE AND POISSON FOR A GYROSTAT IN A NEWTONIAN FORCE FIELD

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Abstract. The scope of the present paper is to provide analytic solutions to the problem of the attitude evolution of a symmetric gyrostat about a fixed point in a central Newtonian force field when the potential function is $V^{(2)}$.

We assume that the center of mass and the gyrostatic moment are on the axis of symmetry and that the initial conditions are the following: $\psi(t_0) = \psi_0$, $\theta(t_0) = \theta_0$, $\phi(t_0) = \phi_0$, $\omega_1(t_0) = 0$, $\omega_2(t_0) = 0$ and $\omega_3(t_0) = \omega_3^0$.

The problem is integrated when the third component of the total angular momentum is different from zero ($B_1 \neq 0$). There now appear equilibrium solutions that did not exist in the case $B_1 = 0$, which can be determined in function of the value of l_3^r (the third component of the gyrostatic momentum).

The possible types of solutions (elliptic, trigonometric, stationary) depend upon the nature of the roots of the function $g(u)$. The solutions for Euler angles are given in terms of functions of the time t. If we cancel the third component of the gyrostatic momentum ($l_3^r = 0$), the obtained solutions are valid for rigid bodies.

Key words: Dynamics of rigid bodies and gyrostats, analogous case to that of Lagrange and Poisson, analytic solutions.

1. Introduction

The problem of the rotatory motion of a rigid body about a fixed point, subject to external moments, has been studied in numerous articles, by authors such as Euler, Jacobi, Poinsot, Lagrange, Poisson, Kowalesky, only to mention a few of the most classic ones; analytical solutions were found in certain cases.

So as to make apparent the influence of internal motions of the bodies, that do not vary their mass distribution, on the rotation of the bodies, the rotatory motion of gyrostats about a fixed point has also been studied. In particular, of stationary gyrostats, which are those whose relative angular momentum of its mobile part with respect to its rigid part (also known as gyrostatic momentum) is constant in the mobile system (fixed in its rigid part). The problem of the rotatory motion of a gyrostat about a fixed point is more general than that of the rigid body (which is what it is reduced to if the gyrostatic moment cancels itself). It has been dealt with, amongst other authors, by G. Peano, V. Volterra, V. A. Steklov, U. Cassina, M. T. Vacca, in works dedicated above all to the study of the motion of the Earth's poles and the variation of the latitude on the surface of the Earth. These and others are mentioned in E. Leimanis's book (1965).

Just like the problem of rotational motion of a heavy rigid body about a fixed point and of the rigid body with a fixed point in a central Newtonian force field, of potential $V^{(2)}$ (this is achieved by taking only up to the main term of the non Keplerian potential, that is to say up to second order harmonics), the problem is reduced to the obtention of a fourth integral of the motion, besides the three classical integrals. In the case of a heavy rigid body there is a fourth integral of the motion which, moreover, is algebric in the classical cases: of Euler and Poinsot (where the external torque vanishes), Lagrange and Poisson ($I_1 = I_2$, $x_1^0 = x_2^0 = 0$) and Kovalevskaya ($I_1 = I_2 = 2I_3$; $x_3^0 = 0$). In the problem of the rigid body with a fixed point in a central Newtonian force field with a potential $V^{(2)}$, Arkhangelskii (1962, 1963) has proven that, in general, there exists a fourth algebraic first integral of the equations of the motion only in cases of sphericity, analogous to that of Lagrange–Poisson and that of Euler–Poinsot.

The problem of the heavy gyrostat with a fixed point has been considered by Keis (1964), who confirms the validity of the Poincaré theorem for this case; and by Vigueras (1983) who analytically integrates a similar case to that of Lagrange–Poisson, for gyrostats. Where the problem of the motion of a gyrostat with a fixed point in a central Newtonian force field with a potential $V^{(2)}$ in Andoyer variables, is concerned, it has been considered in Tsopa (1979, 1981) and integrated into quasi regular cases, by perturbations methods. More general problems, such as that of the roto-translatory motion of n gyrostats, or that of the roto-translatory motion of a gyrostat in a central Newtonian force field, the Earth's rotation using as model of the Earth a symmetric gyrostat, have been considered by R. Cid and A. Vigueras (1985, 1990) and M. E. Sansaturio and A. Vigueras (1988), and they have given qualitative results and approximate integrations.

Returning to the problem of a gyrostat with a fixed point in a central Newtonian force field with a potential $V^{(2)}$, Vigueras (1987) puts to the test the Liouville integrability in three cases, which are reduced to those given by Arkhangelskii if the gyrostatic moment cancels itself.

In this study, a problem that generalizes that of Lagrange–Poisson and deals with gyrostats under the potential $V^{(2)}$, is integrated analytically; concretely, let us suppose that the center of mass and the gyrostatic momentum are on the axis of symmetry and that the initial conditions, in the terms of Euler angles and angular velocities, are the following:

$$\psi(t_0) = \psi_0 \qquad \theta(t_0) = 0 \qquad \phi(t_0) = \phi_0$$

$$\omega_1(t_0) = 0 \qquad \omega_2(t_0) = 0 \quad \omega_3(t_0) = \omega_3^0 \,.$$

In the particular case of the previous one, in which the third component of the total angular momentum cancels itself (this could be achieved by judiciously choosing the third component of the gyrostatic momentum, artificial satellites equipped with symmetrical rotors, come to mind as an exampl); has been dealt with by Cavas and Vigueras (1992), taking into consideration all the possible subcases, that are characterised in function of the distance r from the centre of attraction P to the fixed point O, as well as the geometry of the body and the initial conditions, stating the explicit analytical expression of the solution in each one of them. Independently from the equilibrium solutions, more possibilities of motion become apparent than in the similar case of a heavy gyrostat, which is due to the effects derived from the corresponding gravitational moment of the new term included in the $V^{(2)}$ potential.

Presently, the general case will be studied, that in which the third component of the total angular momentum does not cancel itself ($B_1 \neq 0$), all subcases which are characterised in function of the discriminant Δ of the equation $S'(u) = 0$, will be considered, and the corresponding solution in each subcase will be indicated. There now appear equilibrium solutions that did not exist in the case $B_1 = 0$.

Obviously the obtained results are valid for rigid bodies without having to do more than $l_3^r = 0$ in the corresponding expressions. Thus, the rigid body, similar to that of Lagrange and Poisson, in a central Newtonian force field of potential $V^{(2)}$, has been integrated, as a particular case of the problem dealt with.

However, for a rigid body, whether Δ is positive, negative or zero, depends on the geometry of the body and the initial conditions; on the other hand, for a gyrostat for which l_3^r can be chosen, the sign of the factor Δ can be determined in function of the value of l_3^r.

2. Presentation of the Problem

The problem faced consists in obtaining the equations of the motion of a gyrostat S fixed in one of its points O belonging to its rigid part, submitted to the Newtonian attraction of another point P (or to a rigid body with a spherical distribution of masses), when the relative motion of its mobile part with respect to its rigid part is supposedly known and the mutual potential V is approximated by $V^{(2)}$. With the origin in the fixed point O, two systems of reference are considered: a fixed one $OX_1X_2X_3$, in such a way that the point P lies in the negative part of the axis OX_3, at a constant distance $r = |\mathbf{OP}|$ and another mobile one $Ox_1x_2x_3$, fixed in the body, and whose axes are directed along the principal axes of inertia of the gyrostat at O. Thus, supposing that the gyrostat is symmetrical and that not only the gyrostatic moment but also the position vector of its center of mass, expressed in the mobile system, $\mathbf{l}_r = (l_1^r, l_2^r, l_3^r)$ and $\mathbf{r}_0 = (x_1^0, x_2^0, x_3^0)$ are constant vectors lying on the axis of symmetry (Ox_3), that is to say:

$$l_1^r = l_2^r = 0 \qquad x_1^0 = x_2^0 = 0 \qquad I_1 = I_2$$

according to the angular momentum theorem and the kinematic equations of Poisson, the equations of the motion in the mobile system are:

$$I_1\dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3 + \omega_2 l_3^r = m_0 x_3^0 k_2 + m_1(I_3 - I_1)k_2k_3$$

$$I_1\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 - \omega_1 l_3^r = -m_0 x_3^0 k_1 + m_1(I_1 - I_3)k_1k_3 \qquad (2.1)$$

$$I_3\dot{\omega}_3 = 0$$

$$\dot{\mathrm{k}}_1 + \omega_2 k_3 - \omega_3 k_2 = 0$$

$$\dot{\mathrm{k}}_2 + \omega_3 k_3 - \omega_1 k_3 = 0$$

$$\dot{\mathrm{k}}_3 + \omega_1 k_2 - \omega_2 k_1 = 0$$

where $m_0 = mg$, $m_1 = 3g/r$, $m =$ mass of the gyrostat, $g = GM/r^2$, $G =$ gravitational constant, $M =$ mass of the fixed point P, $I_1 = I_2$ and I_3 are the principal moments of inertia of S at O, $\mathbf{w} = (w_1,\ w_2,\ w_3)$ the instantaneous angular velocity of rotation of the mobile system with respect to the fixed one and $\mathbf{k} = (k_1,\ k_2,\ k_3)$, the unitary vector of the OX_3 axis expressed in the mobile system. The afore mentioned equations admit the following first integrals:

$$k_1^2 + k_2^2 + k_3^2 = 1$$

$$\mathbf{k}(\mathbf{l} + \mathbf{l}_r) = c \text{ (cte)} \tag{2.2}$$

$$\tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + mg(x_1^0 k_1 + x_2^0 k_2 + x_3^0 k_3) +$$

$$+\tfrac{3}{2}\,\frac{g}{r}(I_1 k_1^2 + I_2 k_2^2 + I_3 k_3^2) = h \text{ (cte)}$$

$$\omega_3 = \omega_3^0 \text{ (cte)} .$$

Particularly the third equation (Jacobi's integral) and the second (an integral that expresses the constant character of the projection of the total angular momentum on the third fixed axis) can be expressed as follows:

$$\omega_1^2 + \omega_2^2 = A_0 + A_1 k_3 + A_2 k_3^2$$

$$\omega_1 k_1 + \omega_2 k_2 = B_0 - B_1 k_3 \tag{2.3}$$

A_0, A_1, A_2, B_0 and B_1 being constant, given by the following formulae:

$$A_0 = \left(2h_0 - I_3(\omega_3^0)^2\right)/I_1\ ,\quad A_1 = (-2m_0 x_3^0)/I_1\ ,\quad A_2 = 3g(I_1 - I_3)/rI_1$$

$$B_0 = c/I_1\ ,\quad B_1 = \left(I_3\omega_3^0 + l_3^r\right)/I_1\ . \tag{2.4}$$

Using the expressions of the components of $\boldsymbol{\omega}$ and $\mathbf{k}$ in Euler variables, the integrals (2.2) can be reduced to the equations:

$$\dot{\psi}^2 \operatorname{sen}^2\theta + \dot{\theta}^2 = A_0 + A_1\ \cos\theta + A_2\ \cos^2\theta$$

$$\dot{\psi} \operatorname{sen}^2\theta = B_0 - B_1\ \cos\theta\ . \tag{2.5}$$

The change of the variable $u = \cos\theta$, allows the separation of variables, thus giving:

$$\dot{u}^2 = (A_0 + A_1 u + A_2 u^2)(1 - u^2) - (B_0 - B_1 u)^2 = g(u)$$

$$\dot{\psi} = (B_0 - B_1 u)/(1 - u^2) \qquad (2.6)$$

$$\dot{\phi} = \omega_3^0 - (B_0 - B_1 u)\, u/(1 - u^2)\,.$$

2.1. ADDITIONAL HYPOTHESES

Before continuing, let us suppose that the initial conditions are as follows:

$$\psi(t_0) = \psi_0 \qquad \theta(t_0) = \theta_0 \qquad \phi(t_0) = \phi_0$$

$$\omega_1(t_0) = 0 \qquad \omega_2(t_0) = 0 \qquad \omega_3(t_0) = \omega_3 \qquad (2.7)$$

(being $u_0 = \cos\theta_0$ different from 1 and -1, because these values correspond to two equilibrium configurations of the problem).

Thus, we are going to procede to the analytical integration of the problem considering when the gyrostat (its rigid part) turns, in the initial instant, around the axis of symmetry at a constant angular velocity ω_3^0; the orientation of the gyrostat being arbitrary.

However, for a rigid body, whether Δ is positive, negative or zero, depends on the geometry of the body and the initial conditions; on the other hand, for a gyrostat for which l_3^r can be chosen, the sign of the factor Δ can be determined in function of the value of l_3^r. Therefore from (2.3) and (2.7), it can be deduced:

$$A_0 = -(A_1 u_0 + A_2 u_0^2)\,, \quad B_0 = B_1 u_0\,. \qquad (2.8)$$

Furthermore, if we suppose that the center of mass is situated on top of the fixed point or coincides with it $x_3^0 \geq 0$ (in the case $B_1 = 0$, the hypothesis was $x_3^0 > 0$) and that $I_1 > I_3$; therefore the equations of the motion would be as follows:

$$\dot{u}^2 = (u - u_0)\left[\left(A_1 + A_2(u + u_0)\right)(1 - u^2) - B_1^2(u - u_0)\right] = g(u)$$

$$\dot{\psi} = B_1(u_0 - u)/(1 - u^2)$$

$$\dot{\phi} = \omega_3^0 - B_1(u_0 - u)u/(1 - u^2)\,. \qquad (2.9)$$

3. Analytical Resolution of the General Case $B_1 \neq 0$

So as to obtain the Euler variables in function of time, by integration of the differential Equations (2.9), we can procede to the study of the roots of $g(u) = 0$; for this purpose, let use define

$$s(u) = (1 - u^2)\,(A_1 + A_2u_0 + A_2u) - B_1^2(u - u_0) \tag{3.1}$$

with which we can write

$$g(u) = (u - u_0)\; s(u) \tag{3.2}$$

when studying the behaviour of the function $s(u)$, it is obvious that

$$s(-1) = B_1^2(1 + u_0) > 0 \qquad s(1) = -B_1^2(1 - u_0) < 0$$

$$s'(u) = -3A_2u^2 - 2(A_1 + A_2u_0)u + A_2 - B_1^2 \tag{3.3}$$

$$s''(u) = -6A_2u - 2(A_1 + A_2u_0)\,.$$

By virtue of (3.3) we know that $s(u)$ admits at least one root in $(-1, 1)$, let us call a_1 the smallest of the roots of $s(u)$ in the mentioned interval.

When studying the roots of the equation of second grade, $s'(u) = 0$, we can find the possible maxima and minima of $s(u)$, which will depend on the value of their discriminant

$$\Delta = (A_1 + A_2u_0)^2 + 3A_2(A_2 - B_1^2)\,. \tag{3.4}$$

Likewise, from $s''(u)$ we can see that $s(u)$ presents an inflexion in $u = -(u_0 + A_1/A_2)/3$.

Next, we will study the behaviour of $s(u)$ according to the distinct values of the discriminant Δ, and three cases appear:

3.1. Case $\Delta < 0$

The $s(u)$ function is strictly decreasing and presents a unique simple real root a_1, in the interval $(-1,\ 1)$, and a pair of complex conjugate roots. Now, depending on the relative position of a_1 and u_0, the following possibilities become apparent:

$u_0 < a_1$

$u_0 > a_1$

$u_0 = a_1$

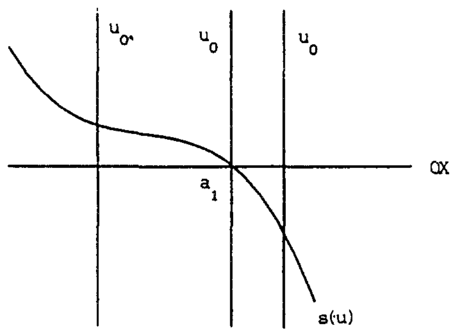

Fig. 1.

3.2. Case $\Delta > 0$

The equation $s'(u) = 0$ presents two real solutions, and we can distinguish the following subcases:

3.2–1. The $s(u)$ admits a unique simple real root a_1, in the interval $(-1,\ 1)$, and a pair of complex conjugate roots. We can consider the same possible relative positions of u_0 and a_1:

$u_0 < a_1$

$u_0 > a_1$

$u_0 = a_1$.

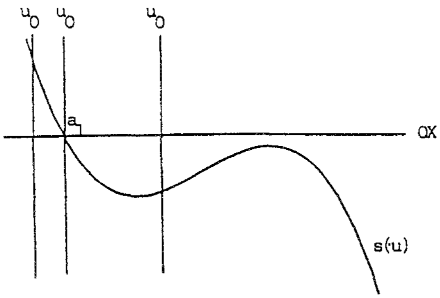

Fig. 2.

3.2–2. That $s(u)$ admits one simple real root a_1 and one double root b_2. Then we can distinguish the following unique possible relative positions of u_0, a_1 and b_2:

$u_0 < a_1 < b_2$

$a_1 < u_0 < b_2$

$a_1 < b_2 < u_0$.

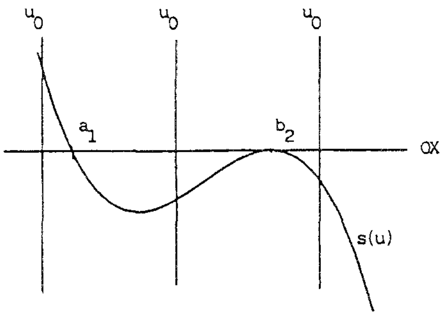

Fig. 3.

3.2–3. That $s(u)$ admits three different real roots $a_1 < a_2 < a_3$. In this subcase we can consider the following possibilities:

$u_0 < a_1 < a_2 < a_3$
$a_1 < u_0 < a_2 < a_3$
$a_1 < a_2 < u_0 < a_3$
$a_1 < a_2 < a_3 < u_0$
$u_0 = a_1 < a_2 < a_3$
$a_1 < u_0 = a_2 < a_3$
$a_1 < a_2 < u_0 = a_3$.

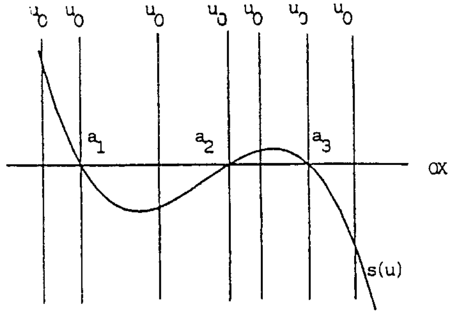

Fig. 4.

3.3. Case $\Delta = 0$

Therefore $s'(u)$ would become factorised by $s'(u) = -3A_2(u - d_1)^2$ let us call $d_1 = -(A_1 + A_2u_0)/3A_2$. The following subcases could present themselves:
3.3–1. That $d_1 = a_1$ and we have the following possible relative positions (the possibility $u_0 = a_1$ must be rejected):

$u_0 < a_1$

$u_0 > a_1$

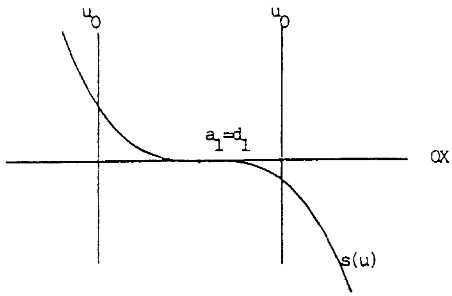

Fig. 5.

3.3–2. That $d_1 \neq a_1$, supposing $d_1 < a_1$ (similar for $d_1 > a_1$) we would have the following subheadings:

$u_0 < a_1$

$u_0 > a_1$

$u_0 = a_1$

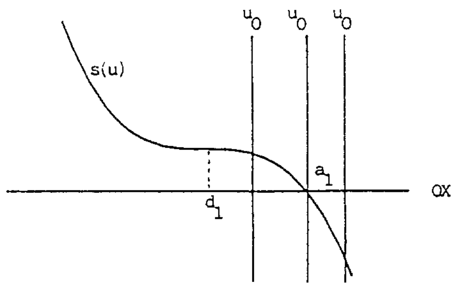

Fig. 6.

The analysis of the previous cases and subcases gives us four generic cases for the function $g(u)$:

(A) That $g(u)$ admits two different real roots (u_0, a_1) and a pair of complex conjugate roots.

By assuming that $a_1 < u_0$ (the case when $a_1 > u_0$ is similar). The motion is possible for values of u situated between a_1 and u_0, $(a_1 \leq u \leq u_0)$, that is to say for θ varying between its initial value θ_0 and a limit value $\theta_1 < \pi$ (such that $a_1 = \cos\,\theta_1$).

From the first of the equations of the motion (2.9) it can be deduced that: (Byrd and Friedman)

$$\sqrt{A_2}\,(t - t_0) = \int_u^u \frac{\mathrm{d}x}{\sqrt{(u_0 - x)\,(x - a_1)\,[(x - a_2)^2 + a_3^2]}} =$$

$$= \alpha\Big[F(\xi, k) - 2\mathcal{K}(k)\Big] \tag{3.5}$$

where

$$\alpha = 1/\sqrt{AB} \qquad \begin{aligned} A^2 &= (u_0 - a_2)^2 + a_3^2 \\ B^2 &= (a_1 - a_2)^2 + a_3^2 \end{aligned}$$

$$\cos\,\xi = \frac{(u_0 - u)B - (u - a_1)A}{(u_0 - u)B + (u - a_1)A} \tag{3.6}$$

$$k^2 = \frac{(u_0 - a_1)^2 - (A - B)^2}{4AB}$$

$\mathcal{K} = \mathcal{K}(k) = F(\pi/2,\ k)$ is the complete elliptic integral of the first kind with modulus k.

Inverting this integral, we obtain:

$$\cos\,\xi = \mathrm{cn}\,(\omega_0(t - t_0) + 2\mathcal{K}) = -\mathrm{cn}\,\omega_0(t - t_0)$$

being $\omega_0 = \sqrt{A_2}$ and cn(–) the cosine amplitude function with the same modulus k.

So as to resolve u, the following expression is reached

$$u = \cos\,\theta = \frac{a_1A + u_0B - (a_1A - u_0B)\,\mathrm{cn}\,\omega(t - t_0)}{A + B - (A - B)\,\mathrm{cn}\,\omega_0(t - t_0)} \tag{3.7}$$

which gives us the nutation θ as a periodic function of the time (with the period $4\mathcal{K}/\omega_0$).

For the seconde equation of the motion (2.9), defining

$R(u) = (u_0 - u)/(1 - u^2)$, results in

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = \frac{B_1 R(u)}{\sqrt{g(u)}}$$

and integrating, the following is obtained

$$\psi - \psi_0 = B_1 \int_u^u \frac{R(x)\,\mathrm{d}x}{\sqrt{g(x)}} =$$

$$= \frac{B_1}{\sqrt{A_2}}\,\alpha \left[\int_0^{u_1} R \left[\frac{u_0 B + a_1 A + (a_1 A - u_0 B)\,\mathrm{cn}\,u}{A + B + (A - B)\,\mathrm{cn}\,u} \right] \mathrm{d}u - C_0 \right] \tag{3.8}$$

where A, B, α, k^2, are those given in (3.6), C_0 is constant and

$$\mathrm{cn}\,u_1 = \cos\xi = \frac{(u_0 - u)B - (u - a_1)A}{(u_0 - u)B + (u - a_1)A}\,. \tag{3.9}$$

In general, the integral in (3.8) can be written in the form

$$\alpha_1 u_1 + \sum_{i=1}^{2} \delta_i \int_0^{u_1} \frac{\mathrm{d}u}{1 + \gamma_i\,\mathrm{cn}\,u} \tag{3.10}$$

being α_1, δ_i, γ_i $(i = 1, 2)$ constants.

Then (Byrd–Friedman)

$$\int_0^{u_1} \frac{\mathrm{d}u}{1 + \gamma_i\,\mathrm{cn}\,u} = \frac{1}{1 - \gamma_i^2} \left[\Pi\left(u_1, \frac{\gamma_i^2}{1 - \gamma_i^2} \right) - \gamma_i f_{1i} \right]. \tag{3.11}$$

Where

$$\Pi\left(u_1, \frac{\gamma_i^2}{1 - \gamma_i^2} \right) \quad (\gamma_i^2 \neq 1)$$

is the incomplete elliptic integral of the third kind with parameter

$$\tau_i^2 = \frac{\gamma_i^2}{\gamma_i^2 - 1}$$

and the same modulus k, and

$$f_{1i} = \left(\frac{\gamma_i^2 - 1}{k^2 + k'^2 \gamma_i^2} \right)^{1/2} \tan^{-1} \left[\left(\frac{k^2 + k'^2 \gamma_i^2}{1 - \gamma_i^2} \right) \mathrm{sd}\,u_1 \right] \quad (\tau_i^2 < k^2)$$

$$= \mathrm{sd}\,u_1 \quad (\tau_i^2 = k^2) \tag{3.12}$$

$$= \frac{1}{2}\left(\frac{\gamma_i^2 - 1}{k^2 + k'^2\gamma_i^2}\right)^{1/2} \ln\left[\frac{\sqrt{k^2 + k'^2\gamma_i^2}\,\mathrm{dn}\,u_1 + \sqrt{\gamma_i^2 - 1}\,\mathrm{sn}\,u_1}{\sqrt{k^2 + k'^2\gamma_i^2}\,\mathrm{dn}\,u_1 - \sqrt{\gamma_i^2 - 1}\,\mathrm{sn}\,u_1}\right] \quad (\tau_i^2 > k^2)$$

being $k' = \sqrt{1 - k^2}$ the complementary modulus and $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, $\mathrm{sd}\,u$ are Jacobian elliptic functions.

If we introduce the expressions (3.10), (3.11), and (3.12) in (3.8) gives us $\psi = \psi(u_1)$ and substituting $u_1 = u_1(t)$ by means of the Equation (3.9), we would have the explicit form of the precession in function of the time t.

Similarly, we would obtain ϕ in function of t, in the following terms:

$$\phi - \phi_0 = \omega_3^0(t - t_0) - \frac{B_1}{\sqrt{A_2}} \int_{u_0}^{u} \frac{x\,R(x)\,\mathrm{d}x}{\sqrt{(u_0 - x)\,(x - a_1)\,[(x - a_2)^2 + a_3^2]}} \tag{3.13}$$

this integral can be expressed in the form

$$\alpha\left\{\int_0^{u_1} \frac{(u_0 B + a_1 A) + (a_1 A - u_0 B)\,\mathrm{cn}\,u}{A + B + (A - B)\,\mathrm{cn}\,u}\right.$$

$$\left. R\left[\frac{(u_0 B + a_1 A) + (a_1 A - u_0 B)\,\mathrm{cn}\,u}{A + B + (A - B)\,\mathrm{cn}\,u}\right] \mathrm{d}u - C_1\right\}$$

where C_1 is constant. The last integral can be put in the form

$$\alpha_{11} u_1 + \sum_{i=0}^{2} \delta_{1i} \int_0^{u_1} \frac{\mathrm{d}u}{1 + \gamma_i\,\mathrm{cn}\,u} =$$

$$= \alpha_{11} u_1 + \sum_{i=0}^{2} \frac{\delta_{1i}}{1 - \gamma_i^2} \left\{\Pi\left(u_1, \frac{\gamma_i^2}{\gamma_i^2 - 1}\right) - \gamma_i f_{1i}\right\} \tag{3.14}$$

being α_{11}, δ_{1i}, γ_i $(i = 0, 1, 2)$ constants and

$$\Pi\left(u_1, \frac{\gamma_i^2}{\gamma_i^2 - 1}\right) \quad (\gamma_i^2 \neq 1)$$

the incomplete elliptic integral of the third kind with parameter $\tau_i^2 = \gamma_i^2/(\gamma_i^2 - 1)$ and modulus k and f_{1i} $(i = 0, 1, 2)$ are analogous functions to those of (3.12).

(B) That $g(u)$ admits either one double real root (which is different from u_0) and two different real roots (u_0, u_1) or one triple real root (which is different from u_0) and the simple real root u_0.

If b_2 is the multiple real root then can be factorised as such $g(u) = A_2(x - b_2)^2\,R(x)$ being $R(x) = (u - u_0)\,(u - u_1)$ (with $u_1 = b_2$ or $u_1 \neq b_2$). Then the equations of motion can be integrated by means of elemental integrals.

(C) That $g(u)$ admits four different real roots u_0, a_1, a_2 y a_3.

Let us suppose that $u_0 < a_1 < a_2 < a_3$ (the remaining cases are similar). Then $g(u) = -A_2(u-u_0)(u-a_1)(u-a_2)(u-a_3)$, where $g(u) \geq 0$ for values of u such that $u_0 \leq u \leq a_1 < a_2 < a_3$; from the first equation of the motion can be deduced that: (Byrd and Friedman)

$$\sqrt{A_2}\,(t-t_0) = \int_{u_0}^{u} \frac{\mathrm{d}x}{\sqrt{(a_3-x)(a_2-x)(a_1-x)(x-u_0)}} =$$

$$= \alpha[\mathcal{K}(k) - F(\xi, k)] \tag{3.15}$$

where

$$\alpha = \frac{2}{\sqrt{(a_3-a_1)(a_2-u_0)}}\,, \quad k^2 = \frac{(a_3-a_2)(a_1-u_0)}{(a_3-a_1)(a_2-u_0)}$$

$$\operatorname{sen}\xi = \frac{\sqrt{(a_2-u_0)(a_1-u)}}{\sqrt{(a_1-u_0)(a_2-u)}}\,, \quad \mathcal{K} = \mathcal{K}(k) = F(\pi/2, k)\,. \tag{3.16}$$

Inverting this integral we obtain

$$u = \frac{a_2(a_1-u_0)\operatorname{sen}^2\xi - a_1(a_2-u_0)}{(a_1-u_0)\operatorname{sen}^2\xi - (a_2-u_0)}\,. \tag{3.17}$$

where

$$\operatorname{sen}\xi = \operatorname{sn}(\omega_0(t-t_0) + \mathcal{K})\,, \quad \omega_0 = \sqrt{A_2}/\alpha\,. \tag{3.18}$$

For the second equation of the motion we have

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = \frac{B_1 R(u)}{\sqrt{g(u)}}\,, \quad \text{where } R(u) = \frac{(u_0-u)}{(1-u^2)}\,. \tag{3.19}$$

Integrating we deduce (Byrd–Friedman)

$$\psi - \psi_0 = B_1 \int_{u_0}^{u} \frac{R(x)\,\mathrm{d}x}{\sqrt{g(x)}} = \frac{B_1}{\sqrt{A_2}} \int_{u_0}^{u} \frac{R(x)\,\mathrm{d}x}{\sqrt{(x-u_0)(a_1-x)(x-a_2)(x-a_3)}} =$$

$$= \frac{B_1}{\sqrt{A_2}}\, g \left[-\int_0^{u_1} R\left[\frac{a_1 - a_2\alpha^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\right] \mathrm{d}u + C_1 \right] \tag{3.20}$$

where

$$g = \frac{2}{\sqrt{(a_3 - a_1)(a_2 - u_0)}} \qquad \alpha^2 = \frac{(a_1 - u_0)}{(a_2 - u_0)} \tag{3.21}$$

$$\mathrm{sn}^2 u_1 = \mathrm{sen}^2 \phi = \frac{(a_2 - u_0)(a_1 - u)}{(a_1 - u_0)(a_2 - u)} \tag{3.22}$$

$$C_1 = \int_0^{\mathcal{K}} R\left[\frac{a_1 - a_2\alpha^2 \,\mathrm{sn}^2 u}{1 - \alpha^2 \,\mathrm{sn}^2 u}\right] \mathrm{d}u \,.$$

The integral in (3.20) can be written in the form

$$\int_0^{u_1} R\left[\frac{a_1 - a_2\alpha^2 \,\mathrm{sn}^2 u}{1 - \alpha^2 \,\mathrm{sn}^2 u}\right] \mathrm{d}u = \eta_0 u_1 + \sum_{i=1}^{2} \eta_i \int_0^{u_1} \frac{\mathrm{d}u}{1 - \gamma_i^2 \,\mathrm{sn}^2 u} \tag{3.23}$$

where η_i are constants, and

$$\int_0^{u_1} \frac{\mathrm{d}u}{(1 - \tau^2 \,\mathrm{sn}^2 u)} = \Pi(u_1, \tau^2) \,.$$

Similarly we would obtain ϕ in function of t in the following terms:

$$\phi - \phi_0 = \omega_3^0 (t - t_0) - \frac{B_1}{\sqrt{A_2}} \int_{u_0}^{u} \frac{R_1(x)\,\mathrm{d}x}{\sqrt{(x - u_0)(a_1 - x)(x - a_2)(x - a_3)}} \tag{3.24}$$

where $R_1(x) = xR(x)$, and we would have to substitute $u = u(t)$ given by the expression (3.17), so as to obtain the proper rotation ϕ in function of the time t. In a similar way the integral in (3.24) can be put in the form

$$\int_0^{u_1} \frac{a_1 - a_2\alpha^2 \,\mathrm{sn}^2 u}{1 - \alpha^2 \,\mathrm{sn}^2 u} R\left[\frac{a_1 - a_2\alpha^2 \,\mathrm{sn}^2 u}{1 - \alpha^2 \,\mathrm{sn}^2 u}\right] \mathrm{d}u =$$

$$= \bar{\eta}_0 u_1 + \sum_{i=1}^{2} \bar{\eta}_i \int_0^{u_1} \frac{\mathrm{d}u}{1 - \gamma_i^2 \,\mathrm{sn}^2 u} + \bar{\alpha} \int_0^{u_1} \frac{\mathrm{d}u}{1 - \alpha^2 \,\mathrm{sn}^2 u} \tag{3.25}$$

being $\bar{\eta}_0$, $\bar{\eta}_i$, $\bar{\alpha}$ constants. The explicit expression of ϕ in function of the time t can be obtained substituting $u_1 = u_1(t)$ by means of Equation (3.22).

(D) That $g(u)$ admits u_0 as a double real root, being u_0 either the smallest or the biggest of theirs real roots.

In this case $g(u) \leq 0$ and the problem presents a solution of equilibrium:

$$\theta(t) = \theta_0 \quad \psi(t) = \psi_0 \quad \phi(t) = \omega_3^0 t + \phi_0 .$$

For the interest that this solution can have, it must be pointed out that there exists this solution if and only if

$$u = u_0 = -A_1/2A_2 = mx_3^0 r/3(I_1 - I_3) .$$

4. Conclusions

We analytically integrate a similar case to that of Lagrange–Poisson, for a gyrostat with a fixed point in a central Newtonian force field of potential $V^{(2)}$, when the third component of the angular moment is different from 0 $(B_1 \neq 0)$ and we give the different types of solutions depending on the roots of the function $g(u)$. This study has been carried out according to the distinct values of the discriminant Δ, given by the expression (3.4).

The solutions arrived at are valid for the case of a rigid body, only having to cancel the third component of the gyrostatic momentum $(l_3^r = 0)$. However, for a rigid body the fact Δ is positive, negative or zero, depends on the geometry of the body and the initial conditions; on the other hand, for a gyrostat with a gyrostatic momentum $\mathbf{l}_r = (0, 0, l_3^r)$, in which it is possible to choose l_3^r, we can determine the sign of the discriminant Δ in function of the values of l_3^r so as to arrive at the desired solution.

Furthermore, the solutions reached at will be of use in the posterior elaboration of analytical theories of the rotational motion of gyrostatic satellites, that can be activated by remote control.

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