Sustainable Development, the Hartwick Rule and Optimal Growth

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Abstract. Defining sustainable development as non-declining utility, the consistency of this concept with the Hartwick rule and optimal growth is explored when resources are exhaustible. A simple proof that a generalized Hartwick rule is necessary and sufficient for constant consumption is derived. The existence of a maximal constant consumption path is shown to depend critically on the elasticity of substitution; if this is less than 1, consumption declines; if it is greater than 1 then consumption is not maximal; if it is equal to 1 (the Cobb-Douglas case) then existence is proved. Consumption can increase along an optimal path if the pure rate of time preference is 0; if it is non-zero then consumption declines.

Key words. Hartwick rule, optimal growth, exhaustible resources.

1. Introduction

Concern about damage to the environment and depletion of resources has made sustainable economic development a concept with both wide currency and wide interpretation, as Pezzey's (1989) exposition demonstrates. Although various criticisms have been levelled at the notion of sustainable development (see, for instance, Nordhaus, 1992a), it is the goal of this paper to explore a particularly simple definition, that per capita utility be non-declining, owing to Pezzey. Given that the sustainability criterion is, in effect, an ethical constraint on the classic economic problem of intertemporal optimization, the key question to be answered is whether, or under what conditions, sustainable development so defined is consistent with optimal growth and finite resources.

If sustainable development means non-declining utility, this leads to two distinct cases to be examined. *Minimal sustainability* is defined to be constant utility over time. If the utility function is a continuous and non-decreasing function of consumption only, then minimal sustainability is equivalent to constant consumption. *Strict sustainability* is defined as increasing utility over time. For the same characteristics of the utility function, this is equivalent to increasing consumption. The analysis that follows therefore breaks down into constant consumption and increasing consumption components.

The initial problem to be examined in this paper is that of finding a development path with maximal consumption that is minimally sustainable in the face of finite resources. Stated this way, it is clear that this is equivalent to a *maximin* programme, which has been widely studied in the literature. The starting ethical position in the preceding work was different, essentially

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a Rawlsian framework in which welfare across time is equal to that of the least well-off generation, but the end goal was the same: maximal constant consumption.

Solow (1974) proved the existence of a path with maximal constant consumption and finite resources, and this was elaborated in Dasgupta and Heal (1979, ch. 7). Both of these results required a Cobb–Douglas production function with the elasticity of output with respect to produced capital being greater than that of natural resources. The famous result in this literature is Hartwick (1977), who showed that the 'Hartwick rule', to invest resource rents, is a sufficient condition for a maximin programme for general production functions. Hartwick (1978) explored the Hartwick rule for several resources and raised unanswered questions about the existence of a maximin path (in particular the path for output) for different values of the elasticity of substitution between capital and resources in a constant elasticity of substitution (CES) production function. Finally Dixit, Hammond and Hoel (hereafter DHH) (1980) showed in a very general framework that an extended Hartwick rule, in which capital accumulation equals unit resource rent times the quantity of resources used plus an arbitrary constant, was necessary and sufficient for the existence of a maximin path.[1]

The substitution possibilities between capital and resources are clearly important in determining whether a maximin path exists. A substantial portion of this paper therefore deals with characterizing the behaviour of both the standard and generalized Hartwick rules under varying values of the elasticity of substitution in a CES production function; this has not been resolved in the literature to date. The first section begins with a simple proof, for general production functions, that the generalized Hartwick rule combined with the Hotelling rule is necessary and sufficient for maximal constant consumption. It ends by tying together the various approaches to the maximin problem in the literature, including the derivation of a weaker condition for maximal consumption under the generalized Hartwick rule than that in DHH (1980) for CES production functions.

The behaviour of a Hartwick-Hotelling programme under varying values of the elasticity of substitution, σ, is related to the debate concerning 'strong' versus 'weak' sustainability (Pearce *et al.*, 1989). The proponents of weak sustainability argue that capital and resources are substitutable, and so identify sustainable development with maintaining total assets (produced capital and natural resources) constant or increasing. The strong sustainability position is that there is a critical quantity of at least some natural resources that must be maintained intact if utility is not to decline in the future – in the limit this would imply a zero elasticity of substitution for these resources. Without doing too much violence to the basic elements of this debate, it is possible to equate 'weaker' sustainability with substitution possibilities that are elastic ($\sigma > 1$), and 'stronger' sustainability with inelastic substitution possibilities ($\sigma < 1$). The result derived below, that constant consumption is not attainable for $\sigma < 1$, therefore implies that the Hartwick rule will not yield constant consumption if you subscribe to the 'stronger' sustainability position.

The second part of the paper examines the conditions under which utility increases in a programme of optimal growth with finite resources, as addressed by Dasgupta and Heal (1979, ch. 10). This paper derives their result in the somewhat more modern garb of optimal control and highlights the critical link between the pure rate of time preference and the possibility of strict sustainability. Optimal control has recently been applied by Hartwick (1990) and Mäler (1991) to elucidate the treatment of natural resources and the environment in national accounting. However, neither of these papers goes on to examine the dynamic behaviour of the system, which is the essential element in this paper.

2. Maximal Constant Consumption Paths

While this section is primarily concerned with the behaviour of the Hartwick rule under different assumptions about the elasticity of substitution of capital and resources, the starting point is a simple proof that a generalized Hartwick rule, to use the terminology of DHH (1980), combined with the Hotelling rule, is necessary and sufficient for constant consumption.

We assume that there is constant population (so that labour can be treated implicitly in the production function), and no disembodied technological growth. The initial endowment is a stock S_0 of resources and K_0 of capital. Output F is produced from capital K and resources R according to the production function,

$$F = F(K, R)$$
$$\text{such that } F_K, F_R > 0, F_{KK}, F_{RR} < 0.$$

Consumption C is defined by the following set of differential equations:

$$\begin{aligned} C &= F - \dot{K} \\ \dot{S} &= -R. \end{aligned} \tag{1}$$

Any efficient programme of production, investment and consumption must satisfy two criteria:

$$\frac{\dot{F}_R}{F_R} = F_K, \tag{2}$$

and

$$\lim_{t \to \infty} S_t = 0. \tag{3}$$

The first of these is the familiar Hotelling rule; in the economy postulated, holders of natural resource stocks must be indifferent between holding resources or the alternative asset, capital, which yields F_K. Any programme that left unexploited natural resources would clearly be inefficient, hence expression (3) which says that the programme must exhaust the initial resource stock.

The generalized Hartwick rule is given by:

$$\dot{K} = F_R(R + v),\ v \text{ constant,} \tag{4}$$

where the return to resources, F_R, is the resource rental rate.

Sufficiency of the generalized Hartwick–Hotelling programme for constant consumption is proved as follows. Applying expressions (2) and (4) we have:

$$\begin{aligned}
\dot{C} &= \frac{\mathrm{d}}{\mathrm{d}t}(F - \dot{K}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(F - F_R(R + v)) \\
&= \dot{F} - \dot{F}_R(R + v) - F_R\dot{R} \\
&= \dot{F} - F_K F_R(R + v) - F_R\dot{R} \\
&= \dot{F} - F_K\dot{K} - F_R\dot{R} \\
&= 0.
\end{aligned}$$

Necessity of the programme for constant consumption is shown by assuming $\dot{C} = 0$. We have,

$$\begin{aligned}
\ddot{K} &= \dot{F} - \dot{C} \\
&= F_R\dot{R} + F_K\dot{K} \\
&= F_R\dot{R} + \frac{\dot{F}_R}{F_R}\dot{K} \\
&= F_R\dot{R} + \dot{F}_R R + \frac{\dot{F}_R}{F_R}(\dot{K} - F_R R).
\end{aligned}$$

Now define $Z = \dot{K} - F_R R$, so that $\dot{Z} = \ddot{K} - F_R\dot{R} - \dot{F}_R R$, and therefore the preceding expression for $\ddot{K}$ can be written as $\dot{Z} = (\dot{F}_R/F_R)Z$. This equation has solution $Z = vF_R$ for constant v, and therefore $\dot{K} = F_R(R + v)$.

Having established that the Hotelling rule and the generalized Hartwick rule are together necessary and sufficient for constant consumption, the next question to be examined is the behaviour of the system under different assumptions about v. For $v \neq 0$ we will explore the generalized Hartwick rule, while the case $v = 0$ will be explored in a sub-section on the standard Hartwick rule.

2.1. THE GENERALIZED HARTWICK RULE

We wish to derive the path for output and consumption under the generalized Hartwick rule. It is clear from the foregoing derivation that the parameter

v is simply a constant of integration – it has no obvious economic interpretation, and it would be disturbing if constant consumption were feasible for any programme that added an arbitrary amount to the quantity of resource extracted.

The first case to be considered is $v < 0$. The efficiency condition that $S \to 0$ implies that $R \to 0$. If $v < 0$, eventually $\dot{K} < 0$, and therefore, assuming capital can be consumed and given the fixed initial endowment K_0, both R and K will tend to 0; assuming that no output is produced purely by labour, constant consumption is impossible.

To take the argument further requires more structure for the production function. Since it is clearly the degree of substitutability between capital and resources that is of key importance in models with exhaustible resources, the important functional form to consider is the class of constant elasticity of substitution (CES) production functions. Defining, as before, σ to be the elasticity of substitution, we have, assuming constant labour force and normalizing per unit of labour,

$$F = (\alpha K^{\frac{\sigma-1}{\sigma}} + \beta R^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta)^{\frac{\sigma}{\sigma-1}},$$
$$\alpha, \beta > 0, \quad \alpha + \beta < 1, \quad \sigma > 0, \quad \sigma \neq 1.$$

Note that F, K, and R are all functions of time, while α, β and σ are fixed parameters. For the case $\sigma = 1$ this reduces to the familiar Cobb–Douglas form.

It will be convenient in what follows to define the following expressions:

$$X = (\alpha K^{\frac{\sigma-1}{\sigma}} + \beta R^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta), \tag{5}$$

and

$$\gamma = \frac{\beta R^{\frac{\sigma-1}{\sigma}}}{X} < 1. \tag{6}$$

With these definitions we can derive.

$$F_R = \beta X^{\frac{1}{\sigma-1}} R^{-\frac{1}{\sigma}} = \beta \frac{F}{X} R^{-\frac{1}{\sigma}} = \beta \left(\frac{F}{R}\right)^{\frac{1}{\sigma}}.$$

The examination of the behavior of the system for $v > 0$ will be divided into three parts, according to the assumptions about the elasticity of substitution. For the Cobb–Douglas function ($\sigma = 1$) we have,

$$C = F - F_R R - F_R v$$
$$= F\left(1 - \beta - \frac{\beta v}{R}\right).$$

Therefore C becomes negative as $R \to 0$, contradicting $\dot{C} = 0$.

If $\sigma < 1$ then,

$$\begin{aligned} C &= F - F_R R - F_R v \\ &= F - \gamma F - v\beta \frac{F}{X} R^{-\frac{1}{\sigma}} \\ &= F\left(1 - \gamma - \frac{v\beta}{\alpha K^{\frac{\sigma-1}{\sigma}} R^{\frac{1}{\sigma}} + \beta R + (1-\alpha-\beta) R^{\frac{1}{\sigma}}}\right). \end{aligned}$$

Again, C becomes negative as $R \to 0$ (note that $\gamma \to 1^-$), contradicting $\dot{C} = 0$.

If $\sigma > 1$ then resources are not essential for production. For large σ, i.e. as $\sigma \to \infty$, $F_R \to \beta$ and $F_K \to \alpha$, so that the Hotelling rule is violated. Efficient production is therefore impossible when capital and resources are perfect substitutes. As derived in the Appendix, the growth rate of output for general σ can be shown to be given by:

$$\dot{F} = \dot{K} \frac{\dot{R}}{R + v}\left(\frac{R(\sigma - 1) - v}{R(\sigma - \gamma) - \gamma v}\right). \tag{7}$$

This expression reduces to the following for the case $v = 0$:

$$\dot{F} = \dot{K} \frac{\dot{R}}{R} \frac{(\sigma - 1)}{(\sigma - \gamma)}. \tag{8}$$

As $\sigma \to 1^+$, $\gamma \to \beta$ and, from expression (7),

$$\dot{F} \to \dot{K} \frac{\dot{R}}{R + v}\left(\frac{-v}{R(1 - \beta) - \beta v}\right).$$

We distinguish two cases according to the initial conditions in the preceding expression. If in the initial period $R_0(1 - \beta) - \beta v > 0$, then for some time beyond this period,

$$R = \frac{\beta}{1 - \beta} v,$$

at which point output is infinite (since the growth rate is positive and infinite – recall that $\dot{R} < 0$ because of efficiency condition (3)). The programme is not feasible.

Alternatively, if $R_0(1 - \beta) - \beta v < 0$ in the initial period, then $\dot{F} < 0$. Note that

$$\begin{aligned} \frac{F}{R} &= \left(\alpha K^{\frac{\sigma-1}{\sigma}} + \beta R^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta\right)^{\frac{\sigma}{\sigma-1}} R^{-1} \\ &= \left(\alpha\left(\frac{K}{R}\right)^{\frac{\sigma-1}{\sigma}} + \beta + \frac{1 - \alpha - \beta}{R^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{\sigma}{\sigma-1}}, \end{aligned}$$

is unbounded as $R \to 0$. Consumption is given by,

$$C = F - F_R(R + v)$$
$$= (1 - \gamma)F - \beta v \left(\frac{F}{R}\right)^{\frac{1}{\sigma}}.$$

Therefore if $R_0(1 - \beta) - \beta v < 0$, consumption becomes negative as $R \to 0$.

These results for the generalized Hartwick rule are summarized in Table I. Any non-zero choice of v, roughly speaking, leads to declining consumption, infinite output or a violation of the Hotelling rule, depending on the value of the elasticity of substitution σ.

Table I. Results for the Generalized Hartwick Rule $\dot{K} = F_R(R + v)$.

	$\sigma < 1$	$\sigma = 1$	$\sigma > 1$ $\sigma \approx 1$	$\sigma \to \infty$
$v < 0$	$F \to 0$ $C \to 0$ *as* $t \to \infty$	$F \to 0$ $C \to 0$ *as* $t \to \infty$	$F \to 0$ $C \to 0$ *as* $t \to \infty$	Hotelling Rule violated
$v > 0$	$C_T < 0$ *for* $T < \infty$	$C_T < 0$ *for* $T < \infty$	Either $\dot{F}(K_T, R_T) = \infty$ *for* $T < \infty$ or $C_T < 0$ *for* $T < \infty$	Hotelling Rule violated

2.2. THE STANDARD HARTWICK RULE

Having shown the generalized Hartwick rule to be infeasible, at least for CES production functions, in what follows we employ the widely known form of the Hartwick rule,

$$\dot{K} = F_R R \tag{9}$$

i.e., that investment equal resource rents. Now the question to be explored is the behaviour of output, consumption and investment, under the standard Hartwick–Hotelling programme, for different values of the elasticity of substitution. We proceed by considering three cases, according to whether this elasticity is less than, equal to, or greater than 1.

We first consider the case $\sigma < 1$. In this instance the marginal product of resources is bounded since,

$$\begin{aligned} F_R &= \beta(\alpha K^{\frac{\sigma-1}{\sigma}} + \beta R^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta)^{\frac{1}{\sigma-1}} R^{-\frac{1}{\sigma}} \\ &= \beta\left(\alpha\left(\frac{K}{R}\right)^{\frac{\sigma-1}{\sigma}} + \beta + \frac{1 - \alpha - \beta}{R^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{1}{\sigma-1}}. \end{aligned} \tag{10}$$

Thus, since K is non-decreasing under the Hartwick rule.

$$F_R \to \beta^{\frac{\sigma}{\sigma-1}} \quad \text{as} \quad R \to 0.$$

This in turn implies that K is bounded because

$$K_T = K_0 + \int_0^T F_R R\,dt \quad \text{and} \quad \int_0^\infty R\,dt = S_0.$$

Consumption is given by,

$$\begin{aligned} C &= F - F_R R \\ &= X^{\frac{\sigma}{\sigma-1}} - \beta X^{\frac{1}{\sigma-1}} R^{-\frac{1}{\sigma}} R \\ &= X^{\frac{1}{\sigma-1}} (X - \beta R^{\frac{\sigma-1}{\sigma}}) \\ &= (\alpha K^{\frac{\sigma-1}{\sigma}} + \beta R^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta)^{\frac{1}{\sigma-1}} (\alpha K^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta). \end{aligned} \tag{11}$$

Therefore, since K is bounded, consumption tends to 0 as R tends to 0 when $\sigma < 1$.

This derivation can be compared with Dasgupta and Heal (1979, Ch. 7), who show that if the elasticity of substitution is less than 1, then F/R is bounded, implying total input is bounded, and therefore that constant consumption is impossible.

The preceding derivation and the result from Dasgupta and Heal (1979) contradict Theorem 3 from Hartwick (1978). One component of this theorem implies that $\dot{F} > 0$ iff $\sigma < 1$. Recalling expression (8), note that $\gamma \to 1^-$ as $R \to 0$, so that, while F may initially be increasing (i.e. for $\gamma < \sigma < 1$), eventually it must decrease. In fact, since K is bounded, eventually $\dot{K} \to 0$, so that $\dot{F} \to 0^-$, contradicting the first part of the theorem. Because total output is bounded, we know that $F \to 0$ in the long run.

Next we consider the case $\sigma > 1$. Since resources are not essential in this instance, both Solow (1974) and Dasgupta and Heal (1979) dismiss this case. The behaviour of the Hartwick–Hotelling programme under these conditions needs to be clarified.

Since $\sigma > 1$ and $\gamma < 1$, expression (8) implies that $\dot{F} < 0$. Because consumption is constant, this in turn implies that $F_R R \to 0$ as $R \to 0$, since

$$F \to (\alpha K^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta)^{\frac{\sigma}{\sigma-1}} \quad \text{as} \quad R \to 0.$$

A further conclusion from the preceding expression is that K must be bounded since $\dot{F} < 0$. This is in spite of F_R being unbounded as $R \to 0$, as is obvious from expression (10).

What value of σ maximizes consumption? Because $F \to \alpha K + \beta R + 1 - \alpha - \beta$ as $\sigma \to \infty$, the Hotelling rule is violated as capital and resources become perfect substitutes. For finite values of the elasticity of substitution we have, following from expression (11):

$$\frac{\partial C}{\partial \sigma} = X^{\frac{1}{\sigma-1}} \ln(X) \frac{(-1)}{(\sigma-1)^2} (\alpha K^{\frac{\sigma-1}{\sigma}} + 1 - \alpha - \beta)$$

$$+ X^{\frac{\sigma-1}{\sigma}} \alpha K^{\frac{\sigma-1}{\sigma}} \ln(K) \frac{1}{\sigma^2}$$

$$= X^{\frac{1}{\sigma-1}} \left(\alpha K^{\frac{\sigma-1}{\sigma}} \left(\frac{\ln(K)}{\sigma^2} - \frac{\ln(X)}{(\sigma-1)^2} \right) - (1 - \alpha - \beta) \frac{\ln(X)}{(\sigma-1)^2} \right).$$

The critical issue is therefore the behaviour of $\frac{\ln(X)}{(\sigma-1)^2}$.

Using l'Hôpital's rules, we take the derivatives of numerator and denominator with respect to σ, $\frac{\frac{\partial \ln(X)}{\partial \sigma}}{2(\sigma-1)} = \frac{\frac{\partial X}{\partial \sigma}}{2X(\sigma-1)}$. Now $\frac{\partial X}{\partial \sigma} = \frac{\alpha K^{\frac{\sigma-1}{\sigma}} \ln(K)}{\sigma^2} + \frac{\beta R^{\frac{\sigma-1}{\sigma}} \ln(R)}{\sigma^2} \to \ln(K^{\alpha} R^{\beta})$ as $\sigma \to 1^+$. Consumption is therefore a declining function of $\sigma > 1$. Constant consumption under the Hartwick rule is consequently maximized as $\sigma \to 1^+$. The Cobb–Douglas production function yields maximal consumption.

Because resources are not essential for elasticities of substitution greater than 1, one strategy for achieving maximal consumption might be to consume all of the resource in the initial period. The derivation in the Appendix shows, however, that such a strategy will not yield constant consumption under the Hartwick rule.

The operation of the standard Hartwick–Hotelling programme, where investment is precisely equal to current resource rents, is summarized for CES production functions in Table II. Only the Cobb–Douglas production function, for which the elasticity of substitution is equal to 1, yields minimal sustainability at the maximum rate of consumption.

Table II. Results for the Standard Hartwick Rule $\dot{K} = F_R R$.

$\sigma < 1$	$\sigma = 1$	$\sigma > 1$ $\sigma \approx 1$	$\sigma \to \infty$
$F \to 0$ $C \to 0$ *as* $t \to \infty$	$\dot{F} = 0$ $\dot{C} = 0$ *C is maximal*	$\dot{F} < 0$ $\dot{C} = 0$ *C is not maximal*	Hotelling Rule violated

2.3. LINKS TO THE LITERATURE

As noted in the introduction, the literature on the Hartwick rule has several strands to it. This section identifies the common threads in the literature and provides a generalization of one result in DHH (1980).

The solution of the system for a Hartwick–Hotelling programme is remarkably simple in the Cobb–Douglas case, for constant consumption C_0:

$$F = K^{\alpha}R^{\beta}, \quad \alpha, \beta > 0, \quad \alpha + \beta < 1,$$
$$\dot{K} = F_R R = \beta F,$$

implying,

$$C_0 = (1 - \beta)F \text{ constant} \Rightarrow F \text{ constant} \Rightarrow \frac{\dot{R}}{R} = -F_K$$

from Appendix (A-1),

so that $$K = K_0 + \frac{\beta C_0}{1 - \beta} t, \text{ and}$$

therefore, $$R = \left(\frac{C_0}{1 - \beta}\right)^{\frac{1}{\beta}} \left(K_0 + \frac{\beta C_0}{1 - \beta} t\right)^{-\frac{\alpha}{\beta}}$$

The condition for the *existence* of a solution to the system is therefore $\alpha > \beta$, i.e. the elasticity of output with respect to capital must be greater than that with respect to resources, since R must have a finite integral equalling S_0. Performing the integration yields the value for maximal consumption,

$$C_0 = (1 - \beta)\,(\alpha - \beta)^{\frac{\beta}{1-\beta}}\, S_0^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}. \tag{12}$$

To tie the literature together, it is worth describing the solutions of Solow (1974) and Dasgupta and Heal (1979) to the maximin problem, which did not use the Hartwick rule explicitly. Both choose the Cobb–Douglas production function after rejecting CES functions where total output is bounded ($\sigma < 1$) and where resources are not essential ($\sigma > 1$). For this production function maintaining constant consumption C_0 implies that,

$$\begin{aligned} K &= K_0 + mt \\ R &= (C_0 + m)^{\frac{1}{\beta}} (K_0 + mt)^{-\frac{\alpha}{\beta}} \end{aligned} \tag{13}$$

where $m = \dot{K}$ is a constant. Both point out that efficiency requires that the integral of the above expression for R exist and be equal to S_0, so that the condition $\alpha > \beta$ is required. Performing this integration yields:

$$C_0 = m^{\beta}\left(\frac{\alpha - \beta}{\beta} S_0\right)^{\beta} K_0^{\alpha-\beta} - m.$$

Dasgupta and Heal then maximize this expression with respect to m to yield the optimal constant consumption.

$$C_0 = (\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})\left(\frac{\alpha - \beta}{\beta}\right)^{\frac{\beta}{1-\beta}} S_0^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}.$$

which simplifies to expression (12).

Solow (1974) takes a different approach to deriving the optimum. By constructing phase diagrams for the problem, the following expression for the parameter m in the system of Equations (13) is arrived at directly,

$$m = \dot{K} = \frac{\beta C_0}{1 - \beta}, \tag{14}$$

for some fixed C_0. The equation for R in expression (13) is then integrated, and set equal to S_0 to arrive at the maximal level of consumption, as given by expression (12).

Hartwick's (1977) key insight, that expression (14) embodies the rule 'invest resource rents', is not derived explicitly in the paper. Instead the sufficiency of this rule for a programme of constant consumption is proved for a general production function.

It is also worth linking what has been presented so far to the much more general framework employed by DHH (1980).[2] If we assume the existence of a competitive output price path p and a positive constant exhaustion rent μ (i.e. assuming efficient resource extraction), then expression (4) may be re-written as,

$$\begin{aligned} p\dot{K} &= pF_R(R + v) \\ &= \mu\,(R + v) \\ &= -\mu\dot{S} + \mu v, \end{aligned}$$

where μv is constant. Rearranging terms then gives the analog of the 'generalized Hartwick rule' of DHH:

$$p\dot{K} + \mu\dot{S} = \mu v.$$

The authors prove that this rule is necessary and sufficient for constant utility in their general framework. They go on to show that any path with $v < 0$ is infeasible.

A point of particular interest in the Dixit, Hammond and Hoel paper is their proof that, given any efficient path such that $v = 0$, any other path with $v > 0$ and a larger capital stock for all $t > 0$ will yield a lower level of utility under the generalized Hartwick rule. Expressing their proposition in terms of the CES production function analysis presented so far, we can say that

$$F_{R'_0}(R'_0 + v) > F_{R_0}R_0 \text{ for } v > 0,$$

is a necessary condition for $C'_0 < C_0$, where the 'unprimed' variables represent their values for $v = 0$ (this is a necessary condition because the capital stock can only be greater than its previous value for all $t > 0$ if its rate of change in the initial period is greater).

Changes in v will clearly affect R_0, because the capital stock level will be altered under the generalized Hartwick rule, and the efficient path must both exhaust the resource and satisfy the Hotelling rule. For infinitesimal changes in v it is shown in the Appendix that the DHH condition for maximal utility is equivalent to

$$\frac{\partial R_0}{\partial v} < \frac{\sigma}{1 - \gamma - \sigma}, \tag{15}$$

while a more direct derivation of this condition, using the machinery of CES production functions presented so far, yields

$$\frac{\partial R_0}{\partial v} < \frac{\sigma}{1 - \gamma}, \tag{16}$$

Recalling that the case $\sigma > 1$ was the most problematic in terms of analyzing its behaviour with respect to changes in v, and noting that $\gamma < 1$, we can conclude that the DHH condition is stronger than necessary for this case, since expression (15) is negative and expression (16) is positive. At least for infinitesimal changes in v the weaker condition (16) yields maximal consumption under the Hartwick–Hotelling programme when the elasticity of substitution is greater than 1.

This concludes the derivation of results concerning constant consumption. We now turn to the consideration of the conditions governing rising consumption and utility, i.e. strict sustainability.

3. Optimal Paths with Rising Consumption

Given two programmes for which $\dot{U} > 0$, it would be rational to prefer the programme for which the sum (or, in continuous time, the integral) of utility over time is the greater. Thus, at least for the purposes of exploring some simple models, we will assume that sustainability is consistent with a Utilitarian ethic as applied to current and future generations. The interesting question, it turns out, is whether there is a pure rate of time preference in the models. If there is, the Utilitarian ethic would suggest that we wish to maximize the present value of utility, where future utility is discounted at some rate r.

In what follows we will assume an explicit form of the utility function,

$$U(C) = -C^{-(\eta - 1)}, \quad \eta > 1. \tag{17}$$

This is one of the class of functions for which the elasticity of the marginal utility of consumption, η, is constant.

In the analysis that follows the Cobb–Douglas production function will be used (for reasons not just of tractability, but bearing in mind the results of the previous section as well) and the efficiency of any programme will again require that the resource be exhausted, as given in expression (3).

Finding the optimal programme can now be expressed as an optimal control problem:

$$\max_{C,R} \int_0^\infty U(C)e^{-rt}dt,$$

subject to: $\dot{K} = F - C$, $\dot{S} = -R$.

The present value Hamiltonian function for this problem is,

$$\begin{aligned}\mathcal{H} &= Ue^{-rt} + \gamma_1\dot{K} + \gamma_2\dot{S} \\ &= Ue^{-rt} + \gamma_1(F - C) - \gamma_2 R,\end{aligned}$$

where γ_1 and γ_2 are the co-state variables. The optimum programme must satisfy the following first order conditions:

$$\frac{d\mathcal{H}}{dC} = U'e^{-rt} - \gamma_1 = 0, \tag{18}$$

$$\frac{d\mathcal{H}}{dR} = \gamma_1 F_R - \gamma_2 = 0, \tag{19}$$

$$\frac{d\mathcal{H}}{dK} = \gamma_1 F_K = -\dot{\gamma}_1 \tag{20}$$

$$\frac{d\mathcal{H}}{dS} = 0 = -\dot{\gamma}_2 \tag{21}$$

From expression (19) we see that $\gamma_1 = \gamma_2/F_R$. By substituting this into expression (20) we obtain $\gamma_2(F_K/F_R) = -d/dt\,(\gamma_2/F_R)$ Since expression (21) implies that γ_2 is constant, this simplifies to,

$$\gamma_2 \frac{F_K}{F_R} = \frac{\gamma_2}{F_R^2}\dot{F}_R, \quad \text{or,} \quad F_K = \frac{\dot{F}_R}{F_R}.$$

This is just the Hotelling rule. It is derived directly as an efficiency condition for the optimal control problem.

It is easily verified that, from expression (17), $U''/U' = -\eta/C$. From expression (18) we see that $U' = \gamma_1 e^{rt}$. Differentiating this expression with respect to time gives,

$$\begin{aligned}U''\dot{C} &= \dot{\gamma}_1 e^{rt} + \gamma_1 r e^{rt} \\ &= -\gamma_1 F_K e^{rt} + \gamma_1 r e^{rt} \qquad \text{(from (20))} \\ &= \gamma_1 e^{rt}(r - F_K) \\ &= U'(r - F_K).\end{aligned}$$

We therefore have the efficiency condition

$$r + \eta\frac{\dot{C}}{C} = F_K. \tag{22}$$

This is the 'Ramsey rule' (after Ramsey (1928)). Efficiency requires that the social rate of return on investment equal the marginal product of capital. For the Cobb–Douglas production function the percentage rate of change in consumption is therefore given by,

$$\frac{\dot{C}}{C} = \eta^{-1}\left(\alpha\frac{R^{\beta}}{K^{1-\alpha}} - r\right). \tag{23}$$

Because resources are essential for production and in finite supply, resource use R will tend asymptotically to 0 in an optimal programme. The only way that the percentage rate of change in consumption, as given in expression (23), can be non-decreasing is if capital stock K eventually tends to 0 as well – i.e. eventually capital must be consumed. But if both capital and resources are declining, them so is output. Since the initial capital stock is fixed, consumption must eventually fall. The optimal path is not sustainable.

By equating the expression for the Hotelling rule and expression (22), it is possible to solve the differential equation to give $e^{-rt}F_R C^{-\eta} = \delta$, for constant of integration δ, and by rearranging terms,

$$C = \delta^{-\frac{1}{\eta}}\left(e^{-rt}\frac{K^{\alpha}}{R^{1-\beta}}\right)^{\frac{1}{\eta}}.$$

If the discount rate for utility, r, is 0 then it is clear from this expression that consumption will grow indefinitely as resource use R declines. As Dasgupta and Heal (1979, Ch. 10) point out, however, the optimum path will only exist if the non-discounted integral of utility is finite, which requires that,

$$\eta \geq \frac{1-\beta}{\alpha-\beta}.$$

The pure rate of time preference is thus a critical element in characterizing the sustainability of development with exhaustible resources. Only if this rate is 0 can strict sustainability be shown to hold – otherwise, consumption eventually falls along the optimum path.

4. Conclusions

If for reasons of intergenerational equity the desired goal is minimal sustainability with maximum consumption, then the Hartwick rule, to invest resource rents, is the keystone. Solow (1986) refers to it as a 'rule of thumb' for growth policy.

This study has drawn together several strands from a diverse literature on the Hartwick rule. The analysis has shown that, given virtually unrestricted production functions, the generalized Hartwick rule in combination with the Hotelling rule is both necessary and sufficient for consumption to be constant. The Cobb–Douglas production function (out of the class of CES production functions), in which the elasticity of substitution between capital and resources is exactly 1, yields consumption that is constant, positive and maximal when a standard Hartwick programme is followed in combination with two efficiency conditions, the Hotelling rule and complete resource exhaustion.

The generalized Hartwick rule, $\dot{K} = F_R(R + v)$ for non-zero v, yields either declining consumption or infinite output for finite values of the elasticity of substitution. The standard Hartwick rule, where $v = 0$, yields either declining consumption or non-maximal consumption for finite elasticities of substitution that are not equal to 1. The derivations in this paper therefore emphasize the 'knife edge' role of the Cobb–Douglas production function. Although the generalized Hartwick rule promises constant consumption for general production functions, the requirement that a maximal constant consumption path *exist* places severe limits on the substitution possibilities inherent in the production function.

Turning to the question of strict sustainability (increasing consumption), there is a strong inference about optimal growth with finite resources. The central issue concerns the discounting of utility: if the pure rate of time preference is greater than 0, then the traditional Utilitarian maximand, the present value of utility, leads to an optimal programme that is not sustainable. It is worth recalling, therefore, Ramsey's (1928) view of discounting utility as 'ethically indefensible'. If, however, the discount rate for utility is 0 then, under fairly weak restrictions, the Utilitarian maximum yields continually increasing utility.

The latter result could have practical consequences, particularly for questions with long time horizons such as greenhouse warming. It is common to use the social rate of return on investment as the discount rate in analyses such as that of Nordhaus (1992b) of controlling greenhouse gases. If we assume a long-run percentage growth rate in per capita consumption of about 2%, and an elasticity of marginal utility of consumption of about 1, then the choice between a zero pure rate of time preference and a low value of, say, 2%, can double the discount rate used in the analysis.

Appendix

We begin with a basic result for the rate of change of output under the generalized Hartwick rule:

$$\begin{aligned} \dot{F} = \ddot{K} + \dot{C} &= \dot{F}_R(R + v) + F_R\dot{R} \\ &= \dot{K}\left(F_K + \frac{\dot{R}}{R + v}\right) \\ &= \dot{K}\left(\frac{\dot{F}_R}{F_R} + \frac{\dot{R}}{R + v}\right). \end{aligned} \tag{A-1}$$

Recalling the definition of X (expression (5)) and γ (expression (6)), a few results follow directly from the definition of the CES production function:

$$F_R = \beta X^{\frac{1}{\sigma-1}} R^{-\frac{1}{\sigma}} = \beta \frac{F}{X} R^{-\frac{1}{\sigma}} = \beta \left(\frac{F}{R}\right)^{\frac{1}{\sigma}}, \tag{A-2}$$

$$F_{RK} = \frac{\alpha\beta}{\sigma} X^{\left(\frac{1}{\sigma-1}-1\right)} R^{-\frac{1}{\sigma}} K^{-\frac{1}{\sigma}} = \frac{1}{\sigma}\frac{\beta R^{-\frac{1}{\sigma}}}{X} F_K = \frac{1}{\sigma}\frac{\gamma}{R} F_K. \tag{A-3}$$

$$F_{RR} = \frac{1}{\sigma}\frac{F_R}{R}(\gamma - 1) < 0. \tag{A-4}$$

Note as well that $\gamma < 1$ and that $\gamma \to \beta$ as $\sigma \to 1$.

The first item to be derived is the expression for $\dot{F}$. We begin with $\dot{F}_R = F_{RR}\dot{R} + F_{RK}\dot{K}$. Therefore,

$$\begin{aligned}\frac{\dot{F}_R}{F_R} &= \frac{1}{\sigma}\frac{\dot{R}}{R}(\gamma - 1) + F_{RK}(R + v) \\ &= \frac{1}{\sigma}\frac{\dot{R}}{R}(\gamma - 1) + \frac{1}{\sigma}\frac{\gamma}{R}(R + v)F_K \\ &= \frac{1}{\sigma}\frac{\dot{R}}{R}(\gamma - 1) + \frac{1}{\sigma}\frac{\gamma}{R}(R + v)\frac{\dot{F}_R}{F_R},\end{aligned}$$

so that,

$$\frac{\dot{F}_R}{F_R} = \left(\frac{\dot{R}}{R+v}\right)\frac{(\gamma - 1)(R + v)}{R\sigma - \gamma(R + v)}.$$

From expression (A-1) we therefore derive,

$$\begin{aligned}\dot{F} &= \dot{K}\frac{\dot{R}}{R+v}\left(\frac{(\sigma - 1)(R + v)}{R\sigma - \gamma(R + v)} + 1\right) \\ &= \dot{K}\frac{\dot{R}}{R+v}\left(\frac{R(\sigma - 1) - v}{R(\sigma - \gamma) - \gamma v}\right),\end{aligned}$$

and thus,

$$\dot{F} \to \dot{K}\frac{\dot{R}}{R+v}\left(\frac{-v}{R(1 - \beta) - \beta v}\right) \quad \text{as} \quad \sigma \to 1^+.$$

The next issue to be considered is the behaviour of the Hartwick-Hotelling system when $\sigma > 1$ and, since resources are not essential for this value of the elasticity of substitution, when all of the resource is extracted in the base period. $\dot{C} = 0$ if and only if, given $F = F(K, R)$,

$$F(K_0 + F_R(K_0, S_0), 0) = F(K_0, S_0) - F_R(K_0, S_0)S_0. \tag{A-5}$$

It will simplify the algebra considerably, without unduly affecting the generality of the argument, if we assume for this derivation that $\alpha + \beta = 1$, so that

$$X = \alpha K^{\frac{\sigma-1}{\sigma}} + \beta R^{\frac{\sigma-1}{\sigma}}.$$

Expression (A-5) may now be re-written as,

$$(\alpha(K_0 + \beta X^{\frac{1}{\sigma-1}} S_0^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} = X^{\frac{1}{\sigma-1}}(X - \beta S_0^{\frac{\sigma-1}{\sigma}}),$$

which implies that,

$$\alpha^{\frac{\sigma}{\sigma-1}}(K_0 + \beta X^{\frac{1}{\sigma-1}} S_0^{\frac{\sigma-1}{\sigma}}) = X^{\frac{1}{\sigma-1}}(\alpha K_0^{\frac{\sigma-1}{\sigma}}),$$

and therefore that,

$$\alpha^{\frac{\sigma}{\sigma-1}} K_0 = X^{\frac{1}{\sigma-1}}(\alpha K_0^{\frac{\sigma-1}{\sigma}} - \alpha^{\frac{\sigma}{\sigma-1}} \beta S_0^{\frac{\sigma-1}{\sigma}}).$$

For consumption to be constant this latter expression should be an identity. However, there are clearly choices of K_0 and S_0 for which the right hand side is less than or equal to 0, so the identity does not hold.

Finally, we consider how the level of consumption varies with v in the generalized Hartwick rule. The analysis will be based on infinitesimal positive changes $\mathrm{d}v$, to examine the transition from the standard to the generalized Hartwick rule – negative values have already been ruled out because they lead to declining consumption. In order for the capital stock to be greater for all time (after the initial period) in the transition to the generalized rule, as DHH (1980) hypothesize, a necessary condition is that $\dot{K}$ be greater under the generalized rule. This may be written as,

$$C_0' < C_0 \quad \text{if } \mathrm{d}v > 0 \quad \text{and} \quad F_{R_0'}(R_0' + \mathrm{d}v) > F_{R_0} R_0.$$

However, for infinitesimal $\mathrm{d}v$ we may write,

$$F_{R_0'} = F_{R_0} + \frac{\partial F_{R_0}}{\partial v}\mathrm{d}v, \quad \text{and} \quad R_0' = R_0 + \frac{\partial R_0}{\partial v}\mathrm{d}v.$$

Therefore

$$\begin{aligned} F_{R_0'}(R_0' + \mathrm{d}v) &= \left(F_{R_0} + \frac{\partial F_{R_0}}{\partial v}\mathrm{d}v\right)\left(R_0 + \left(\frac{\partial R_0}{\partial v} + 1\right)\mathrm{d}v\right) \\ &= F_{R_0}R_0 + \frac{\partial F_{R_0}}{\partial v}R_0\mathrm{d}v + F_{R_0}\left(\frac{\partial R_0}{\partial v} + 1\right)\mathrm{d}v, \end{aligned}$$

where terms in $\mathrm{d}v^2$ have been dropped because they will go to zero in the limit. The question is therefore reduced to whether the sum of the second two terms in the preceding expression is greater than 0. Note that, since we are dealing with the initial period, K_0 is independent of v, and therefore the relationship we wish to test for these terms may be written as,

$$F_{RR_0}\frac{\partial R_0}{\partial v}R_0\mathrm{d}v + F_{R_0}\left(\frac{\partial R_0}{\partial v}+1\right)\mathrm{d}v > 0.$$

Recalling expression (A-4), this may be written for CES production functions as,

$$\frac{\gamma-1}{\sigma}\frac{\partial R_0}{\partial v}\,\mathrm{d}v + \left(\frac{\partial R_0}{\partial v}+1\right),$$

so that, after dividing by dv, the Dixit, Hammond and Hoel condition reduces to

$$\frac{\partial R_0}{\partial v} > \frac{\sigma}{1-\gamma-\sigma}\,. \tag{A-6}$$

Because K_0 is given and independent of v, both C_0 and R_0 are functions of v under the generalized Hartwick rule. A more direct attack on this problem is therefore to evaluate

$$\frac{\partial C_0}{\partial v} = \frac{\partial}{\partial v}\left(F_0 - F_{R_0}\left(R_0 + \mathrm{d}v\right)\right)$$

$$= -F_{RR_0}\frac{\partial R_0}{\partial v}\left(R_0 + \mathrm{d}v\right) - F_{R_0}.$$

In order for C_0 to vary negatively with v we require the latter expression to be less than 0, or, again employing expression (A-4) and dividing by F_{R_0},

$$-\frac{\gamma-1}{\sigma}\frac{1}{R_0}\frac{\partial R_0}{\partial v}\left(R_0 + \mathrm{d}v\right) < 1.$$

Taking the limit as dv tends to 0, this reduces to the condition

$$\frac{\partial R_0}{\partial v} < \frac{\sigma}{1-\gamma}\,. \tag{A-7}$$

For the case $\sigma > 1$, expression (A-7) is less restrictive than (A-6) because its right-hand side is positive, while that of (A-6) is negative.

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The advice and comments of David Ulph, Malcolm Pemberton and David Pearce are gratefully acknowledged. The usual caveats apply.

Notes

[1] Dasgupta and Mitra (1983) showed that in a discrete time formulation constant maximal consumption requires that investment be less than resource rents. However, the standard Hartwick result is approached asymptotically as the time step nears 0.

[2] I am grateful to David Ulph for pointing out this connection.

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