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Ambartsumian's Principle of Invariance and the Reflection of Radio Waves from Plane Inhomogeneous Slabs

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Abstract. Using Ambartsumian's principle of invariance we investigate for radio waves the reflection coefficient of a plane inhomogeneous slab. We find that the reflection coefficient, as a function of slab thickness, satisfies the Riccati equation. From this equation we deduce a geometric theorem on the upper and lower bounds of the reflection coefficient. We illustrate the theorem by applying it to several special cases.

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Ambartsumian's principle of invariance, which plays a central role in the theory of radiative transfer in planetary atmospheres [1], provides the basis of a novel method for calculating the reflection of radio waves from plane-stratified slabs [2]. The method stems from the principle's central idea of considering how much the reflection coefficient of a reflector or scatterer is changed by the addition of a thin layer to its surface, and thus leads to a differential equation for the reflection coefficient. The advantage of the method over conventional methods is that from a computational viewpoint it is easier to solve the differential equation for the reflection coefficient than to solve the linear boundary-value problem for the field within the inhomogeneous dielectric body.

To illustrate the method let us calculate the reflection of a plane wave that is normally incident on a plane-stratified lossless dielectric slab. As shown in Fig. 1, the slab lies in the region $0 \leqq z \leqq a$. Since the slab is inhomogeneous, its wave number $k(z)$ is not a constant, but a real function of z. The half-space $z \leqq 0$ is filled with a homogeneous lossless dielectric of constant wave number k_0, and the half-space $z \geqq a$, with a homogeneous lossless dielectric of constant wave number k_1.

We assume that a wave of unit amplitude is incident on the slab, i.e.,

$$\text{incident wave} = e^{ik_0 z} e^{-i\omega t} \qquad (z \leqq 0). \tag{1}$$

Then a reflected wave in the half-space $z \leqq 0$ and a transmitted wave in the half-space $z \geqq a$ will be generated, i.e.,

$$\text{reflected wave} = R e^{-ik_0 z} e^{-i\omega t} \qquad (z \leqq 0), \tag{2}$$

$$\text{transmitted wave} = T e^{ik_1 z} e^{-i\omega t} \qquad (z \geqq a). \tag{3}$$

The problem is to determine the reflection coefficient R and the transmission coefficient T from a knowledge of $k(z)$ and the slab thickness.

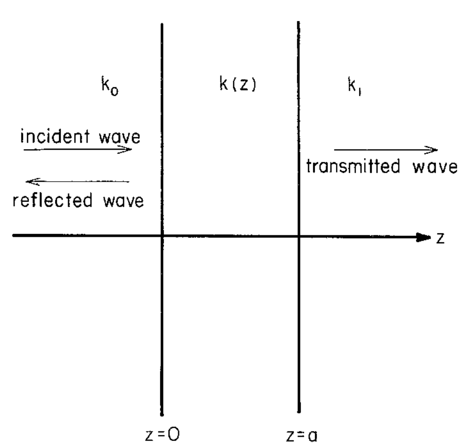

Fig. 1. Plane inhomogeneous slab extending from $z=0$ to $z=a$. k_0 and k_1 are the wave numbers of the homogeneous media to the left and to the right of the slab. $k(z)$ is the wave number within the slab

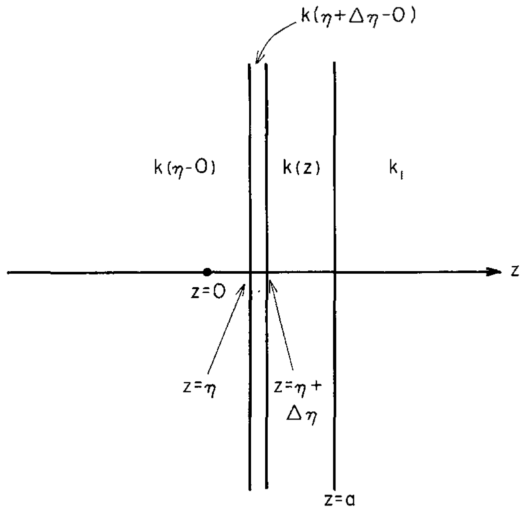

Fig. 2. Addition of thin layer for relating reflection coefficient R at $z=\eta$ to reflection coefficient R at $z=\eta+\Delta\eta$

We now formulate the problem in a way that will enable us to determine the reflection coefficient as a function of the slab thickness. We consider a plane $z=\eta$ lying somewhere within the bounds of the slab, i.e., $0\leqq\eta\leqq a$ (Fig. 2). To the left of this plane, we assume there is a homogeneous half-space of wave number $k(\eta-0)$ where $k(\eta-0)$ denotes the limit of $k(z)$ as z approaches η from below. The wave incident on the plane $z=\eta$ is given by

$$\text{incident wave} = e^{ik(\eta-0)\cdot(z-\eta)} \tag{4}$$

and the wave reflected from it, by

$$\text{reflected wave} = R(\eta)e^{-ik(\eta-0)\cdot(z-\eta)}. \tag{5}$$

When $\eta=0$, (4 and 5) reduce, respectively, to (1 and 2). When $\eta=a$, the problem is the same as the simple problem of determining the reflection coefficient of the plane interface of two homogeneous media and we have

$$R(a)=\frac{k(a-0)-k_1}{k(a-0)+k_1}. \tag{6}$$

To the right of the plane $z=\eta$ we place another plane $z=\eta+\Delta\eta$, where $\Delta\eta$ is a differential of the first order. Since this layer extending from $z=\eta$ to $z=\eta+\Delta\eta$ is thin, its wave number is taken as constant and equal to $k(\eta+\Delta\eta-0)$. The reflection coefficient $R(\eta)$ can be looked upon as the sum of multiple reflections:

$$R(\eta)=R_1(\eta)+R_2(\eta)+R_3(\eta)+\dots, \tag{7}$$

where $R_1(\eta)$ is the contribution of the incident wave reflected at $z=\eta$, $R_2(\eta)$ is the contribution from the transmitted wave which is reflected at $z=\eta+\Delta\eta$ and then transmitted through $z=\eta$, and $R_3(\eta)$ is the contribution from the transmitted wave which is reflected at $z=\eta+\Delta\eta$, at $z=\eta$, and at $z=\eta+\Delta\eta$ and then finally transmitted through $z=\eta$. This method of computing $R(\eta)$, sometimes called the "method of multiple reflections", is particularly useful here because the layer is thin and hence only the first, second, and third order reflections, $R_1(\eta)$, $R_2(\eta)$, $R_3(\eta)$ contribute appreciably. We emphasize that it is necessary and sufficient to include from the start only the first, second, and third order reflections R_1, R_2, R_3 since the higher-order reflections R_4, R_5, etc. will not contribute in the limit $\Delta\eta\to 0$.

By elementary considerations we find that

$$R_1(\eta)=\frac{k(\eta-0)-k(\eta+\Delta\eta-0)}{k(\eta-0)+k(\eta+\Delta\eta-0)},$$

$$R_2(\eta)=\frac{2k(\eta-0)}{k(\eta-0)+k(\eta+\Delta\eta-0)}\times e^{ik(\eta+\Delta\eta-0)\Delta\eta}\cdot R(\eta+\Delta\eta)\times e^{ik(\eta+\Delta\eta-0)\Delta\eta}\times\frac{2k(\eta+\Delta\eta-0)}{k(\eta-0)+k(\eta+\Delta\eta-0)},$$

$$R_3(\eta)=\frac{2k(\eta-0)}{k(\eta-0)+k(\eta+\Delta\eta-0)}\times e^{ik(\eta+\Delta\eta-0)\Delta\eta}\cdot R(\eta+\Delta\eta)\times e^{ik(\eta+\Delta\eta-0)\Delta\eta}\times\frac{k(\eta+\Delta\eta-0)-k(\eta-0)}{k(\eta-0)+k(\eta+\Delta\eta-0)}\cdot e^{ik(\eta+\Delta\eta-0)\Delta\eta}\times R(\eta+\Delta\eta)\times e^{ik(\eta+\Delta\eta-0)\Delta\eta}\cdot\frac{2k(\eta+\Delta\eta-0)}{k(\eta+\Delta\eta-0)+k(\eta-0)}.$$

In the limit $\Delta\eta\to 0$, we have

$$R_1(\eta)=-\frac{dk}{d\eta}\cdot\frac{d\eta}{2k}; \tag{8}$$

and since

$$e^{2ik(\eta+\Delta\eta-0)\Delta\eta}=1+2ik(\eta)d\eta,$$

we get

$$R_2(\eta)=R(\eta+d\eta)+2ik(\eta)d\eta R(\eta+\Delta\eta)$$
$$=R(\eta)+\frac{dR}{d\eta}d\eta+2ik(\eta)d\eta R(\eta); \tag{9}$$

and similarly

$$R_3(\eta)=\frac{dk}{d\eta}\frac{1}{2k}R^2(\eta)d\eta. \tag{10}$$

Substituting (8, 9, and 10) into (7) we obtain

$$\frac{dR}{d\eta}=\frac{dk}{d\eta}\frac{1}{2k}-2ikR-\frac{dk}{d\eta}\frac{1}{2k}R^2. \tag{11}$$

This is the desired differential equation for the reflection coefficient of the inhomogeneous slab. We see that it is a nonlinear equation of the Riccati type, difficult to handle analytically but easy to solve numerically. Starting from the right side of the slab where the end condition (6) holds and progressing in discrete steps to the left, one can solve the equation numerically for the value of R at the left side of the slab.

We have assumed $k(\eta)$ to be sectionally smooth. However, if the medium has a jump discontinuity at any position $z=\eta_1$, the reflection coefficient just to the left of the discontinuity $R(\eta_1-0)$ is related to the reflection coefficient just to the right of the discontinuity $R(\eta_1+0)$ by

$$R(\eta_1-0)=\frac{r(\eta_1-0,\eta_1+0)+R(\eta_1+0)}{1+r(\eta_1-0,\eta_1+0)R(\eta_1+0)}, \tag{12}$$

where $r(\alpha,\beta)=[k(\alpha)-k(\beta)]/[k(\alpha)+k(\beta)]$ [3]. As the integration of (11) proceeds from $\eta=a$ to $\eta=0$ relation (12) must be employed at each discontinuity. If, for example, there is a jump discontinuity at $z=0$, (12) must be used to obtain $R(-0)$ from a knowledge of $R(+0)$.

The reflection coefficient $R(-0)$ can be found analytically in a few special cases [4]. In general, however, numerical methods must be used. In recent years, the reflection coefficient $R(-0)$ has been calculated by the numerical integration of (11) for a number of radio physics problems related to such phenomena as the reflection of radio waves from jet streams [5], the transmission of radar signals through radomes [6], the emission of radio waves from antennas in matter [7], the filtering of millimeter waves by almost periodic structures [8], and the reflection of radio waves from cylindrically or spherically stratified aerosols [9].

In some applications it is sufficient to know only the magnitude of R. That is, in such applications, if we write R as

$$R=A\,e^{i\psi}, \tag{13}$$

where A is the magnitude of R and ψ is its phase, it is sufficient to find only A. To handle applications of this sort, we now shall deduce a theorem that will give in a very simple manner, upper and lower bounds on A in terms of the maxima and minima of $k(z)$.

1. Formulation of the Problem

Substituting (13) into (11) we find that A satisfies

$$\frac{dA}{dz}=(1-A^2)\frac{1}{2k}\frac{dk}{dz}\cos\psi. \tag{14}$$

Using this equation as a point of departure we wish to find upper and lower bounds on $A(+0)$ from a knowledge of $A(a)$ and the maxima and minima of $k(z)$.

It follows from (14) that

$$\int_{A(+0)}^{A(a)}\frac{dA}{1-A^2}=\int_{+0}^{a}\frac{1}{2k}\frac{dk}{dz}\cos\psi dz \tag{15}$$

and hence

$$\frac{1}{2}\ln\left[\frac{1+A(a)}{1-A(a)}\right]\left[\frac{1-A(+0)}{1+A(+0)}\right]=\int_{+0}^{a}\frac{1}{2k}\frac{dk}{dz}\cos\psi dz. \tag{16}$$

To obtain bounds on $A(+0)$, we must obtain bounds on the value I of the integral on the right-hand side of this equation, that is, we must establish bounds on

$$I\equiv\int_{+0}^{a}\frac{1}{2k}\frac{dk}{dz}\cos\psi dz. \tag{17}$$

We shall now find such bounds by noting that $-1\leqq\cos\psi\leqq 1$ and by splitting the integration into sections where $dk/kz>0$ or $dk/dz<0$.

2. Sectionalization

To find bounds on I of (17) we note that $-1\leqq\cos\psi\leqq 1$, k is sectionally smooth, and the range of integration can be divided into sections where $dk/dz>0$ and sections, where $dk/dz<0$. If throughout a section ($z=z_i$ to $z=z_j$) $dk/dz>0$, we see that

$$\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}dz\geqq\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}\cos\psi dz\geqq-\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}dz. \tag{18}$$

On the other hand, if $dk/dz<0$ throughout the section, we have

$$\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}dz\leqq\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}\cos\psi dz\leqq-\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}dz \tag{19}$$

that is,

$$\frac{1}{2}\ln\frac{k(z_j)}{k(z_i)}\geqq\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}\cos\psi dz\geqq\frac{1}{2}\ln\frac{k(z_i)}{k(z_j)} \tag{20}$$

when $dk/z>0$, and

$$\frac{1}{2}\ln\frac{k(z_j)}{k(z_i)}\leqq\int_{z_i}^{z_j}\frac{1}{2k}\frac{dk}{dz}\cos\psi dz\leqq\frac{1}{2}\ln\frac{k(z_i)}{k(z_j)} \tag{21}$$

when $dk/dz<0$.

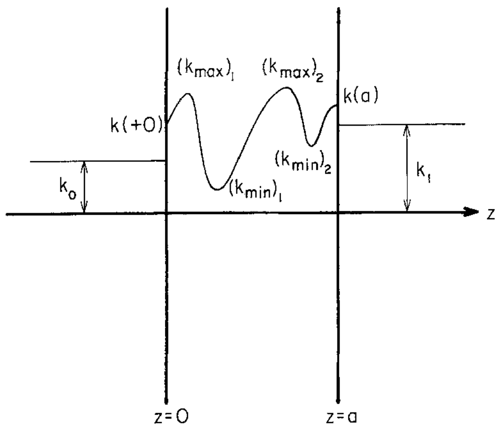

Fig. 3. Sample $k(z)$ showing maxima $(k_{max})_1$, $(k_{max})_2$, and minima $(k_{min})_1$, $(k_{min})_2$

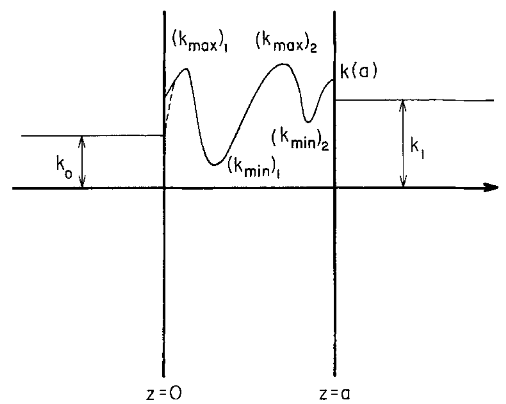

Fig. 4. Smoothing of jumps in $k(z)$. Broken curve shows how jump is smoothed out

From this it follows that

$$\frac{1}{2}\ln\left(\frac{(k_{max})_1^2(k_{max})_2^2(k_{max})_3^2\cdots}{(k_{min})_1^2(k_{min})_2^2(k_{min})_3^2\cdots}\gamma\right)$$

$$\geqq I \geqq \frac{1}{2}\ln\left(\frac{(k_{min})_1^2(k_{min})_2^2(k_{min})_3^2\cdots}{(k_{max})_1^2(k_{max})_2^2(k_{max})_3^2\cdots}\frac{1}{\gamma}\right), \tag{22}$$

where $(k_{max})_1$, $(k_{max})_2$, etc. and $(k_{min})_1$, $(k_{min})_2$, etc. denote the maxima and minima within the slab $(0<z<a)$ excluding the end points (Fig. 3). Here γ is a factor that depends on the behaviour of k as z approaches from within the slab the end points $z=0$ and $z=a$, viz.

$$\begin{aligned}\gamma &= k(+0)\,k(a) && \text{if } k(z) \text{ increases as } z\to 0 \text{ and } z\to a,\\ &= \frac{1}{k(+0)k(a)} && \text{if } k(z) \text{ decreases as } z\to 0 \text{ and } z\to a,\\ &= \frac{k(+0)}{k(a)} && \text{if } k(z) \text{ increases as } z\to 0 \text{ and decreases as } z\to a,\\ &= \frac{k(a)}{k(+0)} && \text{if } k(z) \text{ decreases as } z\to 0 \text{ and increases as } z\to a.\end{aligned} \tag{23}$$

For brevity we denote by L the argument of ln in (22), i.e. we write

$$L \equiv \frac{(k_{max})_1^2(k_{max})_2^2(k_{max})_3^2\cdots}{(k_{min})_1^2(k_{min})_2^2(k_{min})_3^2\cdots}\gamma. \tag{24}$$

Thus we see that

$$\frac{1}{2}\ln L \geqq I \geqq \frac{1}{2}\ln\frac{1}{L}. \tag{25}$$

3. Upper and Lower Bounds

From (16) and (25) it follows that the upper and lower bounds on $A(+0)$ are given by

$$\frac{k(a)-Lk_1}{k(a)+Lk_1} \leqq A(+0) \leqq \frac{Lk(a)-k_1}{Lk(a)+k_1} \tag{26}$$

when $k(a)\geqq k_1$, and

$$\frac{k_1-Lk(a)}{k_1+Lk(a)} \leqq A(+0) \leqq \frac{Lk_1-k(a)}{Lk_1+k(a)}, \tag{27}$$

when $k(a)\leqq k_1$. To find from $A(+0)$ the quantity of physical interest, $|R(-0)|$, we must take into account any jump that k may have at the boundary $z=0$. Here, a most convenient way of doing this is to replace the $k(z)$ that has a jump at $z=0$ with a new $k(z)$ that has its jump smoothed out as shown in Fig. 4. We see that the new $k(z)$ is identical with the original $k(z)$ except at the jump at $z=0$; at this jump the new $k(z)$ is obtained from the original $k(z)$ by replacing the jump with a smooth connecting curve that closely resembles the jump and yet has a non-infinite slope.
Accordingly, from (26) and (27) we have

$$\frac{k(a)-Lk_1}{k(a)+Lk_1} \leqq |R(-0)| \leqq \frac{Lk(a)-k_1}{Lk(a)+k_1} \tag{28}$$

when $k(a)\geqq k_1$, and

$$\frac{k_1-Lk(a)}{k_1+Lk(a)} \leqq |R(-0)| \leqq \frac{Lk_1-k(a)}{Lk_1+k(a)}, \tag{29}$$

when $k(a)\leqq k_1$. Here L is given by (24) and $k(z)$ is the new $k(z)$.
If $k(z)$ happens to have an internal jump we can handle it by the same smoothing procedure. Consequently, the new $k(z)$ can be taken to be jump-free not only at $z=0$ but also throughout the slab.

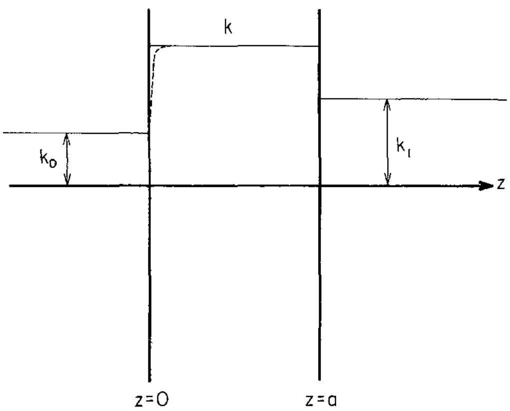

Fig. 5. Homogeneous slab of wave number k. Broken line smoothes out jump at $z=0$

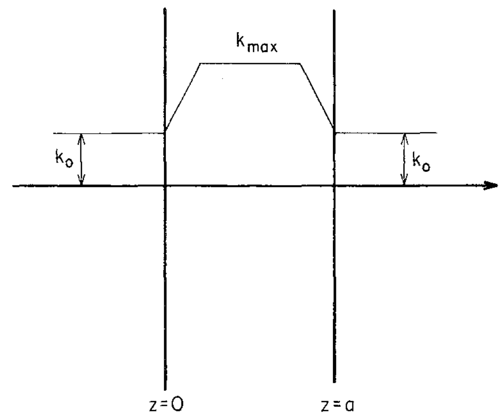

Fig. 7. Slab with trapezoidal distribution

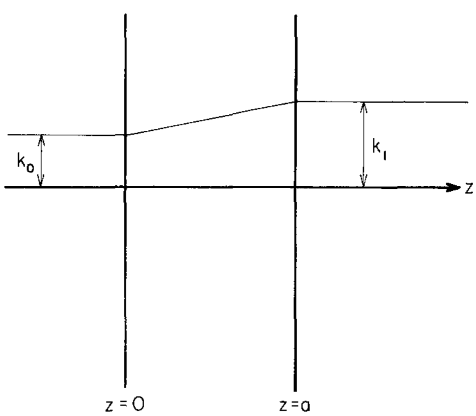

Fig. 6. Slab with ramp distribution

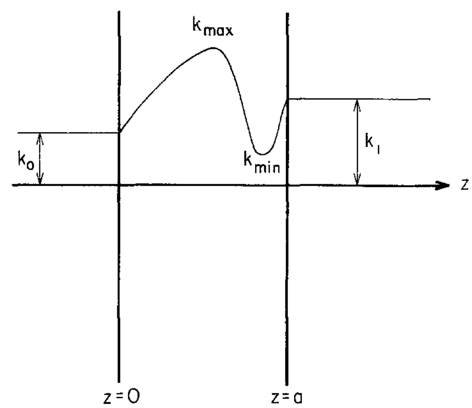

Fig. 8. Slab with serpentine distribution

Since the transmission coefficient T of (3) and the reflection coefficient R of (2) are related to each other by the rule

$$|R|^2 + \frac{k_1}{k_0}|T|^2 = 1\,, \tag{30}$$

which expresses the conservation of energy, we can find the upper and lower bounds of $|T|$ from the bounds on $|R(-0)|$ as given by (28) and (29).

4. The Measure L

From (28) and (29), we know that the upper and lower bounds on $|R(-0)|$ are determined by k_1, the wave number of the homogeneous medium to the right of the slab; by $k(a)$, the wave number of the slab at its right end; and by the measure L of the slab's inhomogeneity, as given by (24).

To show how L is geometrically related to $k(z)$ we examine the following special cases.

We first consider the case of a homogeneous slab whose wave number is k. As shown in Fig. 5, $k > k_1 > k_0$. Since there is a jump at $z=0$, we replace the jump with the broken line shown in the figure and thus obtain the new $k(z)$. By inspection of the figure and by using definition (24) we see that in this case

$$L = \frac{k}{k_0}\,. \tag{31}$$

Similarly, for the case of the ramp, shown in Fig. 6, we see that

$$L = \frac{k_1}{k_0}\,, \tag{32}$$

and for the case of the trapezoid, shown in Fig. 7, we have

$$L = \frac{k_{max}^2}{k_0^2}\,. \tag{33}$$

In Fig. 8 we see a situation where $k(z)$ has one maximum and one minimum. Here

$$L = \frac{k_{max}}{k_{min}}\frac{k_1}{k_0}\,. \tag{34}$$

We wish to emphasize that L can be found for any $k(z)$ by simply inspecting the geometric features of $k(z)$ and entering the pertinent data into (24).

It is clear that $L \geqq 1$. L is equal to 1 only for the trivial situation where $k(z)=k_0$; in all other situations L is greater than 1. It is also clear that if we keep $k(z)$ fixed at $z=0$ and $z=a$ and if we increase the number of wiggles of $k(z)$, then the corresponding L increases. For example, we note that L of Fig. 8 is greater than the L of Fig. 6.

From (24) we see that L depends on the maxima and minima of $k(z)$ and not on their positions along the z axis. This means that we can move the maxima and minima to new positions along the z-axis without causing a change in the value of L.

5. Illustrative Cases

As illustrative examples let us consider the cases shown in Figs. 5–8.

For each of these cases we have already calculated L. By substituting these values of L into (28) we find that

$$|R| \leqq \frac{k^2 - k_1 k_0}{k^2 + k_1 k_0} \tag{35}$$

for the homogeneous slab of Fig. 5,

$$|R| \leqq \frac{k_1 - k_0}{k_1 + k_0} \tag{36}$$

for the ramp distribution of Fig. 6,

$$|R| \leqq \frac{k_{\max}^2 - k_0^2}{k_{\max}^2 + k_0^2} \tag{37}$$

for the trapezoidal distribution of Fig. 7, and

$$|R| \leqq \frac{k_{\max} k_1 - k_{\min} k_0}{k_{\max} k_1 + k_{\min} k_0} \tag{38}$$

for the serpentine distribution of Fig. 8.

The result for the homogeneous slab agrees with what rigorous theory gives [3]. Likewise the results for the ramp and trapezoidal distributions agree with exact theory [10]. However, for the serpentine distribution there is nothing in the literature against which we can check our result.

6. Conclusions

We have investigated the problem of determining for radio waves the reflection coefficient of a plane inhomogeneous slab of lossless dielectric. Using Ambartsumian's principle of invariance as the point of departure, we have shown that the reflection coefficient satisfies a nonlinear equation of the Riccati type which may be solved numerically or may be used to obtain upper and lower bounds on the magnitude of the reflection coefficient. We have shown that such bounds can be established by inspection of the spatial distribution of the slab's wave number.

We have proved the following geometric theorem: The upper and lower bounds on the reflection and transmission coefficients of a lossless inhomogeneous dielectric slab are determined by the slab's measure of inhomogeneity L which depends only on the values of the maxima and minima of the dielectric slab's wave number and not on their relative positions within the slab.

Several cases which illustrate the geometric approach have been worked out and have been found to agree with known results.

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