BRANS–DICKE COSMOLOGY AND THE COSMOLOGICAL CONSTANT: THE SPECTRUM OF VACUUM SOLUTIONS

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Abstract. We study the Brans–Dicke vacuum field equations in the presence of a cosmological term Λ. Considering a Friedmann–Robertson–Walker metric with flat spatial sections ($k = 0$), we provide a qualitative analysis of the solutions and investigate its asymptotic properties. The general solution of the field equations for arbitrary values of w and Λ is obtained.

1. Introduction

Recently a renewed interest seems to exist in the so-called scalar-tensor theories of gravitation of which the Brans–Dicke theory (BDT) is notoriously the most investigated. Part of this interest may be attributed to the recognition of the important role these theories are able to play in the development of the contemporary models of the Universe, such as the *extended inflationary cosmology* program (La and Steinhardt, 1989). Another example of this interest comes from Supergravity via the mechanism of space-time dimensional compactification which generates in a rather natural way the Brans–Dicke scalar fields (Sherk, 1981).

In this paper we consider the Brans–Dicke theory of gravity (Brans and Dicke, 1961) with cosmological constant in the absence of matter. BDT solutions with a nonvanishing cosmological term have been already studied in different contexts (Uehara and Kim, 1982; Cerveró and Estévez, 1983; Lorenz-Petzold, 1984; Pimentel, 1984).

As we shall see in the next section, if we adopt the hypothesis that space is homogeneous and isotropic, then the field equations are reduced to a plane autonomous dynamical system. Therefore, it is possible to carry out a global analysis of the solutions without solving analytically the differential equations. Thus, our first task, before trying to get explicit solutions, will consist mainly of constructing the so-called phase diagrams of the system. Once obtained, the diagrams give us almost all informations about the dynamics of the models, the complete knowledge being provided by working out the general solution.

2. The Field Equations

The Brans–Dicke vacuum field equations with a nonvanishing cosmological term Λ are given by

$$R_{\mu\nu} = -2\Lambda[(w+1)/(2w+3)]g_{\mu\nu} + (w/\phi^2)\phi_{,\mu}\phi_{,\nu} + (1/\phi)\phi_{;\mu;\nu}, \tag{1a}$$

$$\Box\phi = 2\Lambda\phi/(2w+3), \tag{1b}$$

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where w is the scalar field coupling constant (see, for example, Uehara and Kim, 1982).

Considering a Friedmann–Robertson–Walker metric with flat spatial section ($k = 0$) in the form

$$ds^2 = dt^2 - R^2(t)\,[d\chi^2 + \chi^2\,(d\Theta^2 + \sin^2\Theta\, d\Phi^2)]\,,$$

the above equations reduces to

$$\dot{\theta} = -\theta^2/3 - (w+1)\psi^2 - \dot{\psi} + 2\Lambda(w+1)/(2w+3)\,, \tag{2a}$$

$$\dot{\theta} = -\theta^2 - \psi\theta + 6\Lambda(w+1)/(2w+3)\,, \tag{2b}$$

$$\dot{\psi} = -\psi^2 - \psi\theta + 2\Lambda/(2w+3)\,; \tag{2c}$$

where $\theta = 3\dot{R}/R$ describes the expansion of the model; $\psi = \dot{\phi}/\phi$, the overdot denoting time derivative; and ϕ, due to spatial homogeneity, is supposed to be a function of t only. Since in BDT the scalar field is identified to G^{-1}, then $\psi = -\dot{G}/G$ is actually a measure of the time variation of the Newtonian gravitational 'constant' G.

Now, these equations lead to an algebraic relation between the variables θ and ψ:

$$\theta^2/3 + \theta\psi - w\psi^2/2 = \Lambda\,, \tag{3}$$

which can be regarded as a constraint of the dynamical system formed by any chosen pair of the set of Equations (2). In this way, let us choose (2b) and (2c) as defining our planar autonomous dynamical system.

3. The Equilibrium Points

The curves which appear in the phase diagrams represent the parametric solutions $\theta = \theta(t)$, $\psi = \psi(t)$ evolving in time. Generally speaking, it may happen that the dynamical system contains *equilibrium points*, i.e., constant solutions $\theta = \theta_0$, $\psi = \psi_0$, which are the roots of the right-hand side of Equations (2b) and (2c). It turns out that the equilibrium points of the system (2b)–(2c) are given by

$$\theta_0 = 3(1+w)\psi_0\,, \tag{4a}$$

$$\psi_0 = \pm\sqrt{2\Lambda/[(2w+3)(3w+4)]}\,. \tag{4b}$$

In this paper let us assume from the outset that $\Lambda > 0$. We shall return to this point later on with brief comments on the cases $\Lambda = 0$ and $\Lambda < 0$. Thus, if $-\frac{3}{2} < w < -\frac{4}{3}$ we do not have equilibrium points.

One should note that θ_0 and ψ_0 also satisfy the constraint equation (3) and correspond, after straightforward integration, to de Sitter's type of solutions

$$R(t) = R_0 \exp[(1+w)\psi_0 t]\,, \tag{5a}$$

$$\phi(t) = \phi_0 \exp[\psi_0 t]\,. \tag{5b}$$

Incidentally, if $w \to \infty$, we see that ψ_0 and θ_0 tend to zero and $\pm\sqrt{3\Lambda}$, respectively;

and thus (5) becomes identical to de Sitter's solution of general relativity: i.e.,

$$R(t) = R_0 \exp(\pm\sqrt{\Lambda/3}\,t)\,,$$

$$\phi(\mathrm{t}) = \phi_0 = \text{const.}$$

The position of the equilibrium points in the phase diagram depends on the values of w and Λ. For fixed Λ Figures 1(a)–1(f) show how the equilibrium points, represented in the diagrams as A and B, move as w varies from $-\infty$ to $+\infty$.

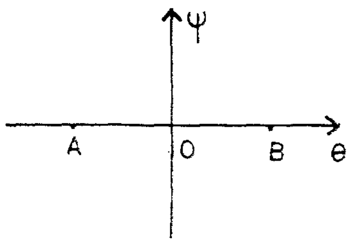

Fig. 1a. $w \to -\infty$.

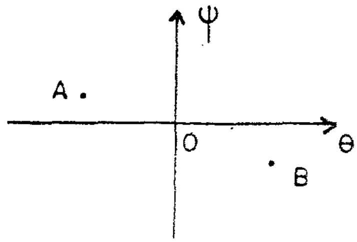

Fig. 1b. $w < -\frac{3}{2}$.

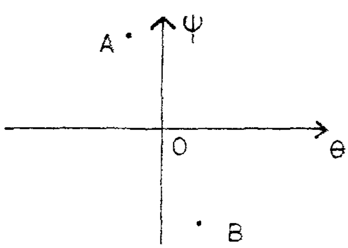

Fig. 1c. $-\frac{4}{3} < w < -1$.

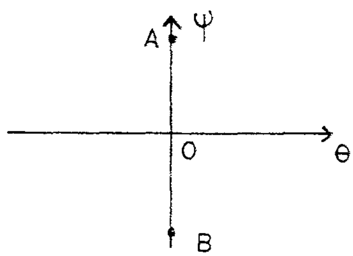

Fig. 1d. $w = -1$.

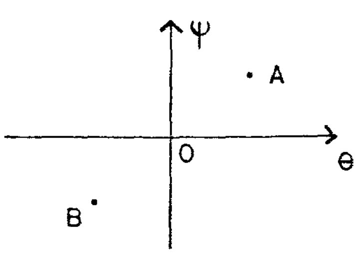

Fig. 1e. $w > -1$.

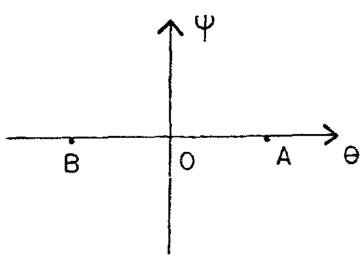

Fig. 1f. $w \to +\infty$.

At the special value $w = -1$ we have $\theta_0 = 0$ and $\psi_0 = \pm\sqrt{2\Lambda}$. Such configuration is immediately recognized as the Minkowski static solution except for the fact that the Newtonian gravitational constant now evolves in time. (We shall return to this point later.)

4. The Phase Diagrams

At this point we should mention that the first integral of the dynamical system defined by Equations (2b) and (2c) is given by

$$C[\psi - (\psi + \theta)/(3w+4)]^2 + 2\Lambda(\psi+\theta)^2/[(2w+3)(3w+4)] = (2\Lambda/(2w+3))^2 , \tag{6}$$

where C is an integration constant*. It is not difficult to see that (6) represents a conic in the plane phase. If we take $C = -2\Lambda(3w+4)/[3(2w+3)^2]$, we obtain the constraint equation (3), which is one of the integral curves of the system.

Thus, our procedure will consist basically of drawing the curve corresponding to the constraint equation (3) for arbitrary values of the parameters w and Λ. This curve contains six, four, or two distinct solutions of the field equations according to $w > -\frac{4}{3}$, $w < -\frac{3}{2}$, or $-\frac{3}{2} < w \leq -\frac{4}{3}$, respectively.

The essential features of the solutions of the Brans–Dicke field equations can be displayed with the aid of the next diagrams. For a fixed value of Λ, we have typically nine distinct diagrams corresponding to Figures 2(a)–2(i).

Let us make some comments on these diagrams. The first one (Figure 2(a)) refers to the configuration of the solutions in the asymptotic limit $w \to -\infty$. There appear to be three solutions: the two equilibrium points corresponding to de Sitter's solutions (constant rate expanding and contracting space-times) graphically represented by the points A and B, and another solution which 'links' them. This third solution starts from A and ends at B, undergoing initially a contracting era ($\theta < 0$), decelerates and begins to

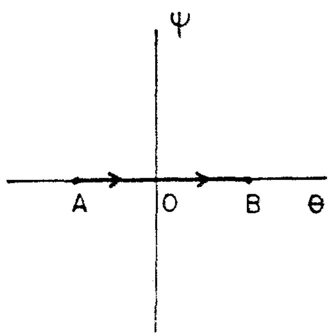

Fig. 2. $w \to -\infty$.

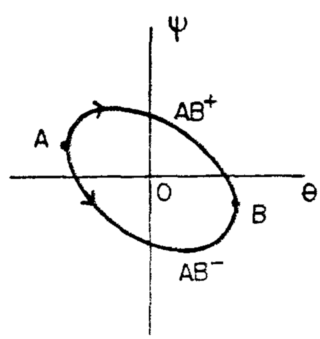

Fig. 2b. $w < -\frac{3}{2}$.

* Equation (6) is not defined for $w = -\frac{4}{3}$ and $w = -\frac{3}{2}$. The case $w = -\frac{3}{2}$ has a solution only if $\Lambda = 0$. On the other hand, $w = -\frac{4}{3}$ must be analysed separately.

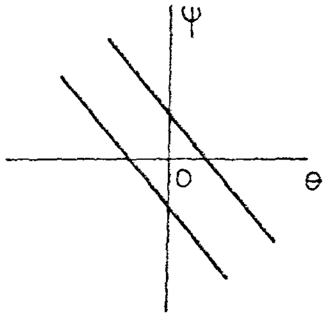

Fig. 2c. $w = -\frac{3}{2}$.

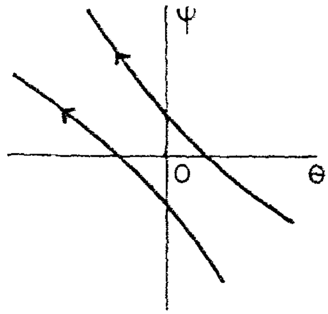

Fig. 2d. $-\frac{3}{2} < w \leq -\frac{4}{3}$.

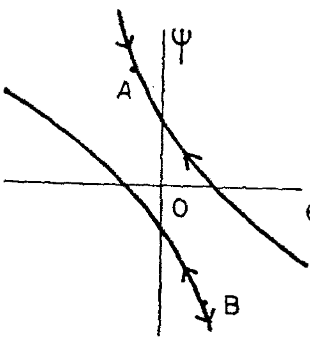

Fig. 2e. $-\frac{4}{3} < w < -1$.

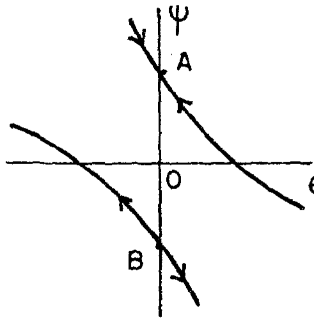

Fig. 2f. $w = -1$.

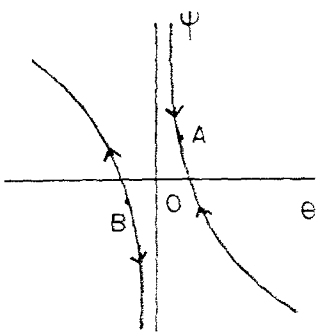

Fig. 2g. $w = 0$.

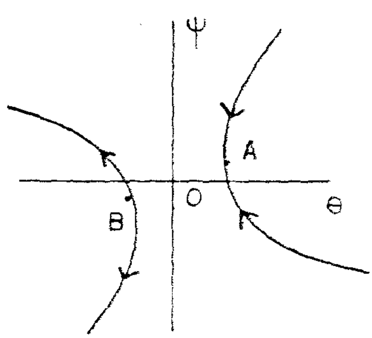

Fig. 2h. $w > 0$.

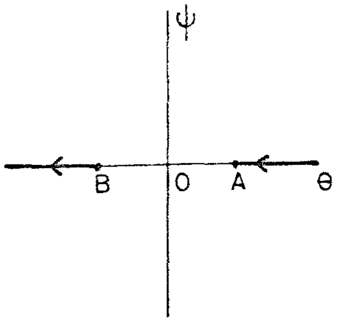

Fig. 2i. $w \to +\infty$.

expand ($\theta > 0$), the turning point corresponding to Minkowski's space-time. All three models share the property of constancy of G (since $\psi = 0$).

The second diagram (Figure 2(b)) is valid for $-\infty < w < -\frac{3}{2}$. The equilibrium points are no longer located at the θ-axis, which means that in the corresponding solutions G varies as time goes by. The model represented by point B describes a universe in which the gravitational strength increases as it expands, in contradiction to Dirac's hypothesis which claims that in an expanding universe G must decrease (Weinberg, 1972). As we shall see, only when $w > -1$ will there appear to be solutions not violating this hypothesis. On the other hand, point A represents a contracting universe in which Newton's gravitational constant decreases. Analogously to the former case, the points A and B are not isolated; rather they are connected by the two distinct solutions AB^+ and AB^- both undergoing different stages during its trajectories. In effect, starting from A in contraction regime they follow, however, quite different paths: AB^+, for instance, describes a model beginning in an ever-increasing contraction regime, reaching a maximum contraction point, after which it decelerates continuously until it begins to expand, approaching point B. As far as G is concerned, following the trajectory of AB^+ we observe that G decreases at the beginning, attains a maximum rate of decrease before entering a region of increasing G, when it violates Dirac's hypothesis mentioned above. AB^-, in turn, describes a space-time starting in a contraction era which gradually changes to an expansion era. It is interesting to note that, before the contraction era has ended, the initially decreasing gravitational constant attains its minimum value and begins to increase. In its final stage towards point B, this universe represented by AB^- experiences a maximum expansion corresponding in the figure to the right vertex of the ellipse. It is worthwhile to mention that all the solutions in the diagram of Figure 2(b) are non-singular and, as we shall see later, this property holds only for $w < -\frac{3}{2}$.

A glance at Equation (3) discloses that the constraint curve assumes the form of an ellipse in the $\theta\psi$-plane for $w < -\frac{3}{2}$. At the critical value $w = -\frac{3}{2}$ this ellipse suddenly breaks off into two parallel lines and the equilibrium points disappear (Figure 2(c)). When $w > -\frac{3}{2}$ the curve becomes an hyperbola, but the equilibrium points do not appear until w is greater than $-\frac{4}{3}$ (Figures 2(d) and 2(e)). Therefore, we conclude that at $w = -\frac{3}{2}$

and $w = -\frac{4}{3}$ the dynamical system undergoes drastic changes with respect to its topological properties.

For w lying in the small interval $-\frac{3}{2} < w \leq -\frac{4}{3}$ we have no equilibrium points and there are two singular solutions (see Figure 2(d)). Practically they behave the same way: both start from a Big Bang ($\theta = +\infty$), decelerate, and contract to a final collapse ($\theta = -\infty$). The gravitational constant starts out increasing, reaches a maximum value and begins to decrease.

For $w > -\frac{4}{3}$ the phase diagrams have the same topological properties in the sense that the constraint curve remains an hyperbola and the two equilibrium points are present. Indeed, the only effect of increasing the value of w is to cause a kind of non-rigid 'rotation' in the $\theta\psi$-plane of the entire pattern of solutions in the clockwise sense (Figures 2(e)–2(h)). In the following we shall discuss some special cases which, in our opinion, deserve some separate comments.

The case $w = -1$ is illustrated by Figure 2(f). Here, the equilibrium points are located on the ψ-axis and, as a consequence, the cosmological solutions represented by these points do not experience any kind of expansion or contraction in its geometry. In other words, we are in the presence of static Minkowskian space-times in which Newton's gravitational constant varies exponentially with time. As was firstly pointed out by O'Hanlon and Tupper (1972), vacuum solutions with a varying scalar field ϕ (or a varying G) contradict Mach's principle in the sense that this quantity is not generated by matter distribution. At this moment the question arises concerning the source of the dynamics of G in the specific case $w = -1$. Looking into Equation (3) we find that in the absence of a cosmological constant there is no possibility of existence of a varying G in a static model. Thus, we are led to a very peculiar situation in which the cosmic evolution of the gravitational constant is not determined by matter nor by geometry; rather it seems to be entirely caused by the presence of Λ. It should be mentioned that a possible connection between G and Λ in a cosmological context has been the object of recent speculation based on quite different reasonings (Pollock, 1984; Tomašek, 1985). On the other hand, from the diagram we see that the solution represented by the point A acts as an attractor of two singular solutions: one of them begins with a Big Bang and goes expanding indefinitely until the expansion vanishes, while the other starts contracting and undergoes an everlasting contracting stage before it becomes static. The model represented by the point B, in turn, is an unstable equilibrium point, and we have two singular solutions (again, one undergoes contraction while the other expands forever) which run away from B. One should add that up to now none of the solutions represented in the diagrams for $w \leq -1$ satisfy Dirac's hypothesis already mentioned.

The next case to be examined is $w = 0$ (Figure 2(g)). Two of the six solutions describe models which expand as G decreases continuously along their whole history. One of these is represented by the curve coming from $(0, +\infty)$ approaching A and the other one by the point A itself. The former experiences a transition from Minkowski's to de Sitter's geometry.

Figure 2(h) is to be representative of all the cases in which the coupling constant w assumes positive values. Basically, if one compares this diagram with the former ($w = 0$)

one could note a qualitative new fact concerning the Big-Bang nature now acquired by the expanding solution which tends to A. Clearly, this solution represents a Big-Bang model whose expansion tends to become stationary as $t \to +\infty$. Surely, of all other vacuum solutions considered up to now, this seems to be the one exhibiting the most desirable characteristics from the point of view of the present observational cosmology, besides the fact that it does not violate Dirac's hypothesis during its whole lifetime.

As the value of w increases the equilibrium points, A and B tend to be located on the θ-axis. In the limit case when $w \to +\infty$ the picture obtained is that of Figure 2(i). It happens that in this extreme case the two branches of the hyperbola corresponding to Equation (3) merge into two lines. These lines, in turn, now correspond to two solutions: one tending to point A and the other going away from point B. It is a well-known fact that in the limit $w \to \infty$ the Brans–Dicke theory becomes indistinguishable from general relativity. Therefore, it is not surprising that the solutions represented by the equilibrium points A and B go over de Sitter's vacuum solutions. Yet it is interesting to call attention to the remaining two solutions tending asymptotically to A and B. The exact form of these solutions as well as the general solution of the field equations (2) are discussed in the next section.

5. The General Solution

In this section we solve the field equations for an arbitrary value of w. Asymptotic solutions such as $w \to \pm\infty$ are also obtained.

To begin with, let us consider Equations (2b) and (2c) which constitute our dynamical system. If we define the new variable $\alpha = \theta + \psi$ and add (2b) to (2c) we get the equation

$$\dot{\alpha} + \alpha^2 - a = 0\,, \tag{7}$$

where $a = 2\Lambda(3w + 4)/(2w + 3)$. Two obvious particular solutions of this equation are

$$\alpha = \pm\sqrt{a}\,, \tag{8}$$

if $w < -\frac{3}{2}$ or $w > -\frac{4}{3}$.

Now, if we substitute Equation (8) into the constraint equation (3) gives for θ and ψ the constant values $\theta = 3(w + 1)\psi_0$ and $\psi_0 = \pm\sqrt{2\Lambda/[(2w + 3)(3w + 4)]}$ which are nothing other than the coordinates of the equilibrium points. Evidently, further integration leads to Equations (5).

To solve (7) we have to consider different intervals of variation of w, according to the sign of a:

(i) $w > -\frac{4}{3}$ or $w < -\frac{3}{2}$

For w lying on this interval $a > 0$ and the general solution of Equation (7) is given by

$$\alpha(t) = \sqrt{a}\,(1 + c\tanh\sqrt{a}\,t)\,(c + \tanh\sqrt{a}\,t)^{-1}\,, \tag{9}$$

where c is an integration constant. If we put $\psi = \alpha - \theta$ into the constraint equation we get the explicit dependence of θ on $\alpha(t)$:

$$\theta(t) = (3w+4)^{-1}\left(3(w+1)\alpha(t) \pm \sqrt{3(2w+3)\,[\alpha^2(t)-a]}\right); \tag{10a}$$

and $\psi(t)$ also gets determined from

$$\psi(t) = (3w+4)^{-1}\left(\alpha(t) \mp \sqrt{3(2w+3)\,[\alpha^2(t)-a]}\right). \tag{10b}$$

If we integrate these two last equations leads immediately to the final solutions for the scale factor $R(t)$ and the scalar field $\phi(t)$:

$$R(t) = R_0\{|(c+1)\exp(2\sqrt{a}\,t) + (c-1)|^{w+1}(|\exp(\sqrt{a}\,t) - \sqrt{(1-c)/(1+c)}| \times \times [\exp(\sqrt{a}\,t) + \sqrt{(1-c)/(1+c)}]^{-1})^{\pm\sqrt{(2w+3)/3}}\}^{1/(3w+4)} \times \times \exp[-(w+1)\sqrt{a}\,t\,(3w+4)^{-1}], \tag{11a}$$

$$\phi(t) = \phi_0\{|(c+1)\exp(2\sqrt{a}\,t) + (c-1)|\,(|\exp(\sqrt{a}\,t) - \sqrt{(1-c)/(1+c)}| \times \times [\exp(\sqrt{a}\,t) + \sqrt{(1-c)/(1+c)}]^{-1})^{\mp\sqrt{3(2w+3)}}\}^{1/(3w+4)} \times \times \exp(-\sqrt{a}\,t/(3w+4)), \tag{11b}$$

for $w > -\frac{4}{3}$ and $|c| < 1$;

$$R(t) = R_0\,|(c+1)\exp(2\sqrt{a}\,t) + (c-1)|^{(w+1)/(3w+4)}\exp(-(w+1)\sqrt{a}\,t/(3w+4) \pm \pm (2/3)\ (3w\ +\ 4)^{-1}\sqrt{3\,|2w+3|}\tan^{-1}(\exp(\sqrt{a}\,t)\sqrt{(c+1)/(c-1)})), \tag{12a}$$

$$\phi(t) = \phi_0\,|(c+1)\exp(2\sqrt{a}\,t) + (c-1)|^{1/(3w+4)}\exp(-\sqrt{a}\,t/(3w+4) \mp \mp 2(3w+4)^{-1}\sqrt{3\,|2w+3|}\tan^{-1}(\exp(\sqrt{a}\,t)\sqrt{(c+1)/(c-1)})), \tag{12b}$$

for $w < -\frac{3}{2}$ and $|c| > 1$, where R_0 and ϕ_0 are integration constants*.

(ii) $-\frac{3}{2} < w < -\frac{4}{3}$

For w assuming values in this interval $a < 0$ and the integration of Equation (7) results:

$$\alpha(t) = \sqrt{-a}\,(\cos\sqrt{-a}\,t - c\sin\sqrt{-a}\,t)\,(\sin\sqrt{-a}\,t + c\cos\sqrt{-a}\,t)^{-1}, \tag{13}$$

where, as before, c is an arbitrary integration constant. Equations (10) and (13) imply the following expressions for $R(t)$ and $\phi(t)$:

$$R(t) = R_0(|c\cos(\sqrt{-a}\,t) + \sin(\sqrt{-a}\,t)|^{(w+1)} \times \times |\tan((\sqrt{-a}\,t + \tan^{-1}c)/2)|^{\pm\sqrt{(2w+3)/3}})^{1/(3w+4)}, \tag{14a}$$

* One should be aware that Equations (11a) and (11b) represent four solutions according to the choice of the $\pm$ sign as well as the domain of validity of t is $t < t_0$ or $t > t_0$, where $t_0 = (1/\sqrt{a})\tanh^{-1}(-c)$. If $w < -\frac{3}{2}$, however, then $|c| > 1$ and $c + \tanh(\sqrt{a}\,t) \neq 0$ for all values of t, which implies that in this case the domain of validity of t is $-\infty < t < +\infty$. Thus, for $w < -\frac{3}{2}$ we have two solutions corresponding to the $\pm$ sign in Equations (12a) and (12b).

$$\phi(t) = \phi_0(|c\cos(\sqrt{-a}\,t) + \sin(\sqrt{-a}\,t)| \times \times |\tan((\sqrt{-a}\,t + \tan^{-1}c)/2)|^{\mp\sqrt{3(2w+3)}})^{1/(3w+4)} . \tag{14b}$$

The value $a = 0$ corresponds to setting $w = -\frac{4}{3}$ since $\Lambda \neq 0$. In this case $\alpha(t) = 1/(t + c)$ is the solution of Equation (7), where again c is an arbitrary constant. Then, a trivial calculations shows that*

$$\theta(t) = 2(t + c)^{-1} - 3\Lambda(t + c) , \tag{15a}$$

$$\psi(t) = -(t + c)^{-1} + 3\Lambda(t + c) , \tag{15b}$$

$$R(t) = R_0\,|(t + c)|^{2/3}\exp(-\Lambda(t^2/2 + ct)) , \tag{16a}$$

$$\phi(t) = \phi_0\,|(t + c)|^{-1}\exp(3\Lambda(t^2/2 + ct)) . \tag{16b}$$

Let us make some comments on the solutions showed above. Each of the solutions may be easily identified in the diagrams displayed in Section 4. Starting with the case for $w > -\frac{4}{3}$ we observe that we actually have six solutions (two of which correspond to the equilibrium points). Except for the equilibrium points all the solutions are singular, not being defined for the entire range $-\infty < t < +\infty$. Furthermore, if $|t|$ tends to infinity it is not difficult to see that Equations (11a) and (11b) goes over into Equations (5), which represent de Sitter-type solutions, just confirming what was concluded from the simple analysis of Figures 2(e)–2(h) carried out in Section 4. In the limit of general relativity, i.e., $w \to \infty$, the general solution (11) tends to

$$R(t) = R_0\exp(-\sqrt{\Lambda/3}\,t)\,|(c + 1)\exp(2\sqrt{3\Lambda}\,t) + (c - 1)|^{1/3} , \tag{17a}$$

$$\phi(t) = \phi_0 = \text{const.} \tag{17b}$$

It is worth mentioning that this solution clearly approaches de Sitter's solution of general relativity when $t \to \pm\infty$. Nevertheless, (17a) is not a vacuum solution of Einstein equations with cosmological constant, and we think this fact deserves a further comment.

Now let us turn our attention to the case $w < -\frac{3}{2}$. As was already pointed out from the analysis of the diagrams, this solution which holds for any value of $w < -\frac{3}{2}$ and positive Λ is singularity-free. In addition to the solutions corresponding to the equilibrium points A and B, there are two solutions which connect them. As can be directly seen from Equations (12a) and (12b), all the models describe eternal universes which eventually undergo stationary regimes of expansion or contraction, running backward or onward in time. In the limit $w \to \infty$, Equations (12a) and (12b) turns into Equations (17a) and (17b), but in this case it should be noted that there is no singularity.

* Again, it should be understood that Equations (16a) and (16b) define two distinct solutions according to $t < -c$ or $t > -c$.

In the narrow interval $-\frac{3}{2} < w < -\frac{4}{3}$ we have no equilibrium points (see Figure 2(d)). Thus the solution does not become close to a de Sitter-type solution at any time of its existence. Rather, it shows the peculiarity of developing two singularities: one in the past and the other in the future, these occurrences being separated by a finite period of time. This may be inferred immediately from the investigation of the domain of validity for the variable t in Equation (13). Clearly, one has to restrict the variation of the cosmological time to the finite range $t_0 < t < t_0 + \pi/\sqrt{-a}$, where $t_0 = (1/\sqrt{-a})\tan^{-1}(-c)$.

Finally, let us consider the special solution for $w = -\frac{4}{3}$. Analogous to the former case, the solutions given by Equations (16) do not tend to a de Sitter's geometry. The two curves appearing in Figure 2(d) correspond to two different domains of variation of t: $t < -c$ and $t > -c$.

6. Final Comments

In this paper we have focused the Brans–Dicke vacuum solutions with a positive cosmological constant Λ. Exactly the same mathematical methods apply to examine $\Lambda = 0$ or $\Lambda < 0$. The case $\Lambda = 0$ was completely solved by O'Hanlon and Tupper (1972) for $w \geq -\frac{3}{2}$. It can be easily verified that the latter may be directly obtained from Equations (3) and (7). Indeed, if we follow this procedure we arrive at the following equations for θ and ψ:

$$\theta(t) = (3w + 4)^{-1}\,[3(w + 1) \pm \sqrt{3(2w + 3)}]\,\alpha(t)\,, \tag{18a}$$

$$\psi(t) = (3w + 4)^{-1}\,(1 \mp \sqrt{3(2w + 3)})\,\alpha(t)\,, \tag{18b}$$

where $\alpha(t) = (t + c)^{-1}$ is the solution of Equation (7) with $a = 0$. Thus, after straightforward integration of Equations (18a) and (18b) we get

$$R(t) = R_0(t + c)^{(3w+4)^{-1}[w + 1 \pm \sqrt{(2w+3)/3}]}\,, \tag{19a}$$

$$\phi(t) = \phi_0(t + c)^{(3w+4)^{-1}[1 \mp \sqrt{3(2w+3)}]}\,; \tag{19b}$$

which is the solution found by O'Hanlon and Tupper. In connection with this, we should point out that the special de Sitter-like solution for $w = -\frac{4}{3}$, obtained by these authors, can also be recovered by just considering the constant solution of Equation (7) $\alpha = 0$ and then putting $\theta = -\psi$ in Equations (2b) and (2c) which implies $\theta = -\psi =$ constant. If $w < -\frac{3}{2}$ there are no solutions for $\Lambda = 0$, since the constraint equation (3) cannot be satisfied for any value of θ and ψ. The presence of the cosmological constant thus makes possible the existence of solutions for any value of w.

We have not considered $\Lambda < 0$ since this case seems somewhat artificial in the sense that the cosmological constant loses its character of a repulsive gravitational force, which was the original motivation of its inclusion into Einstein's equations. Nevertheless, the same analysis carried out for positive Λ may be repeated for $\Lambda < 0$. Some results can be quickly anticipated: for instance, there will be equilibrium points only for $-\frac{3}{2} < w < -\frac{4}{3}$. Also, the form of the solutions of Equation (7) remain the same, but with the obvious modification in the ranges of variation of w.

We would like to conclude this paper with a brief comment on the asymptotic solutions $w \to \infty$. Except for the solutions corresponding to the equilibrium points (which in this limit lie on the θ-axis), it can be readily verified that these solutions do not satisfy Einstein equations with non-vanishing cosmological constant in the absence of matter. Yet one can show that Equation (17a) does represent a solution of Einstein equations with cosmological constant and a matter distribution corresponding to a perfect fluid satisfying an equation of state $p = \rho$. However, in the limit $w \to -\infty$, ρ is negative, which is not permitted classically. Although the Brans–Dicke field equations reduce to Einstein equations in the limit $w \to \infty$, it may happen, as we have just shown, that the solutions obtained from the Brans–Dicke equations for finite w do not tend to the equivalent solutions of general relativity if we take this limit *a posteriori*. In the specific case we have analysed, everything seems to happen as if the Brans–Dicke solutions (after the limit $w \to \infty$ has been taken) just kept the scalar field 'in memory' through a curious mechanism of generating matter from the vacuum.

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