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Spin-Independent Transport Parameters for Superfluid ^{3}He-B

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The scalar kinetic equation for Bogoliubov quasiparticles in the B phase of superfluid ^{3}He is discussed and the collision integral is represented in a compact form. For the cases of shear and second viscosity and diffusive thermal conductivity the problem is reduced to solving one-dimensional integral equations. The quasiparticle interaction enters via weighted angular averages of the normal state scattering amplitude. The effect of strong coupling renormalization of the gap function is accounted for. The transport coefficients are exactly related to relaxation parameters that describe how the system tends toward local equilibrium. For low temperatures the transport parameters are evaluated exactly, including corrections of order T/T_c. *The results are compared with those of a previous paper in which an approximate form of the collision operator was used, as well as with results of a variational approach and with recent experimental data.*

1. INTRODUCTION

Dissipative parameters of superfluid ^{3}He-B have been investigated experimentally at temperatures down to $T > 0.3T_c$. On the theoretical side considerable effort has been spent in finding analytical results for the transport coefficients at the transition[1–4] and at low temperatures.[5–10] However, there is no systematic representation of transport theory. In particular, it is desirable to have a theory that shows how the known angular dependence of the transport problems under consideration can be separated off exactly, leaving one-dimensional integral equations in the energy variable along the lines of the treatment of Fermi liquid transport theory by Jensen *et al.*[12] and Sykes and Brooker.[13] These integral equations can be solved exactly in the normal phase, in the superfluid phase very close to T_c, and

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at low temperature. In particular, the zero-temperature limits of shear viscosity and thermal conductivity are known, due to the work of Pethick *et al.*[5] In this limit, however, for temperatures $T < 0.2T_c$, to be specific, it has been shown[6,14] that the mean free path of the thermal excitations of the system, the Bogoliubov quasiparticles, becomes comparable with a typical size of the sample container. Therefore, a bulk theory that only accounts for quasiparticle scattering as the dominant dissipative process is not applicable in the low-temperature limit. It is therefore desirable to have exact results for temperatures well above $0.2T_c$.

It turns out that corrections to the zero-temperature expressions for all relevant transport parameters in first order in T/T_c can be given exactly in terms of weighted angular averages of the normal state quasiparticle scattering amplitude. As will become clear later, the inclusion of finite-temperature corrections $\sim T/T_c$ to the various transport and relaxation parameters extends the region of applicability to temperatures up to, say, $T \approx 0.5T_c$.

The first calculation of finite-temperature corrections for shear viscosity and thermal conductivity by Pethick *et al.*[5] were restricted to the *s*-wave approximation for the quasiparticle scattering amplitude. More recently, Hara *et al.*[9] have calculated the T/T_c correction to the shear viscosity within the variational approach.

Most experimental data in the hydrodynamic regime are on the shear viscosity,[15–19] but there are also recent and partly preliminary data on the second viscosity and thermal conductivity[19,20] with which we can compare our theory. Finally, there are ultrasound attenuation data on the temperature-dependent width of the order parameter collective mode,[21] which provide an independent check of the theoretical prediction for the relaxation rate of Bogoliubov quasiparticles.

The purpose of this paper is threefold. Our first aim is to derive one-dimensional integral equations for all relevant spin-independent transport parameters from the scalar kinetic equation for Bogoliubov quasiparticles, as first been discussed in Refs. 6 and 7 (hereafter referred to as I). This requires in particular the derivation of the explicit form of the Boltzmann collision operator at arbitrary temperatures. A discussion of the exact properties of the collision operator clearly shows the equivalence of the dissipative parameters of shear and second viscosity and thermal conductivity with relaxation times describing how the associated currents tend toward their local equilibrium values.

Our second aim is to solve the integral equations analytically at low temperatures and give exact results for the finite-temperature corrections $\sim T/T_c$ for all the relevant spin-independent transport and relaxation times of the B phase. It should be pointed out that these results can be entirely written in terms of weighted angular averages of the normal state quasiparticle scattering amplitude ("scattering parameters") and the magnitude of the gap at zero temperature ("strong coupling parameter"). As these parameters cannot be calculated quantitatively from first principles yet (although considerable effort has been spent on this problem[22–26]), it is evident that our exact low-temperature results, when compared to experiment, provide new information about the scattering parameters and the zero-temperature gap.

Our third aim is to demonstrate that the simple approximate form of the transport parameters derived in I for arbitrary temperatures agrees well with the exact results at low temperature and is in addition consistent with the variational results. The results for the transport parameters presented in I are therefore seen to be due to a well-controlled approximation, and, because of their simple structure, are suited for the practical computation of dissipative parameters of superfluid ^{3}He-B at arbitrary temperature.

The paper is organized as follows: In Section 2 we derive transport equations for the coefficients of shear viscosity, second viscosity, and thermal conductivity from a scalar kinetic equation for Bogoliubov quasiparticles. Section 3 is devoted to a detailed analysis of the scalar Boltzmann collision operator and the derivation of one-dimensional integral equations with respect to the energy variable for each transport problem. In Section 4 we discuss exact properties of the collision integral, where the conservation and nonconservation properties are particularly pronounced. From the latter we derive expressions for the relaxation times for the normal fluid density and the normal component of momentum and energy current. In Section 5, the integral equations are solved exactly in the limit of low temperature, including finite-temperature corrections in first order in T/T_c. Finally the results are discussed and compared to other work in Section 6. Various approximations for the scattering parameters are considered and the influence of a possible strong coupling enhancement of the gap at zero temperature on the results for the transport coefficients is studied. The section concludes with a detailed comparison of our theory with recent experimental results on shear viscosity, second viscosity, and thermal conductivity.

2. SCALAR TRANSPORT EQUATIONS FOR BOGOLIUBOV QUASIPARTICLES

Consider a pair-correlated Fermi liquid such as superfluid ^{3}He subject to a (plane wave) external perturbation of frequency ω and wave vector $\mathbf{q}$. It has been shown in Section 1 that if ω and $v_F q$ (v_F is the Fermi velocity) are small compared to the gap frequency $\Delta/\hbar$ (so-called macroscopic limit), the condensate is in local equilibrium with the external perturbation. Dissipative processes are then associated with the dynamics of the thermal excitations of the system alone, the so-called Bogoliubov–Valatin quasiparticles (BQP). They are characterized by momentum $\hbar\mathbf{k}$, spin $\hbar\boldsymbol{\tau}/2$, energy $E_k=[\xi_k^2+|\Delta(T)|^2]^{1/2}$, and an equilibrium distribution $\nu_k^0=|\exp(E_k/k_BT)+1|^{-1}$. Here $\boldsymbol{\tau}=\{\tau^x,\tau^y,\tau^z\}$ is the vector of Pauli spin matrices, $\xi_k=\hbar^2k^2/2m^*-\mu$ is the normal quasiparticle energy measured relative to the chemical potential μ, and $\Delta(T)$ is the gap, which is isotropic in the Balian–Werthamer (BW) state, believed to represent superfluid ^{3}He-B.

The transport of mass, momentum, energy, and spin may be described in terms of a BQP density matrix $\nu_{k\sigma,\sigma'}(\mathbf{r},t)$, representing the state of the quasiparticle system subject to a space ($\mathbf{r}$) and time (t) dependent external driving force. In order to study the linear properties of the system, it is sufficient to restrict oneself to a plane wave external perturbation, $\sim\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$, of small amplitude. One may then consider the BQP distribution function $\delta\nu_{k\sigma\sigma'}(\mathbf{q}\omega)=\nu_{k\sigma\sigma'}(\mathbf{q}\omega)-\nu_k^0\delta_{\sigma\sigma'}$, linearized with respect to global equilibrium. $\delta\nu_{k\sigma\sigma'}$ obeys the following linearized scalar kinetic equation for BQP, derived in I:

$$\omega\,\delta\nu_{k\sigma\sigma'}(\mathbf{q}\omega)-\mathbf{q}\cdot\mathbf{v}_k(\delta\nu_{k\sigma\sigma}+\delta E_{k\sigma\sigma'})$$
$$=-i\left(\frac{\delta\nu'_{k\sigma\sigma'}}{\tau_k^q}-\sum_{p,\sigma''}B^q_{kp\sigma\sigma''}\frac{\delta\nu'_{\mathbf{p},\sigma''\sigma'}}{\varphi_p}\right) \tag{1}$$

In Eq. (1), $\delta E_{k\sigma\sigma'}$ denotes the change in the quasiparticle energy matrix induced by the perturbation, φ_k is the energy derivative of the Fermi function

$$\varphi_k=-\frac{\partial\nu_k^0}{\partial E_k}=\frac{1}{4k_BT}\operatorname{sech}^2\frac{E_k}{2k_BT}$$

and $\mathbf{v}_k=\nabla_kE_k=(\xi_k/E_k)\hbar\mathbf{k}/m^*$ is the group velocity of BQP. The rhs of Eq. (1) is the linearized scalar collision integral, from which the (out-scattering) term containing the relaxation rate $\tau_q(E_k)$ of BQP has explicitly been extracted. The second (in-scattering) term is governed by a scalar collision operator B^q_{kp} acting on the deviation of the BQP distribution from "local" equilibrium, $\delta\nu'_k=\delta\nu_k-\delta\nu_k^{\rm loc}$, where $\delta\nu_k^{\rm loc}=-\varphi_k\,\delta E_k$. The explicit form of τ_k^q and B^q_{kp} will be specified later.

We turn now to the superfluid component of ^{3}He-B, which, in the macroscopic limit may be described by only a few phase variables associated with the spontaneously broken symmetries of the condensed state: the global phase variable ϕ (gauge symmetry) and the rotation axis $\hat{\mathbf{n}}$ and angle θ describing the relative orientation of the spin coordinates with respect to the orbital ones (relative spin–orbit symmetry). In equilibrium the condensate is then characterized by an off-diagonal mean field (unitary gap matrix):

$$\Delta^0_{k\sigma\sigma'} = \Delta(T)\hat{\mathbf{d}}_0(\hat{\mathbf{k}}) \cdot \boldsymbol{\sigma}_{\sigma\sigma'}; \qquad \boldsymbol{\sigma} = i\tau^i\tau^2 \tag{2}$$

where $\Delta(T)$ is the magnitude of the gap and $\hat{\mathbf{d}}^0(\hat{\mathbf{k}})$ is a unit vector in spin space rotated from $\hat{\mathbf{k}}$ by an angle $\theta_0 = \cos^{-1}(-1/4)$ ("Leggett angle") about some arbitrary axis $\hat{\mathbf{n}}$:

$$\hat{\mathbf{d}}_0(\hat{\mathbf{k}}) = \mathbf{R}(\hat{\mathbf{n}}, \theta_0)\hat{\mathbf{k}}\, e^{i\phi_0} \tag{2a}$$

with $\mathbf{R}$ an (orthogonal) rotation matrix. In the spin-independent case the linear response of the superfluid component to the perturbation can be written as

$$\delta\Delta_{k\sigma\sigma'}(\mathbf{q}\omega) = \delta\mathbf{d}(\mathbf{q}\omega) \cdot \boldsymbol{\sigma}_{\sigma\sigma'} \tag{3}$$

where

$$\delta\mathbf{d}(\mathbf{q}\omega) = \Delta(T)\hat{\mathbf{d}}_0(\hat{\mathbf{k}}) \cdot 2i\delta\phi(\mathbf{q}\omega) \tag{3a}$$

with $\delta\phi(\mathbf{q}\omega) = \phi(\mathbf{q}\omega) - \phi_0$ (the factor of 2 in the definition of the phase variable $\delta\phi$ is a matter of convention).

In the following, we restrict ourselves to the spin-independent case, leaving the evaluation of spin-transport parameters to separate papers.[10,27]

In order to identify and calculate the mass transport parameters of ^{3}He-B one starts from the linearized phenomenological two-fluid hydrodynamic equations,[28,29] which can of course be derived from an entirely microscopic point of view.[30,31] One has the conservation laws of mass

$$\omega\,\delta\rho = \mathbf{q} \cdot \mathbf{g} = \mathbf{q} \cdot (\rho_n \mathbf{v}^n + \rho_s \mathbf{v}^s) \tag{4a}$$

momentum

$$\omega \mathbf{g} = \mathbf{q}(\boldsymbol{\pi}^{\text{loc}} + \boldsymbol{\pi}') \tag{4b}$$

and energy

$$\omega\,\delta\varepsilon = \mathbf{q}(\mathbf{j}_\varepsilon^{\text{loc}} + \mathbf{j}_\varepsilon') \tag{4c}$$

and a quasiconservation law for the additional hydrodynamic variable $\delta\phi$ representing the broken gauge symmetry. From $\delta\phi$ one defines the superfluid velocity

$$\mathbf{v}^s = \frac{\hbar}{m} i\mathbf{q}\,\delta\phi \tag{4d}$$

which obeys the acceleration equation

$$\omega \mathbf{v}^{\mathrm{s}} = \frac{1}{m}\mathbf{q}(\delta\mu^{\mathrm{loc}} + \delta\mu') = \frac{1}{m}\mathbf{q}\delta\mu \tag{4e}$$

where $\delta\mu := i\hbar\omega\,\delta\phi$ is the associated shift in the chemical potential (of Cooper pairs). The dissipative responses to gradients in a normal velocity field $\mathbf{v}^{\mathrm{n}}(\mathbf{r}t) = v^{\mathrm{n}} \exp[\mathrm{i}(\mathbf{qr} - \omega t)]$

$$\Pi'_{ij} = -i\eta(q_j v_i^{\mathrm{n}} + q_i v_j^{\mathrm{n}} - \tfrac{2}{3}\delta_{ij}\mathbf{q}\mathbf{v}^{\mathrm{n}}) \tag{5a}$$

a temperature field $T(\mathbf{r}t) = \delta T \exp[i(\mathbf{qr} - \omega t)]$

$$\mathbf{j}'_\varepsilon = -i\kappa\mathbf{q}\,\delta T \tag{5b}$$

and normal to superfluid counterflow

$$\frac{1}{m}\delta\mu' = -i\zeta_3\mathbf{q}\rho_{\mathrm{s}}(\mathbf{v}^{\mathrm{s}} - \mathbf{v}^{\mathrm{n}}) \tag{5c}$$

are characterized by the coefficients of shear viscosity η, diffusive thermal conductivity κ, and second viscosity ζ_3; here ρ_{n} and ρ_{s} are the mass densities of normal and superfluid fraction, and $\rho = \rho_{\mathrm{n}} + \rho_{\mathrm{s}}$ is the total mass density. In writing down Eqs. (5a)–(5c) we have dropped terms containing the second viscosities $\zeta_1 = \zeta_4$, ζ_2, since these dissipative parameters can be shown to be at least of order $(T_c/T_{\mathrm{F}})^2$ due to the small possible deviations from particle–hole symmetry in a superfluid Fermi liquid. Our first aim now is to derive constituent relations between the dissipative parts of the currents and the external fields, allowing for a classification of the spin-independent transport parameters in terms of certain moments of the microscopic scalar collision integral. The external fields may, in our case, be considered to consist of (1) a temperature field $T(\mathbf{r}, t)$ coupling to the energy (or entropy) current, and (2) a velocity field $v^{\mathrm{n}}(\mathbf{r}, t)$ coupling to the momentum current.

In this case it is readily verified that the local equilibrium distribution of BQP is of the form[6]

$$\delta\nu_k^{\mathrm{loc}} = -\varphi_k\,\delta E_k + \varphi_k\left(\mathbf{p}\cdot\mathbf{v}^{\mathrm{n}} + \frac{E_k}{T}\delta T\right) \tag{6}$$

where δE_k is the change in the quasiparticle energy. Now in order to establish the relevant spin-independent transport equations, we start from Eq. (1) and perform an expansion to lowest order in the gradients and in the deviation of the normal component from local equilibrium $\delta\nu' \sim \omega\tau_k^q$ according to the Chapman–Enskog procedure. Doing this, we may replace $\delta\nu_k$ on the lhs of Eq. (1) by its local equilibrium form (6) and obtain the following transport equations:

$$\frac{1}{\tau_k^q}\delta\nu_k' - \sum_{\mathbf{p}} B_{kp}^q \frac{\delta\nu_p'}{\varphi_p} = i\varphi_k \frac{\xi_k}{E_k}\left[\frac{1}{mN_{\mathrm{F}}}\mathbf{q}\cdot(\mathbf{g}-\rho\mathbf{v}^{\mathrm{n}}) - \mathbf{q}\cdot\mathbf{v}_k\frac{E_k}{T}\delta T - \frac{1}{2}p_i v_{kj}\left(q_j v_i^n + q_i v_j^n - \frac{2}{3}\delta_{ij}\mathbf{q}\mathbf{v}^{\mathrm{n}}\right)\right] = -\delta I_k \tag{7}$$

Finally, the dissipative parts of the various currents in Eqs. (5a)–(5c) can be expressed as averages over the deviation of the BQP distribution from local equilibrium in the following form:

$$\Pi_{ij}' = 2\sum_k \hbar k_i v_{kj}\frac{\xi_k}{E_k}\delta\nu_k'$$

$$\mathbf{j}_\varepsilon' = 2\sum_k E_k \mathbf{v}_k \frac{\xi_k}{E_k}\delta\nu_k'$$

$$\delta\mu' = -\frac{1}{N_{\mathrm{F}}}2\sum_k \frac{\xi_k}{E_k}\delta\nu_k' \tag{8}$$

In order to have expressions for the transport parameters η, κ, and ζ_3 we separate off the angular dependence of the distribution function $\delta\nu_k$ in Eq. (7) by making the following ansatz:

$$\frac{\delta\nu_k'}{\varphi_k} = i\frac{\xi_k}{E_k}\frac{\tau_{\mathrm{N}}^0}{mN_{\mathrm{F}}}\phi_\zeta(E_k)\mathbf{q}\cdot(\mathbf{g}-\rho\mathbf{v}^{\mathrm{n}}) - i\mathbf{q}\cdot\mathbf{v}_k\tau_{\mathrm{N}}^0\frac{E_k}{T}\phi_\kappa(E_k)\,\delta T - i\frac{1}{2}\frac{\xi_k}{E_k}p_i v_{kj}\tau_{\mathrm{N}}^0\phi_\eta(E_k)(q_j v_i^{\mathrm{n}} + q_i v_j^{\mathrm{n}} - \tfrac{2}{3}\delta_{ij}\mathbf{q}\cdot\mathbf{v}^{\mathrm{n}}) \tag{9}$$

Here $\phi_\zeta(E_k)$, $\phi_\kappa(E_k)$, and $\phi_\eta(E_k)$ are dimensionless functions of the energy variable E_k only (and of course of temperature and pressure) and completely characterize the transport parameter under consideration. This is seen by inserting (9) into the expressions (8) for the currents, performing the angular integrations, and defining weighted energy averages according to

$$\langle\cdots\rangle_\varphi := \int_{-\infty}^{\infty} d\xi_k\varphi_k\cdots = \frac{1}{4kT}\int_{-\infty}^{\infty} d\xi_k\,\mathrm{sech}^2\frac{E_k}{2kT}\cdots \tag{10}$$

Thus the problem of calculating the shear viscosity

$$\eta = (1/15)v_{\mathrm{F}}^2 p_{\mathrm{F}}^2 N_{\mathrm{F}}\langle(\xi^2/E^2)\phi_\eta(E)\tau_{\mathrm{N}}^0\rangle_\varphi \tag{11a}$$

the second viscosity

$$\zeta_3 = (1/m^2 N_F)\langle(\xi^2/E^2)\phi_\zeta(E)\tau_N^0\rangle_\varphi \tag{11b}$$

and the diffusive thermal conductivity

$$\kappa = (n/m^* T)\langle \xi^2 \phi_\kappa(E)\tau_N^0\rangle_\varphi \tag{11c}$$

consists in deriving and solving scalar integral equations for the dimensionless functions ϕ_η, ϕ_ζ and ϕ_κ. This is provided by a detailed analysis of the collision integral appearing as the rhs of Eq. (1), which will be the topic of the next section.

3. SCALAR COLLISION INTEGRAL

We start our analysis of the collision integral on the rhs of the kinetic equation for Bogoliubov quasiparticles

$$\delta I_k\{\delta\nu'_k\} := -\frac{\delta\nu'_k}{\tau_k^q} + \sum_p B_{kp}^q \frac{\delta\nu'_p}{\varphi_p} \tag{12}$$

with a brief discussion of the structure of the Bogoliubov quasiparticle relaxation rate $1/\tau_k^q$, which has already been given in I. Essentially $1/\tau_k^q$ is given by an expression similar to that of normal (Landau) quasiparticles:

$$\frac{1}{\tau_k^q} = \frac{(\pi k T)^2}{8N_F^2} \sum_{\substack{\mathbf{k}_2,\ldots,\mathbf{k}_4 \\ \mu_2,\ldots,\mu_4}} (2\pi)^3 \delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4) W_s^{\text{out}}(12;34) f(1234) \tag{13}$$

Here $W_s^{\text{out}}(12;34)$ is the superfluid transition rate, which can entirely be expressed in terms of the dimensionless normal state (singlet and triplet) scattering amplitude $A_{0,1}(1234)$,

$$W_s^{\text{out}}(12;34) = W - W_I\left(\frac{\mu_3}{E_1E_3} + \frac{\mu_2\mu_4}{E_2E_4}\right)\Delta^2 + W_D\frac{\mu_2\mu_3\mu_4}{E_1E_2E_3E_4}\Delta^4 \tag{14}$$

Here

$$W = \tfrac{1}{4}\pi(A_0^2 + 3A_1^2) \tag{15a}$$

is the normal state transition rate, the angular average of which determines the normal state quasiparticle lifetime. W_I and W_D are new transition rates characterizing quasiparticle scattering in the presence of the triplet condensate,

$$W_I = \tfrac{1}{4}\pi[(A_0 + A_1)(B_0 + B_1) + 4A_1B_1]c_3 \tag{15b}$$

$$W_D = -W_2c_2^2 + W_3c_3^2 + W_4c_4^2 \tag{15c}$$

Here

$$B_{0,1}(1234) = A_{0,1}(\mathbf{k}_1, -\mathbf{k}_4, \mathbf{k}_3, -\mathbf{k}_2)$$
$$W_2 = \tfrac{1}{4}\pi(A_1^2 - A_0^2)$$
$$W_3 = \tfrac{1}{2}\pi(A_1 + A_0)A_1$$
$$W_4 = \tfrac{1}{2}\pi(A_1 - A_0)A_1$$

and

$$c_j = \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_j; \qquad j = 2, 3, 4$$

The function $f(1234)$ contains the energy δ-function and the usual fermion occupation factors,

$$f(1234) = \frac{2\pi}{\varphi_k}\delta(E_1 + \mu_2 E_2 - \mu_3 E_3 - \mu_4 E_4)\frac{1}{(2\pi kT)^3}$$
$$\times f(E_1) f(\mu_2 E_2)[1 - f(\mu_3 E_3)][1 - f(\mu_4 E_4)] \qquad (16)$$

In Eq. (14) the "particle–hole" variables $\mu_i = \pm 1$, $i = 2, 3, 4$, reflect the pole structure ($\omega = \pm E_k/\hbar$) of the single-particle spectral function.

As the momenta $\mathbf{k}_1, \ldots, \mathbf{k}_4$ of scattering quasiparticles are essentially restricted to the Fermi surface, it is convenient to write the scattering amplitudes in terms of Abrikosov angles θ and ϕ, which may be defined as

$$\cos\theta = c_2 = \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2, \qquad \cos\phi = \frac{c_3 - c_4}{1 - c_2}$$

In terms of these variables, the summation on momenta $\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$ can be shown to be convertible into an integration over energy variables ξ_2, ξ_3, ξ_4, making use of the momentum δ-function:

$$\sum_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \cdots = \frac{N_F^2}{32 E_F}\int_{-\infty}^{\infty} d\xi_2\, d\xi_3\, d\xi_4 \int_0^{2\pi} \frac{d\Phi_2}{2\pi}\langle\cdots\rangle_a$$

where Φ_2 is the azimuthal angle in the local coordinate system with $\hat{\mathbf{k}}_1$ as the z axis, and the brackets $\langle\cdots\rangle_a$ denote the following angular average with respect to the variables θ and ϕ:

$$\langle\cdots\rangle_a = \frac{1}{2}\int_0^{\pi} \frac{d\theta \sin\theta}{\cos(\theta/2)} \int_0^{2\pi} \frac{d\phi}{2\pi} \cdots = 2\int_0^1 d\cos\frac{\theta}{2} \int_0^{2\pi} \frac{d\phi}{2\pi} \cdots$$

If we finally extract the normal state quasiparticle lifetime at the Fermi surface

$$\tau_N^0(\mathrm{T}) = \frac{32 E_F \hbar}{(\pi kT)^2 \langle W\rangle_a} \qquad (17)$$

from the expression for τ_k^q, the Bogoliubov quasiparticle collision rate may be written in the compact form

$$\frac{1}{\tau_k^q}=\frac{1}{\tau_N^0}\{I_0(E_k)-\gamma_0[I_1(E_k)+I_2(E_k)]+\delta_0 I_3(E_k)\} \tag{18}$$

with dimensionless isotropic relaxation rates

$$\begin{gathered}I_n(E_k)=\sum_{\mu_2\cdots\mu_4}\int_0^\infty d\xi_2\,d\xi_3\,d\xi_4\,f(1234)k_n(1234)\\ k_n(1234)=\delta_{n,0}+\left(\frac{\mu_3}{E_1E_3}\delta_{n,1}+\frac{\mu_2\mu_4}{E_2E_4}\delta_{n,2}\right)\Delta^2+\frac{\mu_2\mu_3\mu_4}{E_1E_2E_3E_4}\Delta^4\delta_{n,3}\end{gathered} \tag{18a}$$

Thus, in the superfluid state, the angular dependence of quasiparticle scattering cannot entirely be separated from the energy and temperature dependence. Hence the new superfluid scattering parameters γ_0 and δ_0 in Eq. (18) are defined as ratios of angular averages of transition rates W, W_I, and W_D:

$$\gamma_0=\langle W_I\rangle_a/\langle W\rangle_a,\qquad \delta_0=\langle W_D\rangle_a/\langle W\rangle_a \tag{18b}$$

We turn now to the second (in-scattering) part of the collision integral [Eq. (12)], which in the spin-independent case can be written in the compact form

$$\sum_p B_{kp}^q\frac{\delta\nu_p'}{\varphi_p}=\frac{\varphi_k}{\tau_N^0}\sum_{\mu_2\ldots\mu_4}\int_0^\infty d\xi_2\,d\xi_3\,d\xi_4\,f(1234)\times\int_0^{2\pi}\frac{d\phi_2}{2\pi}\sum_{j=2}^{4}{}'\,p_{\mu_j}^s\frac{W_s^{\text{in}}(1234)\dfrac{\delta\nu_{\mathbf{k}j}^{(s)\prime}(\xi_j)}{\varphi_{kj}}}{\langle W\rangle_a} \tag{19}$$

Here

$$\sum_{j=2}^{4}{}'\,A_j\equiv -A_2+A_3+A_4$$

s denotes the parity of the distribution function $\delta\nu_k'$ with respect to the operation $\mathbf{k}\to-\mathbf{k}$:

$$\delta\nu_{\mathbf{k}}^{(s)\prime}=\tfrac{1}{2}(\delta\nu_{\mathbf{k}}'+s\,\delta\nu'_{(-\mathbf{k})}) \tag{20}$$

and the projector p_μ^s is defined by

$$p_\mu^s=\frac{1+\mu}{2}-s\frac{1-\mu}{2}$$

In order to give an explicit expression for the transition rate $W^{\text{in}}(1234)$ one has to distinguish the cases in which the distribution function is even or odd with respect to the replacement of the energy variable $\xi_k \to -\xi_k$. Introducing the parity t by

$$\delta\nu_{\mathbf{k}}^{(t)} = \tfrac{1}{2}[\delta\nu_{\mathbf{k}}(\xi_k) + t\,\delta\nu_{\mathbf{k}}(-\xi_k)] \tag{21}$$

one obtains for the two cases $t = \pm 1$

$$\begin{aligned} \sum_{j=2}^{4}{}' \, p_{\mu_j}^{s} W_s^{\text{in}}(1234) \frac{\delta\nu'_{\mathbf{k}j}(\xi_j)}{\varphi_{kj}} &= W_s^{\text{out}}(1234) \sum_{j=2}^{4}{}' \, p_{\mu_j}^{s} \frac{\delta\nu'_{\mathbf{k}j}(\xi_j)}{\varphi_{kj}} \frac{1+t}{2} \\ &+ \frac{\xi_1}{E_1}\left[\sum_{j=2}^{4}{}' \, p_{\mu_j}^{-s} \frac{\xi_j}{E_j} \frac{\delta\nu'_{\mathbf{k}j}(\xi_j)}{\varphi_{kj}} W - k_2 p_{\mu_3}^{s} \frac{\xi_3}{E_3} \frac{\delta\nu'_{\mathbf{k}3}(\xi_3)}{\varphi_{k3}} W_{\text{I}}\right] \frac{1-t}{2} \end{aligned} \tag{22}$$

where the quantities W_s^{out} and k_n have been defined by Eqs. (14) and (18a), respectively.

In the backscattering integral we thus have to classify with respect to the parities s and t. As it turns out, the case $t = +1$ is important for the discussion of the conservation properties of the collision integral, whereas an inspection of the ansatz (9) for the distribution function, describing the transport problems of interest, shows that $t = -1$ in all these cases (viscosities: $s = +1$; thermal conductivity: $s = -1$). In order to be able to introduce integral operators one now has to interchange energy variables in Eq. (19) using the symmetry of the function $f(1234)$ such that the distribution function depends on ξ_2 only. The angular dependence of $\delta\nu'_k$ is simplified by first expanding in spherical harmonics and then performing the ϕ_2 integration, with the result:

$$\int_0^{2\pi} \frac{d\phi_2}{2\pi} \delta\nu'_{kj}(\xi_2) = \sum_l (2l+1) \int \frac{d\Omega_p}{4\pi} P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}})\, \delta\nu'\hat{\mathbf{p}}(\xi_2) P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}_j) \tag{23}$$

One is then left with the following set of new angular averages (with respect to θ and ϕ) of the scattering cross sections W, W_{I}, and W_{D}:

$$\begin{aligned} \lambda_l^s &:= \frac{1}{\langle W \rangle_{\text{a}}} \langle W[-sP_l(c_2) + P_l(c_3) + P_l(c_4)] \rangle_{\text{a}} \\ \gamma_l &:= \frac{1}{\langle W \rangle_{\text{a}}} \langle W_{\text{I}} P_l(c_3) \rangle_{\text{a}} \\ \alpha_l^s &:= \frac{1}{\langle W \rangle_{\text{a}}} \langle W_{\text{I}}[-sP_l(c_2) + P_l(c_4)] \rangle_{\text{a}} \\ \delta_l^s &:= \frac{1}{\langle W \rangle_{\text{a}}} \langle W_{\text{D}}[-sP_l(c_2) + P_l(c_3) + P_l(c_4)] \rangle_{\text{a}} \end{aligned} \tag{24}$$

Now we introduce integral kernels by first realizing that the collision integral does not depend on the parities s (momentum) and t (energy) separately but only on the product

$$r = s \cdot t$$

Thus we introduce

$$P_n^{(r)}(\xi_1, \xi_2) = \sum_{\mu_2} (rp_{\mu_2}^{-r}) \sum_{\mu_3\mu_4} \int_0^\infty d\xi_3\, d\xi_4\, f(1234) k_n(1234) \tag{25}$$

with k_n from Eq. (18). With the aid of the integral kernels (25) we can define linear integral operators via

$$(\mathcal{P}_n^{(r)}, \Phi)(\xi_1) = \int_0^\infty d\xi_2\, P_n^{(r)}(\xi_1, \xi_2)\Phi(\xi_2) \tag{26}$$

with an appropriately chosen function of energy $\Phi(\xi)$. Inspection of Eq. (18) together with (22) and the manipulations described above show that only the following two linear combinations of integral operators $\mathcal{P}_n^{(r)}$ completely characterize the backscattering term of the scalar collision integral:

$$\begin{aligned} R_n^{(r)} &= \lambda_l^r \mathcal{P}_0^{(r)} - \alpha_l^{(r)}[\mathcal{P}_1^{(r)} + \mathcal{P}_2^{(r)}] - \gamma_l[\mathcal{P}_1^{(r)'} + \mathcal{P}_2^{(r)'}] + \delta_l^{(r)} \mathcal{P}_3^{(r)} \\ S_l^{(r)} &= \lambda_l^{(r)} \mathcal{P}_0^{(r)} - \gamma_l \mathcal{P}_2^{(r)'} \end{aligned} \tag{27}$$

where the primed integral operators $\mathcal{P}_n'$ are trivially generated from $\mathcal{P}_n$ by

$$\mathcal{P}_n' = \mathcal{P}_n\{k_n(1234) \to k_n(E_1, -\mu_3 E_3, -\mu_2 E_2, \mu_4 E_4)\}$$

which is due to the exchange of variables mentioned above.

The final result for the backscattering integral now has the compact form

$$\begin{aligned} \sum_p B_{kp}^q \frac{\delta\nu_p'}{\varphi_p} \\ = \frac{\varphi_k}{\tau_N^0} \sum_l (2l+1) \int \frac{d\Omega_p}{4\pi} P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) \left(R_l^{(-r)}, \frac{\delta\nu_p'^{(r)}}{\varphi_p} \right); \qquad t = +1 \end{aligned} \tag{28a}$$

if $\delta\nu_p'$ is even in the energy variable ξ_p, and

$$= \frac{\xi_1}{E_1} \frac{\varphi_k}{\tau_N^0} \sum_l (2l+1) \int \frac{d\Omega_p}{4\pi} P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) \left(S_l^{(-r)}, \frac{\xi_p}{E_p} \frac{\delta\nu_p'^{(r)}}{\varphi_p} \right); \qquad t = -1 \tag{28b}$$

if $\delta\nu_p'$ is odd with respect to $\xi_p \to -\xi_p$.

We can now go back to the transport equation (7) of the preceding section. Together with the ansatz (9) for the distribution function and the explicit form of the collision integral (28b), one obtains, after having performed the angular integrations, the following set of integral equations for the dimensionless functions ϕ_η (shear viscosity)

$$\tau_N^0\phi_\eta(E)=\tau_q(E)\left[1+\left(S_2^{(+)},\frac{\xi^2}{E^2}\phi_\eta\right)(E)\right] \tag{29a}$$

for ϕ_ζ (second viscosity)

$$\tau_N^0\phi_\zeta(E)=\tau_q(E)\left[1+\left(S_0^{(+)},\frac{\xi^2}{E^2}\phi_\zeta\right)(E)\right] \tag{29b}$$

and for ϕ_κ (thermal conductivity)

$$\tau_N^0\phi_\kappa(E)=\tau_q(E)\left[1+\frac{1}{E}\left(S_1^{(-)},\frac{\xi^2}{E}\phi_\kappa\right)(E)\right] \tag{29c}$$

Now the problem of evaluating the mass transport parameters is completely determined because the viscosities and thermal conductivity are expressed as energy averages over the functions ϕ [Eqs. (11)], which are solutions of the integral equations (29). Thus we are left with the solution of the integral equations for ϕ_η, ϕ_ζ, and ϕ_κ. Before turning to this problem it is necessary to study some important symmetry and conservation properties of the scalar collision integral for Bogoliubov quasiparticles derived above. This is the topic of the following section.

4. EXACT PROPERTIES OF THE SCALAR COLLISION INTEGRAL

It should first be noted that the integral operators $\mathcal{P}_n^{(+)}$ defined by Eq. (26) have the property of reducing to the dimensionless relaxation rates $I_n(E)$ making up the Bogoliubov quasiparticle relaxation rate (18) when applied to a constant c:

$$(\mathcal{P}_n^{(+)},c)(E)=(\mathcal{P}_n'^{(+)},c)(E)=cI_n(E) \tag{30}$$

The collision integral for different parities of the distribution function is given by certain linear combinations of integral operators $\mathcal{P}_n^{(+,-)}$, the coefficients being new angular averages over the normal state quasiparticle scattering amplitude, defined via Eq. (24). The scattering parameters λ_2^+ and λ_1^-, apart from the quantity $\langle W\rangle_a$, are known to determine the pressure dependence of the normal state shear viscosity and thermal conductivity, respectively. The parameters γ_l, $\delta_l^{(+,-)}$, and $\alpha_l^{(+,-)}$ only occur in the superfluid phase, where they account for the possibility of more complicated scattering processes in the presence of p-wave Cooper pairs. For lowest l values, using the definition (24), these parameters are collected in Table I. From this, one immediately observes that the integral operator $R_l^{(r)}$ defined by Eq.

TABLE I

Special Values for the Scattering Parameters from Eq. (24)

l	λ_l^+	λ_l^-	α_l^+	α_l^-	δ_l^+	δ_l^-	γ_l
0	1	3	0	$2\gamma_0$	δ_0	$3\delta_0$	γ_0
1	1	λ_1^-	$\gamma_0-\gamma_1$	α_1^-	δ_0	δ_1^-	γ_1

(27) for lowest l reduces to the relaxation rate for Bogoliubov quasiparticles when applied to a constant c:

$$\frac{1}{\tau_\mathrm{N}^0}(R_0^{(+)}, c)(E_1) = \frac{1}{\tau_\mathrm{N}^0}(R_1^{(+)}, c)(E_1) = \frac{c}{\tau_q(E_1)} \tag{31a}$$

and when applied to the Bogoliubov energy E_k:

$$\frac{1}{\tau_\mathrm{N}^0}(R_0^{(-)}, E)(E_1) = \frac{E_1}{\tau_q(E_1)} \tag{31b}$$

In order to study the conservation (or nonconservation) properties of the collision integral we write down equations of motion for the quasiparticle contributions to densities and currents starting from the kinetic equation (1). For the homogeneous case ($\mathbf{q}=0$) one generally obtains

$$\omega 2 \sum_\mathbf{k} g(\hat{\mathbf{k}}) a(\xi_k)\, \delta\nu_k = -i2 \sum_\mathbf{k} g(\hat{\mathbf{k}}) a(\xi_k)\, \delta I_k$$

$$g(\hat{\mathbf{k}}) a(\xi_k) = \left\{ \begin{array}{ll} \hbar\mathbf{k} & \text{momentum density} \\ E_k & \text{energy density} \\ \dfrac{\xi_k}{E_k} m & \text{mass density} \\ \dfrac{\xi_k}{E_k} E_k \mathbf{v}_k & \text{energy current} \\ \dfrac{\xi_k}{E_k} p_i v_{kj} & \text{momentum current} \end{array} \right\} \text{ of BQP} \tag{32}$$

In order to calculate the appropriate moments of the collision integral according to the rhs of Eq. (32) we multiply Eq. (28) by some arbitrary function of energy $a(\xi)$ and the spherical harmonics $Y_{lm}(k)$ for fixed l and

m. After some algebra involving exchange of variables we find

$$\sum_{k_1,\mu_1} a(\xi_1)\, Y_{lm}(k_1)\, \delta I_k(\mu_1 E_1)$$

$$= \sum_{k_1,\mu_1} Y_{lm}(\hat{\mathbf{k}}_1)\, \delta\nu'_{k_1}\left[-\frac{1}{\tau_k^q}+\frac{1+(-1)^l s}{2}\frac{p_{\mu 1}^r}{\tau_N^0}(R_l^{(-r)}, a)(\mu_1 E_1)\right] \quad (33a)$$

if $\delta\nu'$ is even in energy ($t=+1$) and

$$\sum_{k_1,\mu_1} a(\xi_1)\, Y_{lm}(\hat{\mathbf{k}}_1)\, \delta I_k(\mu_1 E_1)$$

$$= \sum_{k_1,\mu_1} Y_{lm}(\hat{\mathbf{k}}_1)\, \delta\nu'_{k_1}$$

$$\times\left[-\frac{1}{\tau_k^q}+\frac{1+(-1)^l s}{2}\frac{\xi_1}{E_1}\frac{p_{\mu 1}^r}{\tau_N^0}\left(S_l^{(-r)}, \frac{\xi}{E}a\right)(\mu_1 E_1)\right] \quad (33b)$$

if $\delta\nu'$ is odd in the energy variable ($t=-1$). In deriving Eqs. (33a) and (33b) we have made use of the symmetry properties $\tau_q(E)=\tau_q(\mu E)$, $\mu=\pm 1$, for the Bogoliubov quasiparticle relaxation rate and

$$(\mathcal{P}_n^{(r)}, \Phi)(E_1)=p_{\mu_1}^{-r}(\mathcal{P}_n^{(r)}, \Phi)(\mu_1 E_1)$$

(with some arbitrary function of energy Φ) for the integral operators $\mathcal{P}_n^{(r)}$.

From Eq. (33a) together with Eq. (32) one obtains for the case $a(\xi)=\hbar k_F$, $l=1$, $t=+1$, and $s=-1$ *momentum conservation* of the normal component:

$$\omega 2\sum_{\mathbf{k}} \hbar\mathbf{k}\, \delta\nu_k = 0 \quad (34)$$

and for the case $a(\xi_k)=E_k$, $l=0$, $s=1$, and $t=1$ *energy conservation* of the normal component:

$$\omega 2\sum_{\mathbf{k}} E_k\, \delta\nu_k = 0 \quad (35)$$

As a consequence of the conservation properties (34) and (35), the collision integral vanishes in general when applied to some function $\mathbf{a}\cdot\mathbf{k}+bE_k$ with arbitrary constants $\mathbf{a}$ and b. We may therefore replace $\delta\nu'$ under the collision integral by $\delta\nu-\delta\nu^{\text{loc}}$, with $\delta\nu^{\text{loc}}$ given by Eq. (6), which is equivalent to saying that the scalar collision integral describes relaxation toward the true local equilibrium distribution (6).

In contrast to momentum and energy, the quasiparticle mass, energy current, and momentum current are derived from distribution functions that are odd in the energy ξ_k, and an inspection of Eq. (33b) shows that these quantities are not conserved. In order to relate these nonconservation properties of the collision integral to the existence of the nonvanishing dissipative parameters of bulk and shear viscosity and diffusive thermal conductivity we first define transport times according to Eqs. (11a)–(11c):

$$\tau_\zeta := \langle(\xi^2/E^2)\phi_\zeta(E)\tau_N^0\rangle_\varphi \tag{36a}$$

$$\tau_\eta := \langle(\xi^2/E^2)\Phi_\eta(E)\tau_N^0\rangle_\varphi \tag{36b}$$

$$\tau_\kappa := \langle(\xi_k/2kT_c)^2\Phi_\kappa(E)\tau_N^0\rangle_\varphi \tag{36c}$$

Then we note that via the transport equation (7) the collision integral δI_k can be related to these transport times in the following way:

$$i\delta I_k = -i\varphi_k \frac{\xi_k}{E_k}\left(\frac{1}{mN_F}\frac{\delta\rho^{q'}}{\tau_\zeta} + E_k \frac{\mathbf{v}_k \cdot \mathbf{j}_\varepsilon^{q'}}{(n/m^*)(2kT_c)^2\tau_\kappa} + \frac{1}{2}\frac{\mathbf{p}\cdot\boldsymbol{\pi}^{q'}\cdot\mathbf{v}_k}{p_F^2 v_F^2 N_F \tau_\eta/15} + \cdots\right) \tag{37}$$

Here $\delta\rho^{q'} = -mN_F\,\delta\mu'$, $\mathbf{j}_\varepsilon^{q'}$, and $\boldsymbol{\pi}^{q'}$ are the deviations of quasiparticle mass density, energy current, and momentum current, respectively, from their local equilibrium values.

Finally, we take moments of Eq. (37) for these quantities according to the classifications (32) and obtain as a final result the following relaxation equations for the densities of mass, energy current, and momentum current of the normal component:

$$\omega\,\delta\rho^q = 2\omega\sum_k \frac{\xi_k}{E_k} m\,\delta\nu_k = -i\left\langle\frac{\xi^2}{E^2}\right\rangle_\varphi \frac{\delta\rho^{q'}}{\tau_\zeta} \tag{38a}$$

$$\omega\mathbf{j}_\varepsilon^q = 2\omega\sum_k \frac{\xi_k}{E_k}\mathbf{v}_k E_k\,\delta\nu_k = -i\left\langle\frac{\xi^2}{(2kT_c)^2}\right\rangle_\varphi \frac{\mathbf{j}_\varepsilon^{q'}}{\tau_\kappa} \tag{38b}$$

$$\omega\Pi_{ij}^q = 2\omega\sum_k \frac{\xi_k}{E_k} p_i v_{kj}\,\delta\nu_k = -i\left\langle\frac{\xi^2}{E^2}\right\rangle_\varphi \frac{\Pi_{ij}^{q'}}{\tau_\eta} \tag{38c}$$

with transport times τ_ζ, τ_κ, τ_η given by Eqs. (36) and energy averages $\langle\cdots\rangle_\varphi$ defined by Eq. (10).

We conclude this section by pointing out that we have shown two exact properties of the scalar collision integral for Bogoliubov quasiparticles δI_k: First, δI_k leads to conservation of quasiparticle momentum and energy [Eqs. (34) and (35)]. Second, the nonconservation property of δI_k in the cases of mass density, energy current, and momentum current of Bogoliubov quasiparticles [Eqs. (38a)–(38c)] clearly displays the microscopic origin of the relevant dissipative B-phase parameters of shear viscosity

$$\eta = p_F^2 v_F^2 N_F \tau_\eta/15 \tag{39a}$$

second viscosity

$$\zeta_3 = (1/m^2 N_F)\tau_\zeta \tag{39b}$$

and diffusive thermal conductivity

$$\kappa = (n/m^* T)(2kT_c)^2 \tau_\kappa \tag{39c}$$

In deriving Eqs. (38) and (39) we have established exact relations between the transport parameters η, ζ_3, and κ and the relaxation times τ_η, τ_ζ, and τ_κ, which describe how the associated currents tend toward local equilibrium.

5. EXACT SOLUTION OF THE TRANSPORT EQUATIONS AT LOW TEMPERATURE

It is obvious that an exact solution of the integral equations (29a)–(29c) for the transport times of the superfluid phase is not available at arbitrary temperature, as is the case in the normal Fermi liquid, where the eigenfunctions of the collision operator are known.[12,13]

Exact solutions of the superfluid transport equations are known only for a small temperature region very close to the transition temperature (where an expansion to first order in Δ/kT_c can be performed) and in the zero-temperature limit. It is difficult if not impossible to compare both types of results with experiment, because for the former the $\Delta/k_B T_c$ expansion is only valid for reduced temperatures $1 - T/T_c \leq 10^{-3}$, where it is quite difficult to measure the latter because the exponential low-temperature divergence of the Bogoliubov quasiparticle mean free path restricts the applicability of our bulk transport theory to temperatures $T/T_c > 0.2$, as pointed out in I and also in Ref. 14.

It is nevertheless possible to obtain exact solutions of the scalar Boltzmann equation at finite temperatures, starting from the zero-temperature results, which can be shown to be applicable for temperatures well above $T/T_c = 0.2$. This is essentially due to the fact that at low temperatures the fermion occupation factors $f(1234)$ [cf. Eq. (16)] characterizing the energy and temperature dependence of the collision integral restrict the main contribution to ξ integrals to an interval of width $kT \ll \Delta$ around the Fermi energy. This can be seen by expanding the Fermi function at low temperature

$$f(E_k) = \exp\left[-\frac{\Delta}{kT} - \frac{1}{2}\frac{\xi_k^2}{\Delta kT} + \frac{1}{8}\left(\frac{\xi_k^2}{\Delta kT}\right)^2 \frac{kT}{\Delta} + \cdots\right] \tag{40}$$

Therefore typical values of ξ_k are small compared to the energy gap $\Delta(T)$

and one may expand the Bogoliubov quasiparticle energy E_k as

$$E_k = \Delta\left[1 + x^2\frac{kT}{\Delta} - \frac{x^4}{2}\left(\frac{kT}{\Delta}\right)^2 + \cdots\right] \tag{41}$$

where we have introduced a dimensionless energy variable

$$x = \xi_k/(2\Delta kT)^{1/2} \tag{41a}$$

adopting the notation of Pethick *et al.*[5] Using these definitions, we can write any energy average of some function $\Phi(\xi)$, weighted with the derivative of the Fermi function $\varphi_k = -\partial f(E_k)/\partial E_k$, at low temperature as

$$\langle\Phi\rangle_\varphi = \int_{-\infty}^{\infty} d\xi_k\, \varphi_k \Phi(\xi_k) = Y_{00}(T)\left\langle \phi(x)\left(1 + \frac{x^4}{2}\frac{kT}{\Delta}\right)\right\rangle_{\mathrm{G}} \tag{41b}$$

where $Y_{00}(T)$ is the zero-temperature limit of the Yoshida function

$$Y_{00}(T) = \lim_{T\to 0}\langle 1\rangle_\varphi = (2\pi\Delta/kT)^{1/2}\, e^{-\Delta/kT} \tag{42a}$$

and the brackets $\langle\cdots\rangle_{\mathrm{G}}$ denote a normalized Gaussian average:

$$\langle\cdots\rangle_{\mathrm{G}} = \frac{2}{\sqrt{\pi}}\int_0^{\infty} dx\, e^{-x^2}\cdots \tag{42b}$$

In particular one obtains

$$\langle x^{2n}\rangle_{\mathrm{G}} = \frac{\Gamma(n+1/2)}{\Gamma(1/2)}; \qquad n = 0, 1, 2, \ldots \tag{42c}$$

where Γ is Euler's gamma function.

The coherence factor ξ/E causing the Bogoliubov quasiparticle velocity to vanish at the Fermi surface appears as a weighting factor in the expressions for the transport times defined by Eqs. (36). It can be expanded as

$$\frac{\xi^2}{E^2} = \frac{kT}{\Delta}\left(2x^2 - 4x^4\frac{kT}{\Delta} + \cdots\right) \tag{43}$$

Therefore, the mean square of the Bogoliubov quasiparticle velocity, according to Eqs. (42) and (43), vanishes as

$$\overline{v_k^2} = v_{\mathrm{F}}^2\langle\xi^2/E^2\rangle_\varphi/\langle 1\rangle_\varphi = v_{\mathrm{F}}^2\, kT/\Delta \sim T \tag{44}$$

in the zero-temperature limit.

From the arguments given above, it is clear that we can solve the scalar Boltzmann transport equations by an iteration to at least first order in the small parameter $kT/\Delta \sim T/T_c$.

In order to do this, we first expand the argument of the Bogoliubov quasiparticle energy δ-function to first order in kT/Δ, using Eq. (41), as

$$\begin{aligned} &E_1+\mu_2E_2-\mu_3E_3-\mu_4E_4\\ &\quad=(1+\mu_2-\mu_3-\mu_4)\Delta[1+x_1^2+\mu_2x_2^2-\mu_3x_3^2-\mu_4x_4^2\\ &\quad-\tfrac{1}{2}(kT/\Delta)(x_1^4+\mu_2x_2^4-\mu_3x_3^4-\mu_4x_4^4)] \end{aligned} \tag{45}$$

where we have dropped terms of relative order $\exp(-\Delta/kT)$. With (45) the integral operators $\mathscr{P}$ and $\mathscr{P}'$ [defined by Eq. (26)] can be expanded to first order in kT/Δ as

$$\begin{aligned} (\mathscr{P}_n^{(r)},\Phi)&=\frac{8}{3\pi^{3/2}}I_{00}\sum_{\mu_2\cdots\mu_4}(rp_{\mu_2}^{-r})\,\delta_{1+\mu_2,\mu_3+\mu_4}\\ &\times\int_0^\infty dx_2\,\Phi(x_2)\int_0^\infty dx_3\,dx_4\prod_{i=2}^{4}\exp\left(-\frac{x_1^2}{2}+\frac{x_1^4}{4}\frac{kT}{\Delta}\right)k_n\\ &\times\delta\Big\{x_1^2+\mu_2x_2^2-\mu_3x_3^2-\mu_4x_4^2\\ &-\frac{1}{2}\frac{kT}{\Delta}(x_1^4+\mu_2x_2^4-\mu_3x_3^4-\mu_4x_4^4)\Big\} \end{aligned} \tag{46}$$

where I_{00} is the energy-independent, zero-temperature limit of the dimensionless relaxation rate $I_0(E_k)$ [cf. Eq. (18a)]:

$$I_{00}(T)=\lim_{T\to0}I_0(E_k)=\frac{3}{(2\pi)^{1/2}}\left(\frac{\Delta}{kT}\right)^{3/2}e^{-\Delta/kT}=\frac{3}{2\pi}\frac{\Delta}{kT}Y_{00}(T) \tag{46a}$$

and the quantity k_n was defined by Eq. (18a). A similar result for the integral operator $\mathscr{P}'$ is obtained from Eq. (46) by replacing k_n by

$$k_n'=k_n(E_1,-\mu_3E_3,-\mu_2E_2,\mu_4E_4)$$

Next we expand both the integral operators $\mathscr{P}_n^{(r)}$ ($\mathscr{P}_n^{(r)\prime}$) and the function Φ to lowest order in kT/Δ according to

$$\begin{aligned} \Phi(x)&=\Phi_0+\frac{kT}{\Delta}\Phi_1(x)\\ (\mathscr{P}_n^{(r)},\Phi)&=\Phi_0(\mathscr{P}_{n(0)}^{(r)},1)+\frac{kT}{\Delta}[(\mathscr{P}_{n(0)}^{(r)},\Phi_1(x))+\Phi_0(\mathscr{P}_{n(1)}^{(r)},1)] \end{aligned} \tag{47}$$

Here we have anticipated that Φ does not depend on energy at zero temperature. For the evaluation of the zeroth order integral operators we expand $\Phi(x)$ in a power series as

$$\Phi(x)=\sum_k c_kx^{2k} \tag{48}$$

Thus, the integrations in (46) are elementary; the results can be written in the following compact form:

$$(\mathcal{P}^{(r)}_{n(0)}, \Phi)(x)$$
$$= I_{00} \sum_k c_k \frac{\Gamma(k+1/2)}{\Gamma(1/2)} \left[B_k(x)(\delta_{n,0} + \delta_{n,3}) + \frac{1}{3}(\delta_{n,1} + \delta_{n,2}) - \frac{2}{3}(\delta_{n,0} + \delta_{n,1} + \delta_{n,2} + \delta_{n,3}) \frac{1-r}{2} \right]$$

$$(\mathcal{P}^{(r)'}_{n(0)}, \Phi)(x)$$
$$= I_{00} \sum_k c_k \frac{\Gamma(k+1/2)}{\Gamma(1/2)} \left\{ B_k(x)(\delta_{n,0} + \delta_{n,1} + \delta_{n,2} + \delta_{n,3}) - \frac{2}{3}\left[(\delta_{n,1} + \delta_{n,2}) \frac{1+r}{2} + (\delta_{n,0} + \delta_{n,3}) \frac{1-r}{2} \right] \right\} \tag{49}$$

where r is the product of parities s (momentum k) and t (energy ξ) as introduced by Eq. (25). Here $B_k(x)$ is a polynomial of degree $2k$,

$$B_k(x) = \frac{1}{3} + \frac{2}{3} \sum_{l=0}^{k} \binom{k}{l} \frac{\Gamma(k-l+1/2)}{\Gamma(k-1)\Gamma(1/2)} x^{2l} \tag{50}$$

with the important integral property

$$\langle B_k(x) \rangle_G = 1; \qquad k = 0, 1, 2, \ldots \tag{50a}$$

The evaluation of the first-order integral operators $\mathcal{P}^{(r)}_{(1)}$ is somewhat tedious because one has to account for the first order in the energy δ-function as well, but straightforward. The result for the case $r = +1$ reads

$$(\mathcal{P}^{(+)}_{n(1)}, 1)(x)$$
$$= (\mathcal{P}^{(+)'}_{n(1)}, 1)(x)$$
$$= I_{00} \left[\frac{3}{4}(1+x^2)\left(\delta_{n0} + \delta_{n3} + \frac{\delta_{n1} + \delta_{n2}}{3} \right) - \frac{2x^2}{} \frac{\delta_{n1} + \delta_{n2}}{3} - (1+2x^2)\, \delta_{n3} \right] \tag{51}$$

The result for $\mathcal{P}^{(-)}_{n(1)}$ turns out to be only important for the $(kT/\Delta)^2$ correction to the diffusive thermal conductivity and will not be considered here.

Now we have established the complete mathematical basis necessary to solve the integral equations for bulk and shear viscosity and thermal conductivity exactly to first order in kT/Δ.

First, using the relation $(\mathscr{P}_n^{(+)}, 1) = I_n$, we find the following low-temperature results for the dimensionless relaxation rates $I_n(E_k)$ [cf. Eqs. (18a) and (46a)]:

$$\begin{aligned} I_0(x) &= I_{00}\left[1 + \frac{3}{4}\frac{kT}{\Delta}(1+x^2)\right] \\ I_1(x) &= \frac{1}{3}I_{00}\left[1 + \frac{1}{4}\frac{kT}{\Delta}(3-5x^2)\right] \\ I_2(x) &= \frac{1}{3}I_{00}\left[1 + \frac{1}{4}\frac{kT}{\Delta}(-1+3x^2)\right] \\ I_3(x) &= I_{00}\left[1 - \frac{1}{4}\frac{kT}{\Delta}(1+5x^2)\right] \end{aligned} \tag{52}$$

From (52), we find, together with Eq. (18), for the relaxation rate of Bogoliubov quasiparticles at low temperature

$$\frac{1}{\tau_q} = \frac{1}{\tau_q^{(0)}}\left\{1 + \frac{kT}{\Delta}\left[\frac{3}{4}(1+x^2) - (1+2x^2)\frac{\delta_0 - \gamma_0/3}{w_0}\right]\right\} \tag{53}$$

Here

$$\frac{\tau_{\mathrm{N}}^0}{\tau_q^{(0)}} = I_{00}w_0 = \frac{3w_0}{(2\pi)^{1/2}}\left(\frac{\Delta}{kT}\right)^{3/2} e^{-\Delta/kT} \tag{53a}$$

is the zero-temperature limit of the Bogoliubov quasiparticle relaxation rate, the pressure dependence of which beneath the zero-temperature gap is contained in the quantity

$$w_0 = 1 - \tfrac{2}{3}\gamma_0 + \delta_0 \tag{53b}$$

where γ_0 and δ_0 are the superfluid scattering parameters introduced by Eq. (18b).

We turn now to the integral equations (29a)–(29c) for the mass transport parameters, which can be written in compact form as

$$\begin{aligned} &\tau_{\mathrm{N}}^0 E^{(1-r)/2}\Phi_T(x) \\ &\quad = E^{(1-r)/2}\tau_q(E)\left[1 + E^{(r-1)/2}\left(S_l^{(r)}, \frac{\xi^2}{E^2}E^{(1-r)/2}\Phi_T)(x)\right)\right] \end{aligned} \tag{54}$$

where $\Phi_T = \Phi_\eta, \Phi_\zeta$ for parity $r = +1$ and $\Phi_T = \Phi_\kappa$ for parity $r = -1$. Equation (54) is solved iteratively by expanding the integral operators $S_l^{(r)}$

and the solutions Φ_T in powers of kT/Δ:

$$\tau_N^0\left[\Phi_T^{(0)}+\frac{kT}{\Delta}\Phi_T^{(1)}(x)\right]$$

$$=\left[\tau_q^{(0)}+\frac{kT}{\Delta}\tau_q^{(1)}(x)\right]\left[1+\frac{kT}{\Delta}\Phi_T^{(0)}(S_{l(0)}^{(r)}, x^2)(x)\right] \tag{55}$$

It should be noted that the iteration procedure becomes particularly simple in the present case of spin-independent transport as the integral operators S_l are applied to the quantity ξ^2/E^2, which, because of relation (43), is already at least of order kT/Δ. The zeroth-order solutions of Eq. (55) in the three cases of interest are identically given by the zero-temperature limit of the Bogoliubov quasiparticle collision time [cf. Eq. (53a)]:

$$\tau_N^0\Phi_T^{(0)}=\tau_q^{(0)}=\frac{\tau_N^0}{I_{00}(T)w_0} \tag{56}$$

The first-order correction displays the characteristic symmetry of the transport problem under consideration:

$$\tau_N^{(0)}\Phi_T^{(1)}(x)=\tau_q^{(0)}\left[-\frac{3}{4}(1+x^2)+(1+2x^2)\frac{\delta_0-\gamma_0/3}{w_0}\right.$$

$$\left.+\frac{1}{I_{00}W_0}(S_{l(0)}^{(r)}, x^2)(x)\right] \tag{57}$$

For $l=0, 2$ $(r=+1)$ one obtains solutions for the viscosities η and ζ_3; the diffusive thermal conductivity is obtained from the case $l=1, r=-1$. The quantity $(S_{l(0)}^{(r)}, x^2)$ is readily evaluated using Eqs. (49) and (27) for the case $\Phi(x)=x^2$ to give

$$(S_{l(0)}^{(r)}, x^2)=\tfrac{2}{3}I_{00}\cdot\begin{cases}\lambda_l^{(+)}(1+x^2)-\gamma_l x^2; & r=+1\\ \lambda_l^{(-)}x^2-\gamma_l(1+x^2); & r=-1\end{cases} \tag{58}$$

Finally, the transport times τ_T with $T=\eta$, ζ_3, and κ introduced by Eqs. (36) are given as integrals over the low-temperature solutions $\Phi_T(x)$ using Eq. (43) as

$$\tau_T=\frac{kT}{\Delta}Y_{00}\left\langle 2\Phi_T(x)\left(1+\frac{kT}{\Delta}\frac{x^2}{2}\right)\left(x^2-2x^4\frac{kT}{\Delta}\right)\right\rangle_G\tau_N^0 \tag{59}$$

The results appear to be of the general form

$$\tau_T=\tau_T^0\left(1+c_T\frac{kT}{\Delta}\right) \tag{60}$$

TABLE II

Exact Low-Temperature Results for Spin-Independent Transport and Relaxation Times of the BW State as given by Eq. (60)

Transport and relaxation time	Zero-temperature limit	First-order correction	
Quasiparticle collision time at the Fermi surface ($\xi_k = 0$)	$\tau_q^{(0)} = \frac{1}{3w_0}\left[2\pi\left(\frac{\Delta}{kT}\right)^3\right]^{1/2} e^{\Delta/kT}\tau_N^0$	$c_q = \frac{\delta_0 - \gamma_0/3}{w_0} - \frac{3}{4}$	(60a)
Shear viscosity $\eta = \frac{1}{5}nm^* v_F^2 \tau_\eta$	$\tau_\eta^{(0)} = \frac{2\pi}{3w_0}\left(\frac{kT}{\Delta}\right)^2 \tau_N^0 =: \tau_s$	$c_\eta = -4c_q + \frac{\frac{5}{3}\lambda_2^+ - \gamma_2}{w_0}$	(60b)
Second viscosity $\zeta_3 = (1/m^2 N_F)\tau_\zeta$	$\tau_\zeta^{(0)} = \tau_s$	$c_\zeta = -4c_q + \frac{\frac{5}{3} - \gamma_0}{w_0}$	(60c)
Diffusive thermal conductivity $\kappa = (n/m^* T)(2k_B T_c)^2 \tau_\kappa$	$\tau_\kappa^{(0)} = \tau_s$	$c_\kappa = -4c_q + 3 + \frac{\lambda_1^- - \frac{5}{3}\gamma_1}{w_0}$	(60d)

They are collected in Table II together with the result for the Bogoliubov quasiparticle relaxation time at the Fermi surface.

6. DISCUSSION OF RESULTS

6.1. Relaxation Rate of Bogoliubov Quasiparticles and Fundamental Transport and Relaxation Times of the BW State

The transport and damping parameters of superfluid ^{3}He-B are closely related to certain energy averages of the relaxation rate $1/\tau_q(E)$ for Bogoliubov quasiparticles. Therefore we begin this section with a summary of the most important results for this quantity.

The collision rate $1/\tau_q(E)$ defined by Eq. (18) characterizes the time scale for collisions between Bogoliubov quasiparticles in a situation where the frequency ω and the wave vector $\mathbf{q}$ of the external perturbation are small compared to the gap frequency $\Delta/\hbar$ and the inverse coherence length ξ_0^{-1}, respectively.

It is isotropic for the Balian–Werthamer state and can be expressed by four different dimensionless relaxation rates $I_n(E_k, T)$, $n=0,1,2,3$, which depend on the quasiparticle energy $E_k(T)=[\xi_k^2+\Delta^2(T)]^{1/2}$, and temperature:

$$I_q(E)=\frac{\tau_N^0(T)}{\tau_q(E)}=I_0(E)-\gamma_0[I_1(E)+I_2(E)]+\delta_0 I_3(E) \tag{61}$$

Here $\tau_N^0(T)=32E_f\hbar/(\pi k_B T)^2\langle W\rangle_a$ is the normal state quasiparticle lifetime at the Fermi surface ($\xi=0$) and the functions $I_n(E)$ are defined by Eq. (18a). The angular dependence of quasiparticle scattering in the superfluid state cannot be entirely separated from its energy dependence. Thus γ_0 and δ_0 are ratios of angular averages of superfluid scattering amplitudes, which have been introduced via Eq. (18b).

In general the functions $I_n(E)$ can only be evaluated numerically. At low temperatures, however, the methods described in Section 5 can be applied to obtain, up to first order in kT/Δ,

$$\begin{aligned}
I_0(\xi_k)&=I_{00}(T)\left[1+\frac{3}{4}\left(1+\frac{\xi_k^2}{2kT\Delta}\right)\frac{kT}{\Delta}\right]\\
I_1(\xi_k)&=\frac{1}{3}I_{00}(T)\left\{1+\left[\frac{3}{4}\left(1+\frac{\xi_k^2}{2kT\Delta}\right)-\frac{\xi_k^2}{kT\Delta}\right]\frac{kT}{\Delta}\right\}\\
I_2(\xi_k)&=\frac{1}{3}I_{00}(T)\left\{1+\left[\frac{3}{4}\left(1+\frac{\xi_k^2}{2kT\Delta}\right)-1\right]\frac{kT}{\Delta}\right\}\\
I_3(\xi_k)&=I_{00}(T)\left\{1+\left[\frac{3}{4}\left(1+\frac{\xi_k^2}{2kT\Delta}\right)-\left(1+\frac{\xi_k^2}{kT\Delta}\right)\right]\frac{kT}{\Delta}\right\}
\end{aligned} \tag{62}$$

From Eqs. (62) together with the general form (60) for the transport and relaxation times we find for the inverse lifetime of Bogoliubov quasiparticles at the Fermi surface ($E=\Delta$) at low temperature

$$\frac{1}{\tau_q(E=\Delta)}=\frac{1}{\tau_q^0}\left(1-c_q\frac{kT}{\Delta}\right) \tag{63}$$

where

$$\tau_q^0=\lim_{T\to 0}\tau_q(E)=\tau_N^0/I_{00}w_0 \tag{64}$$

$$w_0=1-\tfrac{2}{3}\gamma_0+\delta_0 \tag{64a}$$

and the coefficient of the first-order correction c_q is given by

$$c_q=\frac{\delta_0-\gamma_0/3}{w_0}-\frac{3}{4} \tag{64b}$$

This coefficient can be analytjcally evaluated within the s-wave approximation (isotropic quasiparticle scattering) for the scattering amplitude to give

$$-c_q=13/28\approx 0.46 \qquad \text{(isotropic scattering)} \tag{64c}$$

The scattering of quasiparticles in ^{3}He-B is, however, far from isotropic. There are various ways of generating more refined approximations for the quasiparticle scattering amplitude. The first extension of the s-wave result can be achieved by including a p term as proposed by Dy and Pethick.[32] The results for the normal state transport parameters evaluated within the sp-wave approximation generally agree better with experiments at vapor pressure than with those at intermediate and high pressure, where they are below the experimental results by about a factor of two. In I we therefore adopted the point of view that scattering parameters that are very sensitive to a pressure variation (such as $\langle W\rangle_a$, λ_2^+, λ_1^-) are to be determined from other experiments.

Meanwhile there have been many new ideas to find better approximations for the scattering amplitude. The first is the so-called "effective potential approximation" by which the angular dependence of the scattering amplitude is approximated in the form

$$A_s(\theta,\phi)=W_s(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_3)+(-1)^s W_s(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_4)$$

Here $s=0$ (1) in case of the singlet (triplet) component, θ is the angle enclosing the momenta $\mathbf{p}_1,\mathbf{p}_2$ of incoming particles, and ϕ is the angle between the planes containing the incoming ($\mathbf{p}_1,\mathbf{p}_2$) and outgoing ($\mathbf{p}_3,\mathbf{p}_4$) momenta, respectively ("Abrikosov angles"). The Legendre expansion of the functions $W_s(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_{3,4})$ contains coefficients that can be related to the

TABLE III

Scattering Parameters for Spin-Independent Transport Problems in Superfluid ^{3}He-B in Various Approximations

	Scattering parameter in given transport problem										
	BQP relaxation rate				Shear viscosity			Thermal conductivity			Second viscosity
Approximation	$\langle W\rangle_a$	γ_0	δ_0	c_q	λ_2^+	γ_2	c_η	λ_1^-	γ_1	c_κ	c_ζ
s-Wave (isotropic) scattering		1/3	7/15	−13/28	1/5	23/105	−173/98	1/3	7/15	11/14	−11/14
sp Wave											
0 bar	66.6	0.119	0.295	−0.539	0.628	0.091	−1.37	1.25	0.268	1.50	−0.886
10 bar	105.9	0.108	0.301	−0.534	0.529	0.092	−1.49	0.966	0.326	1.21	−0.869
20 bar	123.7	0.097	0.303	−0.536	0.511	0.088	−1.51	0.910	0.333	1.16	−0.858
30 bar	126.5	0.094	0.301	−0.532	0.520	0.086	−1.50	0.928	0.327	1.18	−0.858
spd Wave (34.34 bar)[24]	97.2	0.058	0.274	−0.540	0.701	0.101	−1.31	1.305	0.226	1.60	−0.873
Hara[11]											
spd Wave "C"		0.182	0.377	—	—	—	—	1.52	0.315	—	—
spd Wave "D" (21 bar)		0.039	0.485	—	—	—	—	1.92	0.078	—	—
Pfitzner and Wölfle[26]											
0 bar	74.3	0.135	0.268	−0.561	0.679	0.124	−1.39	1.19	0.325	1.31	−0.942
10 bar	109.6	0.156	0.296	−0.546	0.753	0.153	−1.26	1.33	0.341	1.46	−0.915
20 bar	118.6	0.139	0.285	−0.549	0.762	0.148	−1.26	1.33	0.327	1.46	−0.917
30 bar	121.2	0.089	0.252	−0.563	0.738	0.120	−1.32	1.26	0.238	1.41	−0.931

Landau parameters.[33] The first two coefficients of this expansion just reproduce the *sp* approximation of Dy and Pethick. An *spd* approximation generated from the effective potential form of the scattering amplitude has been applied by Ono *et al.*[34] and Hara[11] for the evaluation of the superfluid shear viscosity and thermal conductivity.

A similar treatment with the $l=3$ term included has been performed at melting pressure by Sauls and Serene.[24] For the evaluation of transport parameters in this paper we use scattering parameters taken from a recent paper of Pfitzner and Wölfle.[26] In their work the quasiparticle scattering amplitude is calculated from a generalized Landau–Bethe–Salpeter equation using the polarization potentials of Aldrich and Pines[35] as input. The solution guarantees exchange symmetry explicitly and improves in that respect on a similar calculation by Bedell and Pines.[25]

Table III lists the results for the scattering parameters obtained in various approximations, if available from the original papers.[11,24,26,32] We shall come back to this table whenever the pressure dependence of transport parameters is considered.

Let us now return to the first-order correction c_q of the BQP relaxation rate [Eq. (64b)]. From Table III it is seen that this quantity neither depends very much on pressure nor on the approximation used in its evaluation.

In Fig. 1 we plot the numerical results for the relaxation rates $I_n(E=\Delta)$ taken at the Fermi surface (solid lines) vs. Δ/kT. Also shown in this figure is the relaxation rate I_q of Bogoliubov quasiparticles normalized to the normal state lifetime $\tau_N^0(T)$ (upper solid curve), which has been generated from the I_ns using scattering parameters $\gamma_0=0.1$ and $\delta_0=0.3$, appropriate for intermediate pressure.

A comparison of I_0 with I_q in Fig. 1 shows that the terms in I_q multiplied by γ_0 and δ_0 partially compensate each other such that the contribution of I_0 to the full relaxation rate dominates at all temperatures. The dashed lines close to the I_n are the corresponding low-temperature forms, which are obtained from Eq. (62) by putting $\xi_k=0$. It is seen from this comparison that the validity of the low-temperature approximation for the relaxation rates $I_n(E=\Delta)$ is restricted to values $\Delta/kT \gtrsim 2.5$.

In what follows, we summarize our results for transport and relaxation times, plotting them vs. reduced temperature T/T_c. This requires a specification of the temperature dependence of the gap function $\Delta(T)$. We introduce a "strong coupling" parameter by

$$\delta_{sc}=\Delta(0)/kT_c \tag{65}$$

and, as in I, use the interpolation formula

$$\Delta(T)=\Delta(0)\tanh\frac{\pi}{\delta_{sc}}\left[\frac{2}{3}\frac{\Delta C}{C_N}\left(\frac{T_c}{T}-1\right)\right]^{1/2} \tag{66}$$

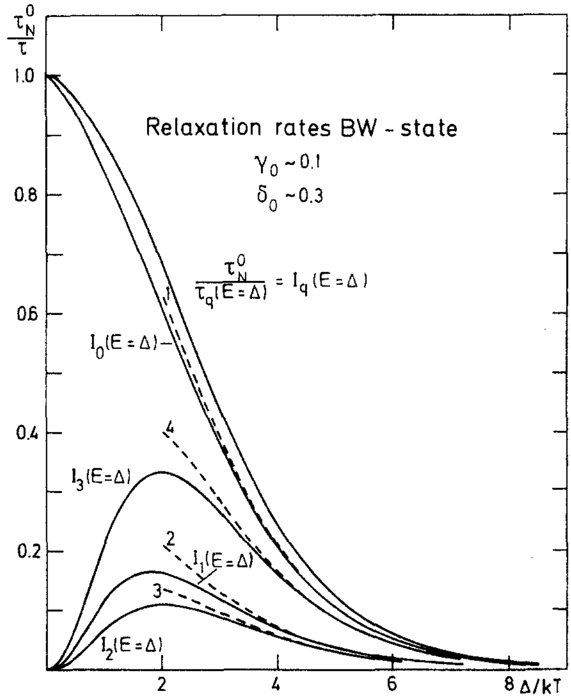

Fig. 1. Dimensionless relaxation rates for Bogoliubov quasiparticles of the BW state at the Fermi surface vs. Δ/k_BT as defined by Eq. (18a). The upper solid line represents the full quasiparticle relaxation rate for $E=\Delta$ according to Eq. (18). The dashed lines are the low-temperature forms of I_n from Eq. (62).

for the gap function, where $\Delta C/C_N$ is the specific heat discontinuity. It turns out that in the weak coupling (BCS) limit, where $\delta_{sc}=1.76$ and $\Delta C/C_N=1.43$, $\Delta(T)$ from (66) is off the values for the gap function tabulated by Mühlschlegel[36] by at most 1.5%.

In Fig. 2 the agreement of the weak coupling gap function $\Delta(T)$ from Eq. (66) (lowest curve) with Mühlschlegel's values (points) is visualized. Also shown in this figure (upper curves) is the gap function, normalized to k_BT_c for three gap renormalization parameters $\delta_{sc}=1.82$ (10 bar), 2.0 (20 bar), and 2.10 (30 bar), the pressure dependence of which is predicted by Bloyet *et al.*[37] from an analysis of a spin relaxation experiment. It should be noted that the zero-temperature gap according to the "weak coupling plus" model of Serene and Rainer[22] turns out to be much less enhanced when the pressure is raised: $\delta_{sc}=1.80$ (10 bar), 1.83 (20 bar), and 1.85

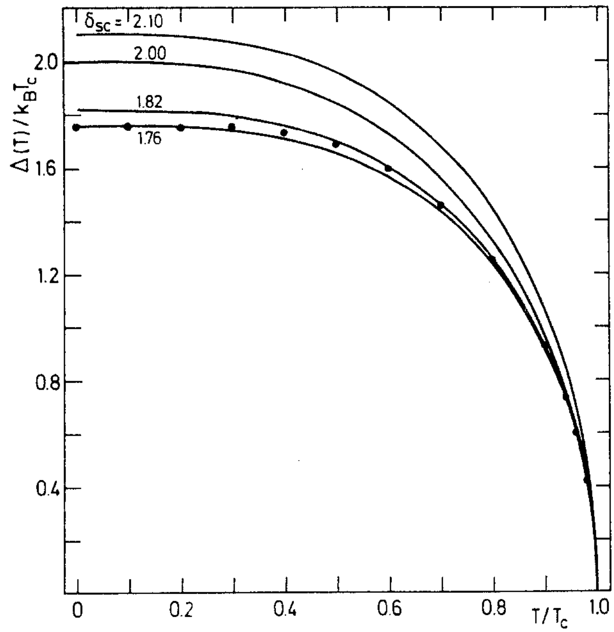

Fig. 2. Gap parameter of ^{3}He-B normalized to $k_B T_c$ for various values of the gap renormalization parameter $\delta_{sc} = \Delta(0)/k_B T_c$ vs. reduced temperature from Eq. (66). Points are values from Mühlschlegel's table,[36] shown for comparison.

(30 bar). There is no argument explaining the discrepancy between the "weak coupling plus" results and the spin relaxation experiment.

As we saw in Section 4, the physically relevant transport times [cf. Eqs. (36a)–(36c)] are expressed as energy averages of the solutions $\phi_T(E) \sim \tau_q(E)$ of the transport integral equations with various weighting factors ("coherence factors" ξ^2/E^2).

In Fig. 3 we therefore plot the energy averages

$$\tau_{Q1} = \langle \tau_q(E) \rangle_\varphi \qquad \text{("quasiparticle")} \tag{67a}$$

$$\tau_{V1} = \langle \tau_q(E)\xi^2/E^2 \rangle_\varphi \qquad \text{("viscosity")} \tag{67b}$$

$$\tau_{T1} = \langle \tau_q(E)\xi^2/4k^2T^2 \rangle_\varphi \qquad \text{("thermal conductivity")} \tag{67c}$$

as a function of T/T_c using the weak coupling parameters $\delta_{sc} = 1.76$ and $\Delta C/C_N = 1.43$ for the gap function according to Eq. (66).

The results for the three energy averages shown in this figure correspond to a crude approximate solution of the transport equations (29a)–(29c) for shear viscosity, second viscosity, and thermal conductivity, in which the integral operators $S_l^{(r)}$, which entirely carry the characteristic symmetry of

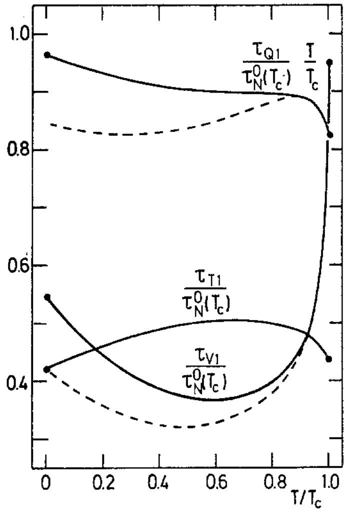

Fig. 3. Dimensionless fundamental transport times of ^{3}He-B defined by Eqs. (67a)–(67c). Solid lines are obtained with $\delta_{sc} = 1.76$ (weak coupling BCS), dashed lines with $\delta_{sc} = 2$. The influence of changing δ_{sc} from 1.76 to 2 on τ_{T1} is weak and not shown in the figure.

the transport problem under consideration, have been put equal to zero ("simple relaxation time approximation"). The results obtained within this approximation correspond to those of Dörfle *et al.*,[4] who started from a Kubo formula approach. They are not consistent with any of the exact conservation (or nonconservation) properties of the scalar collision integral derived in Section 4. But since the momentum and energy current (being related to shear viscosity and thermal conductivity) of Bogoliubov quasiparticles are conserved neither in the superfluid nor in the normal state of the Fermi liquid, the results of this treatment should give at least qualitative agreement with the exact solution of the Boltzmann equation.

In the case of the second viscosity, however, which is closely related to the phenomenon of intrinsic relaxation of the normal fluid fraction, as discussed in I, things look strikingly different. The normal fluid density is not conserved in the superfluid state, and being driven out of its mutual equilibrium with the superfluid component, relaxes back via quasiparticle collisions. The associated relaxation time diverges as the temperature approaches the transition from below,[2] indicating the conservation of quasiparticle number in the normal state. Therefore an approximation to the collision integral, like the one described above, that is not consistent with the conservation properties of the system cannot be expected to yield even qualitative agreement with the exact solutions of the problem of intrinsic relaxation, i.e., second viscosity. One can easily verify that the second viscosity evaluated within the simple relaxation time approximation stays constant as $T \to T_c$ instead of displaying a divergence proportional to $(1 - T/T_c)^{-1/2}$, a property of the exact solution.[2]

It should be noted that the transport times τ_{V1} and τ_{T1}, representing shear viscosity and diffusive thermal conductivity in the simple relaxation time approximation, remain finite in the whole temperature range. The quantity τ_{Q1}, on the other hand, diverges like T^{-1} as $T \to 0$ and thus describes an important property of the two components of the spin diffusion tensor in ^{3}He-B, to which τ_{Q1} can be shown to be the simple relaxation time approximation.[10,27]

The dashed curves in Fig. 3 correspond to the same transport times evaluated with a strong coupling parameter $\delta_{sc} = 2$. The effect of renormalizing the zero-temperature gap on the thermal conductivity transport time τ_{T1} turns out to be negligible in the whole temperature range and is therefore not shown in the figure.

In Fig. 4 we plot energy averages of the inverse lifetime of Bogoliubov quasiparticles defined by

$$\tau_{Q2} = \frac{\langle 1 \rangle_\varphi}{\langle 1/\tau_q(E) \rangle_\varphi} \tag{68a}$$

$$\tau_{V2} = \frac{\langle \xi^2/E^2 \rangle_\varphi}{\langle [1/\tau_q(E)]\xi^2/E^2 \rangle_\varphi} \tag{68b}$$

$$\tau_{T2} = \frac{\langle \xi^2 \rangle_\varphi}{[1/\tau_q(E)]\xi^2 \rangle_\varphi} \tag{68c}$$

vs. reduced temperature. A common factor $(T_c/T)^{1/2} \cosh^2(\Delta/2kT)$ has been extracted from these quantities in order to render them finite in the whole temperature range.

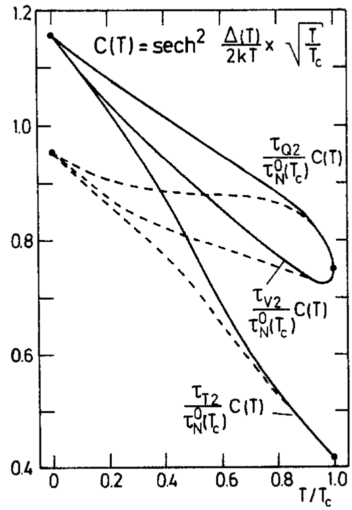

Fig. 4. Dimensionless relaxation rates of ^{3}He-B from Eqs. (68a)–(68c) multiplied by $C(t) = \{\mathrm{sech}^2[\Delta(T)/2k_BT]\}(T/T_c)^{1/2}$ vs. reduced temperature. Solid lines are obtained with $\delta_{sc} = 1.76$, dashed lines with $\delta_{sc} = 2.00$.

In particular, the quantity τ_{Q2} can be interpreted as a characteristic energy-independent collision time of Bogoliubov quasiparticles. It enters the approximate form of the collision integral in the "two relaxation time model" used to solve exactly the single plane boundary problem.[38] Furthermore, together with the averages τ_{V1}, τ_{T1}, and the relaxation times (67a)–(67c) it specifies the "separable kernel approximation" (SKA) introduced in I to evaluate quantitatively spin-independent transport parameters of superfluid ^{2}He-B at arbitrary temperature. We shall discuss the properties of this approximation later.

In Fig. 4 dashed lines again correspond to a strong coupling parameter $\delta_{sc} = 2.0$. The strong dependence of the low-temperature transport and relaxation times on the gap at low temperature is evident in this figure.

6.2. The Damping of Order Parameter Collective Modes

It turns out that one particular (frequency-dependent) energy average of the Bogoliubov quasiparticle relaxation rate can be directly compared with experiment. The width of the so-called "squashing" and "real squashing" peaks in the zero-sound attenuation, as derived by Koch and Wölfle,[39] is related to the lifetime

$$\tau_H(\omega, T) = \frac{1}{2}\left\{\int_{-\infty}^{\infty} d\xi_k \frac{\Delta^2}{2E_k^2} \tanh\frac{E_k}{2kT}\left[1 - \left(\frac{\hbar\omega}{2E_k}\right)^2\right]^{-1}\right\} \times\left\{\left\langle \frac{\Delta^2}{E_k^2}\frac{1}{\tau_q(E_k)}\left[1 - \left(\frac{\hbar\omega}{2E_k}\right)^2\right]^{-1}\right\rangle_\varphi\right\}^{-1} \tag{69}$$

with $\omega = (12/5)^{1/2}\Delta$ for the "squashing" mode and $\omega = (8/5)^{1/2}\Delta$ for the "real squashing" mode.

In Fig. 5 the prediction for $\tau_H(\omega = (12/5)^{1/2}\Delta, T)$ is compared with experimental data taken at 14 bar by Halperin.[21] We have plotted τ_H vs. reduced temperature using scattering parameters appropriate for intermediate pressure and a value for the normal state quasiparticle lifetime $\tau_N^0(T_c) = 1.3 \times 10^{-7}$ sec. The agreement is seen to be good in the whole range of temperatures for a value $\delta_{sc} = 2$ (solid line). The dashed curve is obtained with the weak coupling value $\delta_{sc} = 1.76$.

In what follows, we collect our low-temperature results for the bulk shear viscosity, second viscosity, and diffusive thermal conductivity and compare them to results obtained in I by means of the SKA method.

Our general aim in doing this is to show that the SKA results agree sufficiently well with the exact results in the limit of low temperatures. The good agreement with the exact results for $T \to T_c$ has already been pointed out in I. The advantage of the SKA method, as will turn out below, is the transparent structure of the results for the superfluid transport parameters

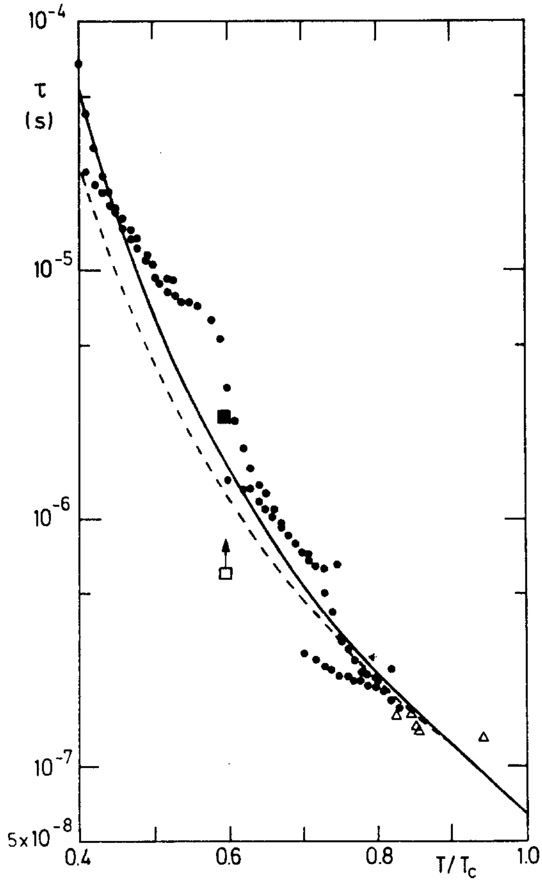

Fig. 5. Lifetime of the order parameter collective modes in ^{3}He-B vs. reduced temperature. The solid (dashed) line is a numerical evaluation of expression (69) with $\hbar\omega = (12/5)^{1/2}\Delta$ and $\delta_{sc} = 2.00$ (1.76). Points are taken from a measurement by Halperin.[21]

as compared to the more complicated variational procedure. The SKA transport parameters can all be expressed as simple rational combinations of some of the six "fundamental" transport times τ_{Qi}, τ_{Vi}, and τ_{Ti}, $i = 1, 2$ (cf. Figs. 3 and 4) with only normal state scattering parameters $(\lambda_1^-, \lambda_2^+)$ entering as nontrivial factors.

6.3. Shear Viscosity

The low-temperature result for the shear viscosity, according to Eq. (60) and Table II, reads

$$\eta = \tfrac{1}{5} n m^* v_F^2 \tau_\eta, \qquad \tau_\eta = \tau_s(1 + c_\eta kT/\Delta) \tag{70}$$

Here $\tau_s = 2\pi/3w_0(kT_c/\Delta)^2\tau_N^0(T_c)$ is the finite zero-temperature limit of the B-phase shear viscosity. The coefficient c_η of the first-order correction in kT/Δ can be expressed in terms of well-known scattering parameters as

$$c_\eta = -4c_q + (\tfrac{5}{3}\lambda_2^+ - \gamma_2)/w_0 \tag{70a}$$

Within the s-wave approximation for the quasiparticle scattering amplitude c_η can be evaluated analytically to give

$$-c_\eta = 173/98 \approx 1.77 \qquad \text{(isotropic scattering)} \tag{70b}$$

in agreement with the exact result of Pethick *et al.*[5] The general result for c_η is, of course, pressure dependent. An inspection of the values for c_η (in Table III) obtained in various approximations shows that this pressure dependence is quite weak. The approximate result for the shear viscosity transport time τ_η using the SKA as derived in I reads, with the abbreviations introduced by Eqs. (67) and (68),

$$\tau_\eta^{\text{SKA}} = \tau_{V1} + \lambda_2 \frac{Y_2}{(1/\tau_{Q2})Y_0/Y_2 - \lambda_2/\tau_{V2}} \tag{71}$$

where we have defined generalized Yoshida functions

$$Y_n(T) = \langle(\xi/E)^n\rangle_\varphi \tag{71a}$$

As pointed out in I (for details see Ref. 30) the separable kernel approximation, although consistent with the conservation properties of the system, does not exactly reproduce the nonconservation equation for the momentum current (38c), i.e., the angular dependence of the collision operator is simply represented by only one scattering parameter λ_2^+, which also determines the normal state shear viscosity.

In order to show the deviation of this approximation from the exact result at low temperature [Eq. (60)] we expand the energy integrals in (67) at low temperatures with the aid of (41b) and (43) to obtain

$$\tau_\eta^{\text{SKA}} = \tau_s\left(1 + c_\eta^{\text{SKA}}\frac{kT}{\Delta}\right); \qquad \tau_s = \frac{2\pi}{3W_0}\left(\frac{kT}{\Delta}\right)^2\tau_N^0 \tag{72}$$

where

$$c_\eta^{\text{SKA}} = c_\eta^{\text{exact}} - (\tfrac{5}{3}\lambda_2 - \delta_2)/w_0 + \lambda_2 \tag{72a}$$

The error introduced by using the SKA thus in general depends on pressure and, according to the results collected in Table III, on the approximation chosen for its evaluation.

In Fig. 6 we plot the reduced shear viscosity vs. reduced temperature for various values of the scattering parameter λ_2^+ and the strong coupling

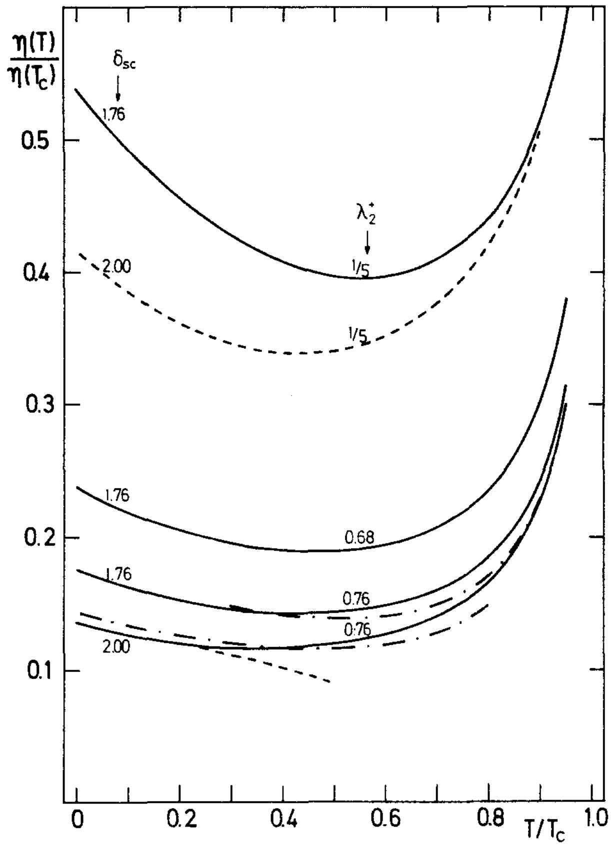

Fig. 6. Reduced shear viscosity of the BW state vs. reduced temperature, for various strong coupling (δ_{sc}) and scattering (λ_2^+) parameters as indicated. Solid lines and the upper dashed line are obtained within the SKA [cf. Eq. (71)]. The lower dashed line is the exact asymptotic result from Eq. (70). Dashed-dotted lines show variational results[34] for comparison, as discussed in the text.

parameter δ_{sc}. The solid curves are obtained within the separate kernel approximation for the collision operator. The upper curves ($\lambda_2 = 1/5$) are obtained if isotropic quasiparticle scattering is assumed. The dashed line shows the influence of changing the zero-temperature gap from its weak coupling value $\delta_{sc} = 1.76$ to $\delta_{sc} = 2$. The three lower solid curves are calculated using scattering parameters of Pfitzner and Wölfle[26] ($\lambda_2 = 0.68$ for $P = 0$ bar and $\lambda_2 = 0.76$ for $P = 21$ bar). The difference between the lowest two solid curves again displays the influence of the gap renormalization $\delta_{sc} = 2$, which is predicted for $P = 21$ bar by Bloyet *et al.*[37] The dashed line that merges into the lowest solid line represents the exact result for $\lambda_2 = 0.76$

and $\delta_{sc}=2$. Finally, the two dashed-dotted lines represent the most recent variational result of Ono *et al.*[34] for 21 bar with $\delta_{sc}=1.76$ (upper curve) and $\delta_{sc}=1.95$ (lower curve).

All these curves obtained in various approximations for different pressures show the characteristic qualitative behavior of the hydrodynamic shear viscosity of the infinitely extended volume: a drastic decrease below T_c followed by a minimum at intermediate temperature and a *finite* low-temperature limit.

One can draw several conclusions from Fig. 6. The first is the good agreement between the exact result for the slope of the shear viscosity with the SKA result at the low-temperature end of the graph. Second, there is rather good agreement with the results of the variational solution of the Boltzmann equation. Keeping in mind, third, that the SKA result deviates from the exact result just below T_c (Ref. 1) by typically 1%, as shown in I, it follows that for practical purposes the simple SKA, which can be generated from the transport times shown in Figs. 3 and 4 and the scattering parameters in Table III, gives a good quantitative description of the shear viscosity at arbitrary temperature. A further important point is that the pressure dependence of the shear viscosity enters via two different theoretical inputs: the scattering parameters λ_2, γ_2, etc., and the gap renormalization ratio $\delta_{sc}=\Delta(0)/k_B T_c$. According to the spin relaxation analysis of Bloyet *et al.*,[37] the parameter δ_{sc} changes by about 14% above its weak coupling value if one raises the pressure from 0 to 21 bar. Our exact theoretical result (70) for the low-temperature viscosity shows that $\eta \sim \delta_{sc}^{-2}$, so there is a 30% change of the shear viscosity of 21 bar due to this effect. If we use the scattering parameters of Pfitzner and Wölfle and δ_{sc} values given by Bloyet *et al.*, our theory predicts that the plateau value of the reduced shear viscosity decreases monotonically with increasing pressure up to melting pressure. Before we compare our theoretical results for the shear viscosity to experiment, some general remarks concerning the experimental analysis have to be made.

The shear viscosity can be measured in various ways, in particular by Andronikashvili or vibrating wire techniques or by using a conventional sound resonator. The existence of container walls makes it necessary to theoretically consider surface corrections to the hydrodynamics of the infinitely extended volume. In the case of the Andronikashvili cell this amounts to studying a stationary Poiseuille flow problem,[40] whereas for the sound resonator one has to consider the transverse surface impedance.[38,40] The extent to which the liquid feels the surface is governed by the viscous mean free path of the thermal excitations. At T_c the viscous mean free path lies typically between 1 and 10 μm, decreases below T_c, has a minimum, and increases exponentially at low temperatures. It has been demonstrated[40,38] that the finiteness of the mean free path leads to a "slip effect" of the normal velocity component at the walls of the container and, associated with it, to a correction of order one mean free path over typical container spacing to the bulk hydrodynamic results.

As one is interested in the shear viscosity representing the infinitely extended volume, these effects have to be accounted for in order to properly eliminate the influence of the surface.

First we compare our theory with the torsional pendulum and spherical viscometer data of Archie *et al.*[16] In Fig. 7 the solid lines represent the theoretical (SKA) prediction for the reduced shear viscosity as a function of T/T_c at four pressures, evaluated with scattering parameters of Pfitzner and Wölfle and δ_{sc} values of Bloyet *et al.* Open symbols represent torsional

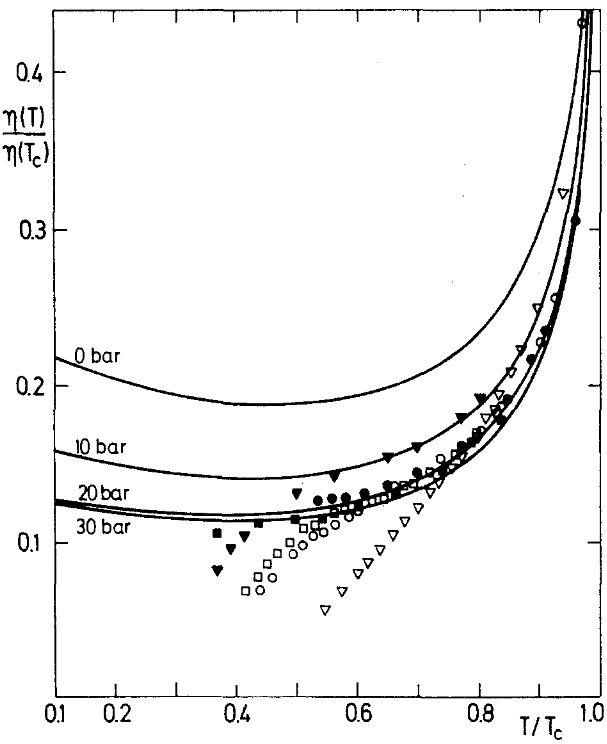

Fig. 7. Shear viscosity of ^{2}He-B vs. reduced temperature at four different pressures. The solid lines are the theoretical prediction obtained within the SKA with strong coupling parameters δ_{sc} from Ref. 37 and scattering parameters taken from Pfitzner and Wölfle.[26] Open symbols are torsion pendulum data, closed symbols are spherical viscometer data of Archie *et al.*[16] Inverted triangles: 5 bar; circles, 20 bar; squares, 30 bar.

oscillator data, and closed symbols data obtained with a spherical viscometer, both with a first-order slip correction included, Inverted triangles are 5-bar data, while circles and squares correspond to 20 and 30 bar, respectively.

The agreement between theory and experiment is good for temperatures above $\sim 0.9T_c$. Below $0.9T_c$ the experimental points are seen to fall below the theoretical prediction ("droop"). The discrepancy between theory and experiment in this respect is larger for the torsion pendulum than for the spherical viscometer data. None of the data sets shows a tendency to go through a minimum. The pressure dependence of the experimental viscosity data at low temperatures does not display a monotonic decrease with increasing pressure as predicted by theory.

In Fig. 8 we compare with the slip-corrected shear viscosity data obtained in a recent vibrating wire experiment by Carless *et al.*[19] The

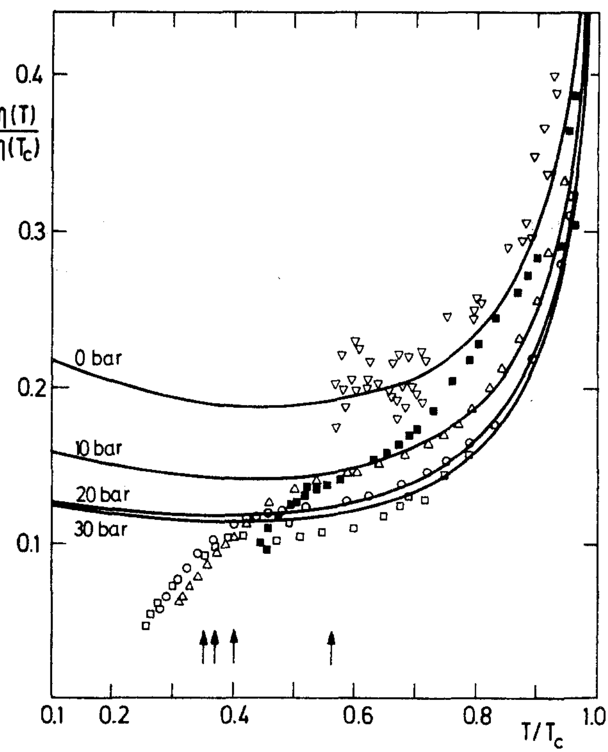

Fig. 8. Shear viscosity of ^{3}He-B as a function of reduced temperature. The solid lines are the same as in Fig. 7. Symbols represent vibrating wire data of Carless *et al.*[19] Inverted triangles, 0.1 bar; closed squares, 2.1 bar; triangles, 10 bar; circles, 20 bar; open squares, 30 bar.

theoretical curves are the same as in Fig. 7. Different symbols correspond to different pressures: inverted triangles, 0.1 bar; closed squares, 2.1 bar; triangles, 10 bar; circles, 20 bar; open squares, 30 bar. At pressures equal to or larger than 10 bar the experimental points agree very well with our theory in their temperature as well as in their pressure dependence for temperatures $\geq 0.5T_c$. Below that temperature the data points again fall below the theoretical prediction ("droop"). It should be emphasized that outside the "droop" regime this experiment for the first time confirms the prediction of hydrodynamic theory for the shear viscosity with respect to both temperature and pressure dependence.

In Fig. 9 we compare our results with the sound resonator data of Eska *et al.*[18] The damping of ordinary hydrodynamic sound is dominated by the surface contribution, which can be expressed by the transverse acoustic

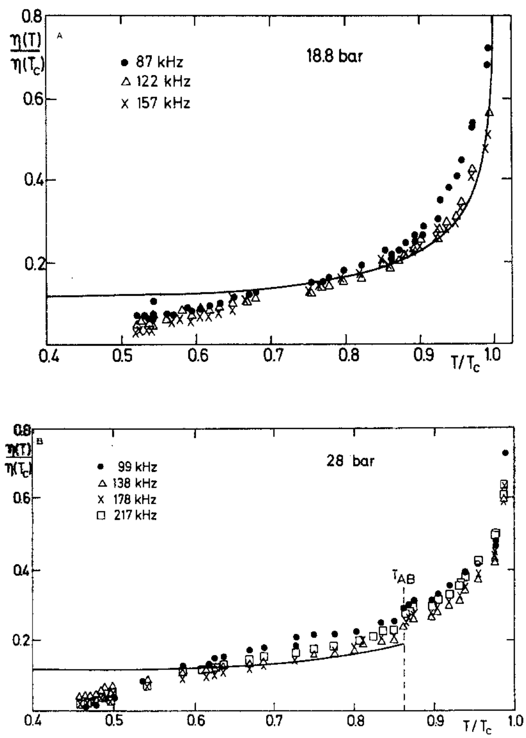

Fig. 9. Shear viscosity of ^{3}He-B vs. reduced temperature. Solid lines are theoretical predictions for (a) 20 and (b) 30 bar, respectively, shown in Figs. 7 and 8. Symbols are from first-sound attenuation measurements by Eska *et al.*[18] at various frequencies as indicated in the figure.

impedance. In order to extract η from the attenuation, the surface impedance has been evaluated in the low-frequency limit ($\omega\tau \to 0$). The solid curves show the theoretical reduced viscosity at 20 and 30 bar, respectively. Different symbols denote different frequencies. At 28 bar the temperature T_{AB} (dashed vertical line) separates A-phase from B-phase data. There is again fairly good agreement between theory and experiment for higher temperatures and a "droop" of the experimental viscosity at the low-temperature end. The slight frequency dependence of the experimental points at lower temperatures seems to indicate that one is already outside the hydrodynamic regime ($\omega\tau \sim 0$) and that here the full expression for the surface impedance[38,40] has to be used for the analysis of sound attenuation. As far as the "droop" of experimental viscosity is concerned in all cases, there is no explanation for this phenomenon. In working out the slip correction to the damping in each case purely diffuse scattering of quasiparticles at the walls of the container has always been assumed. In the normal liquid ^{3}He, this assumption seems to be consistent with experiments on U-tube oscillations[41] and dispersion of ordinary sound.[18]

The discrepancy in the superfluid might be due to a new scattering mechanism of (Bogoliubov) quasiparticles at the container walls, different from the purely diffuse one, which serves to enhance the slip length and reduce the surface impedance, respectively. In any case we believe that this discrepancy is due to a surface effect, because the relaxation time τ_H (cf. Fig. 5) taken from a high-frequency experiment, where the surface corrections should be negligible, does not show any sign of a "droop" at low temperatures.

6.4. Second Viscosity

We continue our discussion of theoretical results, considering the second viscosity ζ_3. This quantity governs the response of chemical potential to normal-to-superfluid counterflow, and is thus a dissipative parameter associated with a new (intrinsic) relaxation mechanism only present in superfluid systems.

Let us start with the low-temperature result for the second viscosity:

$$\zeta = \frac{1}{m^2 N_F} \tau_\zeta, \qquad \tau_\zeta = \tau_s \left(1 + c_\zeta \frac{kT}{\Delta} \right) \tag{73}$$

Here N_F is the density of states for both spin projections at the Fermi surface, and $\tau_s = 2\pi\tau_N^0(T_c)/3w_0\delta_{sc}^2$. Note that the transport time associated with second viscosity has the same zero-temperature limit as the shear viscosity transport time τ_η. As the second viscosity derives from a transport equation similar to that of shear viscosity but with $l = 0$ symmetry, the coefficient of the first-order correction c_ζ is obtained from c_η [cf. Eq. (72a)] just be replacing the scattering parameters λ_2 by $\lambda_0 \equiv 1$ and γ_2 by γ_0:

$$c_\zeta = -4c_q + (\tfrac{5}{3} - \gamma_0)/w_0 \tag{73a}$$

An estimate of the slope c_ζ of the second viscosity ζ_3 at low temperature within the approximation of isotropic scattering yields the analytical result

$$-c_\zeta = 11/14 \approx 0.79 \tag{73b}$$

The variation of c_ζ with pressure according to more refined approximations for the quasiparticle scattering amplitude (cf. Table III) is weak. If one applies the separable kernel approximation to the collision operator in the case of second viscosity, the result reads

$$\tau_\zeta^{\text{SKA}} = \tau_{\text{V}1} + \frac{Y_2}{(1/\tau_{\text{Q}2})\,Y_0/Y_2 - 1/\tau_{\text{V}2}} \tag{74}$$

where the transport times from Eqs. (67) and (68) and the generalized Yoshida functions $Y_n(T)$ [cf. Eq. (71a)] have been used. The error introduced by the SKA at low temperatures as compared to the exact result for τ_ζ can be mathematically expressed as

$$\tau_\zeta^{\text{SKA}} = \tau_s(1 + c_\zeta^{\text{SKA}} kT/\Delta) \tag{75}$$

where

$$c_\zeta^{\text{SKA}} = c_\zeta^{\text{exact}} - (\tfrac{5}{3} - \gamma_0)/w_0 + 1 \tag{75a}$$

In Fig. 10 we plot the second viscosity normalized to $\tau_{\text{N}}^0(T_c)/m^2 N_{\text{F}}$, with the divergence at T_c removed by multiplying with $\Delta(T)/\Delta(0)$, as a

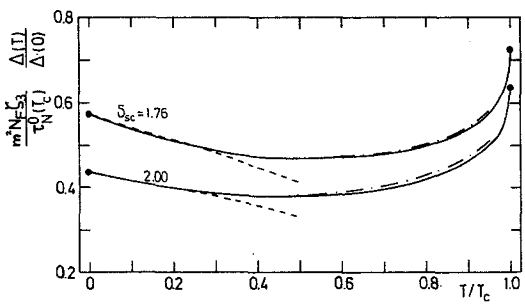

Fig. 10. Normalized second viscosity ζ_3 of the BW state vs. reduced temperature. Solid lines are obtained from a numerical evaluation of Eq. (74) with strong coupling parameters predicted by Bloyet *et al.*[37] and scattering parameters from Pfitzner and Wölfle[26] appropriate for 0 bar (upper curve) and 20 bar (lower curve). Dashed lines indicate the corresponding exact asymptotic results from Eq. (73). Dashed-dotted lines are obtained from (74) with scattering parameters for isotropic scattering.

function of reduced temperature. The solid lines are the SKA results obtained with scattering parameters from Pfitzner and Wölfle[26] (see Table III) and strong coupling parameters from Bloyet *et al.*[37] appropriate for 0 bar (upper curve) and 20 bar (lower curve). These approximate results compare well with the corresponding exact asymptotic results at low temperature [cf. Eq. (73)], which appear as dashed lines in Fig. 10. The reduced form $m^2 N_F \zeta_3 / \tau_N^0(T_c)$ of the second viscosity is quite insensitive to the pressure-dependent input of the scattering parameters (γ_0, δ_0). So the dashed-dotted lines correspond to the SKA results obtained with the assumption of isotropic scattering. Therefore the pressure dependence of the second viscosity ζ_3 is seen to be dominated by, besides strong coupling gap renormalization, that of the effective mass (entering via N_F) and by $\tau_N^0(T_c)$, which decreases by roughly an order of magnitude if one increases the pressure from 0 to 20 bar, say.

Recently, Carless *et al.*[19] have published the first sets of experimental data on the second viscosity at very low pressure. In Fig. 11 their data taken at 1.28 bar (circles) are compared with our theory. We have plotted ζ_3 as a function of reduced temperature at 1.28 (upper curve) and at 21 bar (lower curve). At 1.28 bar we used $\tau_N^0(T_c) = 3.22 \times 10^{-7}$ sec and $N_F = 10.82 \times 10^{38}\ (\text{erg cm}^3)^{-1}$ as proposed in Ref. 19. At 21 bar we took $\tau_N^0(T_c) = 5.4 \times 10^{-8}$ sec and $N_F = 2.04 \times 10^{38}\ (\text{erg cm}^3)^{-1}$. The agreement of the experimental points with the theoretical curve is good in general, the scatter of the data being, however, quite large close to T_c. The second viscosity at 21 bar is predicted to be roughly an order of magnitude smaller than at 1.28 bar, basically due to the strong pressure dependence of $\tau_N^0(T_c)$, and is therefore probably more difficult to determine experimentally.

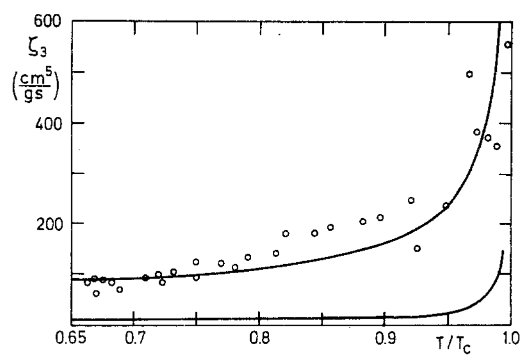

Fig. 11. Second viscosity ζ_3 of ^{3}He-B at 1.28 bar (upper curve) and 21 bar (lower curve) vs. reduced temperature. Open circles refer to the vibrating wire data at 1.28 bar of Carless *et al.*[19]

Finally, it should be noted that our theoretical second viscosity has been successfully used by Brand and Cross[42] in their explanation of the anomalously large U-tube flow dissipation observed by Eisenstein and Packard.[43]

6.5. Diffusive Thermal Conductivity

This section is devoted to the thermal conductivity of superfluid ^{3}He-B. In the low-temperature regime this quantity is given by [cf. Eq. (60d), Table II]

$$\kappa = \frac{n}{m^* T}(2kT_c)^2\tau_\kappa, \qquad \tau_\kappa = \frac{\delta_{sc}}{4}\tau_s\left(1 + c_\kappa \frac{kT}{\Delta}\right) \tag{76}$$

In contrast to the expressions for the viscosities [Eqs. (70) and (73)], the low-temperature result for the thermal conductivity multiplied by the absolute temperature is independent of gap renormalization effects because the parameter δ_{sc}^{-2} precisely drops out, as seen from Eq. (76):

$$\lim_{T\to 0} \tau_\kappa = \frac{\pi}{6w_0}\tau_N^0(T_c) \tag{76a}$$

Therefore the pressure dependence of the low-temperature thermal conductivity should be expected to exclusively depend on the normalized quasiparticle scattering cross section w_0, defined by Eq. (53b).

According to Eq. (60d) (Table II), the coefficient of the first-order correction c_κ in (76) is given by

$$c_\kappa = -4c_q + 3 + (\lambda_1^- - 5\gamma_1/3)/w_0 \tag{76b}$$

We explicitly give the analytical result for c_κ obtained with the assumption of isotropic scattering

$$c_\kappa = 11/14 \tag{76c}$$

which agrees with the result of Pethick *et al.*[5]

An inspection of Table III shows that the low-temperature slope c_κ does not vary much with pressure and that isotropic scattering is certainly a bad approximation for the evaluation of this quantity. Furthermore, Table III shows values for the scattering parameters λ_1^-, γ_1 obtained by Hara[11] at 21 bar from *spd* effective potentials with unknown Landau parameters treated as free parameters. The row labeled "C" ("D") gives results for the scattering parameters determining the thermal conductivity in the case where the variation of the free Landau parameters leads to a maximum (minimum) in the variational $\kappa(T=0)$.

Next we compare the exact low-temperature behavior of the thermal conductivity with the SKA result derived in I, which can be written as

$$\tau_\kappa^{\mathrm{SKA}} = \tau_{\mathrm{T}1} + \frac{\lambda_1^-}{3}\left(\frac{T}{T_c}\right)^2 \left[\frac{1}{T_{\mathrm{Q}2}} \frac{\Delta^2}{(2kT)^2} \frac{Y_0}{Y_2'} + \left(1 - \frac{\lambda_1^-}{3}\right)\frac{1}{\tau_{\mathrm{T}2}}\right]^{-1} Y_2' \quad (77)$$

Here we have introduced

$$Y_n'(T) = \langle (\xi/2kT)^n \rangle_\varphi$$

Expanding $\tau_\kappa^{\mathrm{SKA}}$ at low temperatures to first order in kT/Δ leads to the result

$$\tau_\kappa^{\mathrm{SKA}} = \frac{\delta_{\mathrm{sc}}^2}{4}\tau_s\left(1 + c_\kappa^{\mathrm{SKA}}\frac{kT}{\Delta}\right) \quad (78)$$

with

$$c_\kappa^{\mathrm{SKA}} = c_\kappa^{\mathrm{exact}} - \frac{\lambda_1^- - \frac{5}{3}\gamma_1}{w_0} + \frac{\lambda_1^-}{3} \quad (78a)$$

In Fig. 12 we collect various theoretical results for the normalized quantity $\kappa T/\kappa_c T_c$ as a function of reduced temperature.

The four solid lines are the SKA results for different scattering parameters. According to Table III, $\lambda_1^- = 1/3$ represents the approximation of isotropic scattering, $\lambda_1^- = 1.19$ and 1.33 are the parameters obtained by Pfitzner and Wölfle for 0 and 21 bar, respectively, and 1.92 is the value given by Hara ("D"). For isotropic scattering the upper dashed line shows the influence of changing δ_{sc} from 1.76 to 2. The two dashed lines merging into the 0- and 21-bar curves in the middle are the corresponding exact results evaluated with Eq. (76). Finally, the two dashed-dotted lines show the variational results labeled "C" (upper curve) and "D" (lower curve) of Hara.[11]

The thermal conductivity of the B phase generally varies smoothly with temperature. The question of whether the thermal conductivity κT goes up or down below T_c is seen to crucially depend on the input for the (normal state) scattering parameter λ_1^-.

The agreement of our result obtained within the SKA with the variational calculation of Hara (shown here only for $\lambda_1^- = 1.92$) is quite good in the whole temperature range.

At low temperatures, the SKA apparently underestimates the slope of κT a bit, as can be seen by comparing with the exact results.

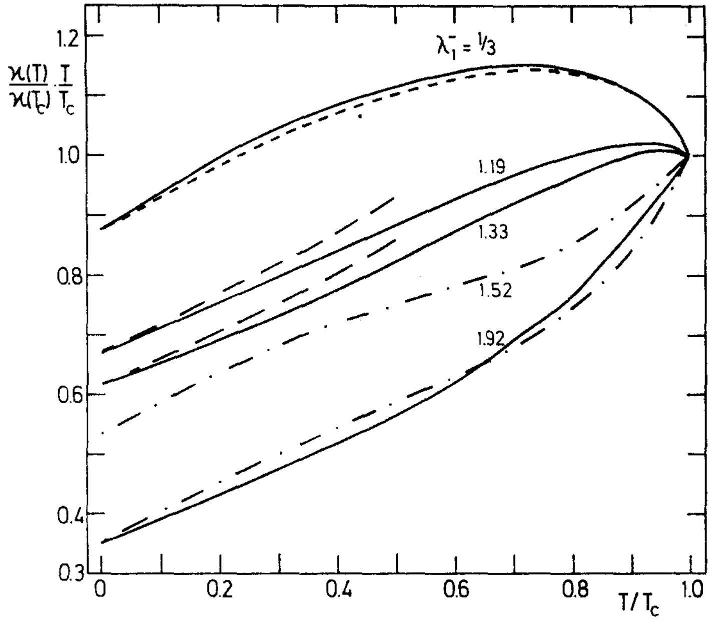

Fig. 12. Normalized thermal conductivity of the BW state vs. reduced temperature. Solid lines and the upper dotted line are the SKA results obtained from Eq. (77) for different values of λ_1^- and δ_{sc} as explained in the text. The two dashed lines are exact asymptotic results from Eq. (76). The dashed-dotted lines show variational results of Hara[11] for comparison.

The influence of strong coupling corrections to the zero-temperature gap on the thermal conductivity is quite weak, as can be seen from the two upper curves, and can be neglected for practical purposes.

Very recently the diffusive thermal conductivity was observed for the first time experimentally by Hook and co-workers[20] using heat pulse techniques. The preliminary result of their experiment at 21 bar is compared with various theoretical curves obtained within the SKA in Fig. 13. We plot the product κT, normalized to its value at T_c vs. reduced temperature. Crosses stand for the experimental points; different theoretical curves are obtained for different scattering parameters λ_1^-. The experimental curve is seen to first go up below T_c and to fall at lower temperatures with a slope much larger than predicted by theory.

If the slight increase of the experimental κT below T_c can be taken seriously, the data are clearly compatible with a scattering parameter $\lambda_1^- \approx 0.9$. This is in contrast to the predictions of Pfitzner and Wölfle ($\lambda_1^- = 1.33$) and Hara ($1.52 < \lambda_1^- < 1.92$) for a pressure of 21 bar.

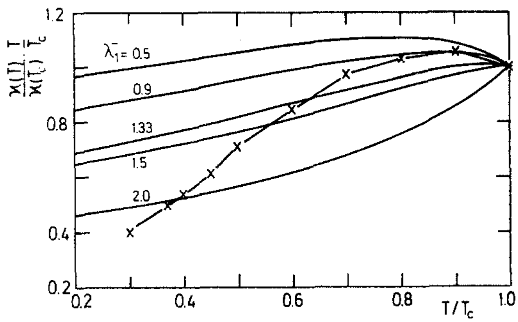

Fig. 13. Normalized thermal conductivity of ^{3}He-B plotted vs. T/T_c. Solid lines are SKA results for parameter values λ_1^- as indicated. Crosses are heat pulse data of Hook and co-workers.[20]

The physical origin of the discrepancy between theory and experiment at lower temperatures is not known. It may, however, have an explanation similar to the slip effect in the context of Poiseuille flow. More experiments at different pressures are in preparation and there is hope that they will contribute to clarifying our physical understanding of the thermal conductivity.

6.6. Conclusion

In this paper we have calculated all relevant spin-independent hydrodynamic transport parameters of the infinitely extended superfluid ^{3}He-B.

The results of I have been completed by the derivation of the exact form of the one-dimensional transport integral equations from which these parameters originate, and the exact solution of these equations at low but *finite* temperatures. The exact results compare well with those obtained in I, where a separable kernel approximation (SKA) for the collision operator has been applied. The SKA results for shear viscosity and thermal conductivity are in good agreement with the results of variational calculations. Thus the SKA is a well-controlled approximation at all temperatures, which in addition has the advantage of describing the temperature dependence of the resulting transport parameters by only a few well-defined relaxation times and the pressure dependence by only one scattering parameter (λ_2^+, shear viscosity; λ_1^-, thermal conductivity) and the gap at zero temperature $\delta_{sc} = \Delta(0)/k_B T_c$.

A comparison of our theory with the experimental data on the shear viscosity η shows that there are two significant temperature regimes. In the regime of higher temperatures the agreement is very good in all cases. At lower temperatures, in the "droop" regime, the experimental viscosities deviate systematically toward lower values and thus fail to show the minimum predicted by theory. We believe that this discrepancy is due to a surface effect, in that it crucially depends on the assumptions on the detailed scattering processes undergone by the thermal excitations at the surface of the container. These problems have to be left for further investigations. Our results for the second viscosity ζ_3 have now been confirmed by experiment. There is also a preliminary measurement on the diffusive thermal conductivity κ, which indicates that the product κT displays a maximum below T_c before it goes down much more rapidly as expected from the theoretical point of view.

There is clearly need for more experiments, in particular to clarify the role of the surface in different geometries. On the theoretical side, effort has to be made in understanding quasiparticle scattering processes from the solid surface.

With the role of the surface in hydrodynamic experiments sufficiently known, our theory as presented in this paper can be used to extract information on important properties, such as the energy spectrum of thermal excitations and the interactions of quasiparticles in the infinitely extended superfluid.

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