Journal of Low Temperature Physics, Vol. 67, Nos. 3/4, 1987

Singular Behavior of the Superfluid ^{3}He-A at $T = 0$ and Quantum Field Theory*

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(Received November 21, 1986)

The interaction of the fermionic quasiparticles with bosonic collective modes in ^{3}He-A recalls the interaction of massless chiral fermions with photons, W-bosons, and gravitational waves in high-energy physics. The chiral anomaly and vacuum polarization are responsible for singular dynamics of ^{3}He-A at $T = 0$.

1. INTRODUCTION

There are a number of peculiarities in the low-temperature dynamics of superfluid ^{3}He-A, which are now understood to have a strong analogy with phenomena intensively studied in particle physics. The basis of the analogy is the fact that the low-temperature behavior of this liquid is defined by quantum field theory, which describes the fermionic quasiparticles (the chiral fermions, which have no mass, due to the gapless spectrum in ^{3}He-A) interacting with bosonic fields (there are at least 18 collective bosonic excitations in both phases of superfluid ^{3}He, corresponding to the oscillations of the order parameter, 3×3 complex matrix $A_{\alpha i}$). Among the bosonic fields there are the collective modes, which recall the photons, W-bosons, and gravitons in particle physics both by their influence on fermionic excitations and by their Lagrangian.

All the peculiarities are the consequence of the zero mass of the fermions resulting from the cancellation of the gap at two points (nodes or boojums) on the Fermi surface at $\mathbf{k} = \pm k_F \mathbf{l}$, where $\mathbf{l}$ is the direction of the orbital angular momentum of Cooper pairs in ^{3}He-A. In particular:

1. This gives rise to the nonanalytical, logarithmically divergent free energy gradient expansion,[1] which has a direct analogy[2,3] with the cancellation of the electronic charge in quantum electrodynamics (QED) due to

*This paper was prepared for the Banff Conference on Quantum Fluids and Solids, October 13–17, 1986.

0022-2291/87/0500-0301$05.00/0

vacuum polarization.[4] The coupling constant between the chiral ^{3}He-A fermions (electrons) and orbital waves (oscillations of the **l** vector, which correspond to photons in QED) drops logarithmically at large distance.

2. The nonconservation of the superfluid current at $T=0$[5,6] results from the chiral anomaly[7] describing the creation of the chiral current from the vacuum, which is possible due to the massless spectrum of fermions. In the language of ^{3}He-A, this anomaly means that the fermionic quasiparticles are created in the process of the dynamics of the ^{3}He-A superfluid vacuum and as a result the vacuum linear momentum transfers to the momentum of excitations. This momentum transfer is regulated by the Schwinger equation[8] for the source of the chiral current J_5 in QED:

$$\partial_\mu J_5^\mu = (e^2/16\pi^2)\, e^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The corresponding gauge field in ^{3}He-A is $\mathbf{A} = k_\mathrm{F}\mathbf{l}$.[6]

3. The same chiral anomaly gives rise to the anomalous supercurrent

$$\mathbf{j}_{\mathrm{an}} = -\tfrac{1}{2} C_0 \mathbf{l}(\mathbf{l} \cdot \mathrm{rot}\, \mathbf{l}), \qquad C_0 = k_\mathrm{F}^3/3\pi^2 \tag{2}$$

which for a long time was the stumbling block on the path to a derivation of a closed system of dynamical equations for ^{3}He-A at $T=0$.

4. The gapless spectrum of fermions gives a nonanalytic, nonzero density of states on the Fermi surface, which produces a nonzero normal density at $T=0$ and a specific heat linear in T in the presence of **l** texture.[5,6,9]

5. The strange behavior of the internal angular momentum in ^{3}He-A (see, e.g., Ref. 5) is also related to the gapless spectrum. This comes from the possibility of the easy transformation of the vacuum angular momentum of Cooper pairs into the angular momentum of fermionic excitations.[10] In the language of QED, this corresponds to the easy creation of electrons and positrons from vacuum if the fermions are massless (on the creation of electron–position pairs in a strong electric field see Ref. 11).

These analogies between ^{3}He-A and particle physics allow one to apply the powerful methods of particle physics to ^{3}He-A. On the other hand, ^{3}He-A can serve as a testing ground for modern theory in high-energy physics. In particular, in spite of the analogy[3] between ^{3}He-A and the standard model of electroweak interaction,[12] the coupling between the W-bosons (spin-orbital waves) and fermions in ^{3}He-A does not display the asymptotic freedom[3] that takes place in the standard model. Also, the origin of the mass of the W-boson is different in these field theories. In addition, the ^{3}He-A dynamics can choose between different possible theories of gravitation. The dynamics of the collective clapping mode, which corresponds to the gravitational wave in Einstein theory, proves to be more consistent with one of the modifications of the theory.[13]

2. FERMIONS AND BOSONS IN ^{3}He-A

The fermionic quasiparticle spectrum in superfluid ^{3}He is obtained from the Bogoliubov equation

$$E\psi = \begin{pmatrix} v_F(k-k_F) & \hat{\Delta} \\ \hat{\Delta}^+ & -v_F(k-k_F) \end{pmatrix} \psi \tag{3}$$

where the spinor matrix $\hat{\Delta}$ is expressed in terms of the order parameter matrix $A_{\alpha i}$:

$$\hat{\Delta} = \hat{g}\hat{\sigma}_\alpha A^i_\alpha k_i/k_F, \qquad \hat{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

($\hat{\sigma}_\alpha$ are the Pauli matrices).

For the equilibrium A-phase state, which is the eigenstate of the spin projection $S_z = 0$ on the spin quantization axis $\mathbf{d}$ and of the orbital momentum projection $L_z = 1$ on the orbital quantization axis $\mathbf{l}$, the order parameter is factorized as

$$A^{(0)i}_\alpha = \Delta_0 d_\alpha (\Delta^i_1 + i\Delta^i_2) \tag{5}$$

where $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ are unit orthogonal vectors with $\mathbf{l} = \boldsymbol{\Delta}_1 \times \boldsymbol{\Delta}_2$, and Δ_0 is the gap amplitude. This results in the following fermionic spectrum in equilibrium:

$$E^2 = v_F^2(k-k_F)^2 + (\Delta_0/k_F)^2(\mathbf{k}\times\mathbf{l})^2 \tag{6}$$

which becomes to zero at two points, $\mathbf{k} = \pm k_F\mathbf{l}$. The low-energy excitations in the vicinity of these two nodes define the main low-temperature properties of ^{3}He-A.

Let us find which bosonic excitations of the order parameter are coupled with these two species of low-energy fermions. Here we consider for simplicity deviations of the order parameter from the equilibrium value $A^{(0)}_{\alpha i}$ in Eq. (5) that do not change its spin structure:

$$A^i_\alpha = d_\alpha(e^i_1 + ie^i_2)k_F \tag{7}$$

Here $\mathbf{e}_1$ and $\mathbf{e}_2$ are arbitrary vectors, which in equilibrium are given by

$$\mathbf{e}^{(0)}_1 = \Delta_0\boldsymbol{\Delta}_1/k_F = \Delta_0\hat{x}/k_F, \qquad \mathbf{e}^{(0)}_2 = \Delta_0\boldsymbol{\Delta}_2/k_F = \Delta_0\hat{y}/k_F$$

(x, y, z or x^1, x^2, x^3 are Cartesian coordinates).

The nodes in the spectrum in the case of the order parameter in Eq. (7) are again at $\mathbf{k} = \pm k_F\mathbf{l}$, if $\mathbf{l}$ is defined as $\mathbf{l} = \mathbf{e}_1 \times \mathbf{e}_2/|\mathbf{e}_1 \times \mathbf{e}_2|$. In the vicinity of the nodes the spectrum of fermions may be written in the following covariant form:

$$E^2 = g^{ij}(k_i - eA_i)(k_j - eA_j) \tag{8}$$

where the metric tensor g^{ij} is expressed in terms of "tetrads" e_1^i, e_2^i, and $e_3^i = v_F l^i$:

$$g^{ij} = \sum_a e_a^i e_a^j \tag{9}$$

while the gauge field $\mathbf{A}$ is expressed in terms of $\mathbf{l}$:

$$\mathbf{A} = k_F \mathbf{l} \tag{10}$$

and the charge e takes the value $+1$ for fermions near the north pole of the Fermi sphere and -1 for the opposite pole. In equilibrium

$$g^{(o)ij} = c_\parallel^2 l^i l^j + c_\perp^2(\delta^{ij} - l^i l^j), \qquad c_\parallel = v_F, \qquad c_\perp = \Delta_0/k_F \tag{11}$$

Equation (8) corresponds to the spectrum of a massless charged particle in gravitational and electromagnetic fields. The correspondence between the collective modes of the order parameter oscillations and the bosonic modes in particle physics is given in Table I, where Q is one of the quantum numbers[16] for bosons in ^{3}He-A and corresponds to spin projection in particle physics. Six modes (orbital waves, clapping modes, sound, and pseudosound) are oscillations of vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. The other 12 modes are related to the deviation of the order parameter $A_{\alpha i}$ from Ansatz (7); four of them, the W-bosons, are shown explicitly and discussed in Section 6, while eight correspond to the generalization of gravitation with tetrads depending on the isospin, these modes giving rise to additional degeneracy (numbers in parentheses in Table I) in a weak coupling approximation. On the collective modes in ^{3}He-A see Refs. 14–16. The modes related to gravitation are discussed in the next section.

TABLE I

Collective Modes in ^{3}He-A and Their Analog in Particle Physics

				Particle physics	
Mode	^{3}He-A Variable	Degeneracy	Q	Variables	Mode
Orbital waves	$\mathbf{l}$	2	± 1	$\mathbf{A}, g^{13}, g^{23}$	Photons
Clapping modes	$e_1^y + e_2^x$, $e_1^x - e_2^y$	2 (6)	± 2	$g^{12}, \frac{1}{2}(g^{11} - g^{22})$	Gravitational waves
Pseudosound	$e_1^x + e_2^y$	1 (3)	0	$\frac{1}{2}(g^{11} + g^{22})$	Additional gravitons
Sound	$e_1^y - e_2^x$	1 (3)	0	—	Torsion wave
Spin-orbital waves	A_{13}, A_{23}	4	± 1	$\mathbf{W}_\alpha$	W-bosons

3. GRAVITATIONAL WAVES

Only three components of the metric tensor g^{ij} are dynamical variables producing the collective oscillations: g^{12}, g^{11}, g^{22}. The components g^{13} and g^{23} correspond to vector **l** oscillations and therefore may be incorporated in the **A** field of the photon. The Lagrangian describing the gravitational waves propagating for simplicity along **l** is as follows[15]:

$$L_G = \frac{N_F}{32}\left\{\sum_{a=1}^{4}\left[\frac{1}{3}c_\parallel^2(\partial_3 f_a)^2 - \left(\frac{\partial f_a}{\partial t}\right)^2\right] + \frac{4}{3}\Delta_0^2(f_1^2+f_2^2+2f_3^2)\right\} \tag{12}$$

$$f_1 = (g^{11}-g^{22})/2g^{(0)11}, \qquad f_2 = g^{12}/g^{(0)11}$$

$$f_3 = (g^{11}+g^{22})/2g^{(0)11} - 1$$

Here N_F is the density of states, the massless variable f_4 corresponds to the sound wave: this is the torsion wave, the oscillations of tetrads without change in g^{ij} (see next section).

The first two waves, the clapping modes, correspond to the ordinary gravitational waves in Einstein theory with the spin projection $Q=\pm 2$. The third wave, pseudosound, with $Q=0$, has no analogy in general relativity. All the gravitons are massive, since the clapping mode and pseudosound are not Goldstone modes. It is interesting that the mass term in Eq. (12) may be considered as the cosmological terms in Einstein theory modified in Ref. 13 in the following way:

$$\begin{aligned} L_{\text{cosm}} &= \Lambda(-g)^{1/2}(\tfrac{1}{2}g_{\mu\nu}^{(0)}g^{\mu\nu} - 1) \\ &\approx \Lambda(-g^{(0)})^{1/2}[1+\tfrac{1}{2}(f_1^2+f_2^2+2f_3^2)] \end{aligned} \tag{13}$$

where $\Lambda = \Delta_0^4/12\pi^2$ and $g_{\mu\nu}^{(0)}$ is the equilibrium metric tensor corresponding to Minkowski space. As distinct from the ordinary cosmological term, $\Lambda(-g)^{1/2}$, Eq. (13) does not lead to the gravitating mass of the equilibrium vacuum.

4. CHIRAL FERMIONS

The massless fermions are described by the Weyl equation, which is the square root of Eq. (8) and may be obtained from the Bogoliubov equation. Using the order parameter with fixed spin structure, Eq. (7), one obtains two Weyl equations for fermions near two nodes[2,3]:

$$\{i\partial_t - \tfrac{1}{2}\tau^a[e_a^j(-i\partial_j - A_j) + (-i\partial_j - A_j)e_a^j]\}\eta = 0 \tag{14a}$$

$$\{i\partial_t + \tfrac{1}{2}\tau^a[e_a^j(-i\partial_j + A_j) + (-i\partial_j + A_j)e_a^j]\}\xi = 0 \tag{14b}$$

Here $\tau^a = (\tau^1, \tau^2, \tau^3)$ are the Pauli matrices corresponding to the Bogoliubov spinors, as distinct from the Pauli matrices σ_α, which describe an ordinary spin. The wave function η describes the fermions near the north pole of the Fermi sphere ($\mathbf{k} \sim k_F \mathbf{l}$). These particles have positive electric charge, $e = 1$, and right chirality, while ξ corresponds to particles near the south pole ($\mathbf{k} \sim -k_F \mathbf{l}$) with $e = -1$ and left chirality.

Equation (14) may be written in covariant form, e.g., Eq. (14a) transforms to ($a = 0, 1, 2, 3$; $\alpha = 0, 1, 2, 3$)

$$e_a^\alpha \gamma^a \nabla_\alpha \eta = 0, \qquad \gamma^a = (1, \tau^1, \tau^2, \tau^3)$$
$$\nabla_\alpha = \partial_\alpha + \tfrac{1}{8}\omega_{\alpha,ab}[\gamma^a, \gamma^b] - i\tilde{A}_\alpha \tag{14c}$$

where the connection

$$\omega_{\alpha,ab} = e_a^\nu(\partial_\alpha e_{\nu b} - \Gamma^\mu_{\alpha\nu} e_{\mu b})$$

is expressed in terms of the Christoffel symbol Γ and the gauge field $\mathbf{A}$ is modified by including the torsion tensor $A_{\gamma,\mu\nu}$:

$$\tilde{A}_\alpha = A_\alpha + \tfrac{1}{8} g_{\alpha\beta} E^{\beta\gamma\mu\nu} A_{\gamma,\mu\nu}, \qquad A_{\gamma,\mu\nu} = e_\gamma^a(\partial_\mu e_{\nu a} - \partial_\nu e_{\mu a})$$

As distinct from the standard Einstein theory, the torsion field is the dynamical variable giving rise to the sound wave (see Table I).

Equations (14) may be generalized to include the deviation of the order parameter $A_{\alpha i}$ from the Ansatz (7), i.e., to take into account the spin structure of the order parameter. Here we consider only four additional collective variables, A_{13} and A_{23}. They form the so-called spin-orbital waves, which are gapless in a weak coupling approximation.[16] These fourfold-degenerate modes correspond to W-bosons in the standard theory of electroweak interaction, since Eqs. (14) become as follows [we shall write only Eq. (14a)]:

$$\{i\partial_t - \tfrac{1}{2}\tau^a[e_a^j(-i\partial_j - A_j - \sigma_\alpha W_j^\alpha) + (-i\partial_j - A_j - \sigma_\alpha W_j^\alpha)e_a^j]\}\eta = 0 \tag{15}$$

Now we use Eqs. (14) and (15) for the investigation of the ^{3}He-A singular behavior at low temperature. The classical Bose fields of the order parameter, corresponding to the electromagnetic, gravitational, and W fields, influence through these equations the fermionic vacuum, producing the vacuum polarization and anomalies. The effect of the Bose fields on the vacuum is crucial due to the zero mass of fermions.

5. NONANALYTIC GRADIENT EXPANSION AND THE CANCELLATION OF ELECTRIC CHARGE

It is well known in QED[4] that the electric charge is screened by the polarization of the fermionic vacuum. If the fermions are massless, the screening is complete, i.e., the electronic charge e_{eff} logarithmically drops to zero at large distances (or at small frequency):

$$e_{\text{eff}}^2 = 3\pi/\ln(\Lambda^2/\omega^2) \tag{16}$$

where Λ is the ultraviolet cutoff parameter.

This may be directly applied to ^{3}He-A. At large $\ln(\Lambda^2/\omega^2)$ the electromagnetic Lagrangian is overwhelmingly defined by the vacuum polarization term

$$L_{\text{em}} = (-g)^{1/2} F_{\mu\nu} F^{\mu\nu} / 16\pi e_{\text{eff}}^2 \tag{17}$$

which is covariant, since it is defined by the electron–photon interaction, described by the covariant equations (14). Substituting

$$F^{\mu\nu} = g^{\mu\eta} g^{\nu\lambda} F_{\eta\lambda}, \qquad \mathbf{A} = k_{\text{F}} \mathbf{l}, \qquad \Lambda = \Delta_0 \tag{18a}$$

and the equilibrium metric tensor

$$g^{(0)00} = -1, \qquad g^{(0)33} = c_\parallel^2, \qquad g^{(0)11} = g^{(0)22} = c_\perp^2, \qquad (-g^{(0)})^{1/2} = (c_\parallel c_\perp^2)^{-1} \tag{18b}$$

and using $c_\perp \ll c_\parallel$, one obtains from the Eqs. (17) and (16) the leading term in the Lagrangian, describing the dynamics of the $\mathbf{l}$ vector:

$$L_{\text{em}} = \frac{k_{\text{F}}^2 v_{\text{F}}}{24\pi^2} \ln \frac{\Delta_0^2}{\omega^2} \left[(\mathbf{l} \times \operatorname{rot} \mathbf{l})^2 - \frac{(\partial_t \mathbf{l})^2}{v_{\text{F}}^2} \right] \tag{19a}$$

The first term in the square brackets of this logarithmically divergent Lagrangian has been obtained by Cross,[1] while the second one corresponds to the logarithmically divergent orbital susceptibility.[17]

In the static case the calculation of the $\mathbf{l}$ texture energy is equivalent to that of the magnetic energy of the fermionic vacuum in the presence of the magnetic field $\mathbf{B} = k_{\text{F}} \operatorname{rot} \mathbf{l}$. Applying the procedure described in Ref. 18 to the case of the anisotropic fermions, one obtains the "magnetic" energy

$$F_m = \frac{k_{\text{F}}^2 v_{\text{F}}}{24\pi^2} (\mathbf{l} \times \operatorname{rot} \mathbf{l})^2 \ln \frac{\Delta_0}{v_{\text{F}} |\mathbf{l} \times \operatorname{rot} \mathbf{l}|} \tag{19b}$$

which also may be found directly from Eq. (19a), where ω should be substituted by the "Larmor" frequency[6]

$$\omega_{\text{L}}^2 = \Delta_0 v_{\text{F}} |\mathbf{l} \times \operatorname{rot} \mathbf{l}| = c_\perp c_\parallel |\mathbf{B}_\perp|$$

This nonanalytic term in the gradient expansion is in accordance with a suggestion made in Ref. 19.

6. PHOTONS AND W-BOSONS

The oscillations of the vector $\mathbf{l}$, the orbital waves, may be obtained from Eq. (19a). These photons are extremely anisotropic since $g^{(0)33} \gg g^{(0)11} = g^{(0)22}$ and their spectrum is

$$\omega_{\mathrm{ph}}^2 = c_\parallel^2 (\mathbf{q} \cdot \mathbf{l})^2, \qquad c_\parallel = v_{\mathrm{F}} \tag{20}$$

It is interesting to find also the coupling constant of the fermions with W-bosons. The calculation of the vacuum energy in the presence of the "colored" magnetic field $F_{ij}^\alpha = \partial_i W_j^\alpha - \partial_j W_i^\alpha$ shows that the leading term in the Lagrangian for W-bosons has the same structure as Eq. (17):

$$L_W = (-g)^{1/2} F_{\mu\nu}^{(\alpha)} F^{(\alpha)\mu\nu} / 16\pi e_{\mathrm{eff}}^2 \tag{21}$$

This means that there is the cancellation of the "weak" charge[3] in ^{3}He-A instead of the asymptotic freedom predicted in the standard model of electroweak interaction. The difference comes from the different vacuum structure in these field theories: the ^{3}He-A vacuum is of pure fermionic origin and does not contain the zero-point oscillations of the bosonic fields, which produce an additional magnetic energy of the vacuum in the standard model.

The spectrum of W-bosons found from Eq. (21) is the same as that of photons. However, this is valid only in the weak coupling approximation. The strong coupling corrections result in the mass

$$\omega_W^2 = c_\parallel^2 (\mathbf{q} \cdot \mathbf{l})^2 + m_W^2, \qquad m_W^2 \sim \Delta_0^2 \delta \tag{22}$$

where δ is the so-called spin-fluctuation parameter, which is responsible for the strong coupling effect; $\delta \ll 1$ at low pressure.

It is interesting that the origin of the mass of the W-bosons in ^{3}He-A is quite different from that in electroweak theory. In ^{3}He-A the W-bosons are massless in the weak coupling approximation due to hidden symmetry[16] and the mass m_W appears due to strong coupling corrections violating the symmetry. In the standard model of electroweak interaction the mass of W-bosons appears due to spontaneously broken symmetry as result of the Higgs phenomenon.

7. CHIRAL ANOMALY AND MOMENTUM CONSERVATION LAW

Another consequence of the chiral and massless nature of the fermions in ^{3}He-A is the chiral anomaly[7,18]—the nonconservation of the fermionic current due to the influence of gauge and gravitational fields on the fermionic vacuum. The source of the fermionic chiral current is given by the Schwinger equation (1) if one neglects the anomaly caused by gravitation, which is of higher order in the order parameter gradients (see below). The source on the the right-hand side of Eq. (1) is a pure derivative; thus, the total current of particles plus vacuum is conserved:

$$\partial_\mu(J_5^\mu + J_{\text{vac}}^\mu) = 0, \qquad J_{\text{vac}}^\mu = -(1/4\pi^2)\, e^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \tag{23}$$

However, J_{vac}^μ is not gauge-invariant, i.e., the fermionic vacuum violates the gauge invariance.

Just the same situation with the nonconservation of the mass current exists in ^{3}He-A. The vacuum supercurrent, i.e., the superfluid mass current at $T=0$,

$$\mathbf{j} = \rho \mathbf{v}_s + \tfrac{1}{2}\,\mathrm{rot}(\tfrac{1}{2}\rho \mathbf{l}) - \tfrac{1}{2} C_0 \mathbf{l}(\mathbf{l}\cdot \mathrm{rot}\,\mathbf{l}) \tag{24}$$

is not conserved. There is a source of the vacuum current (or the source of linear momentum)

$$\partial_t j_i + \partial_j \pi_{ij} = -\tfrac{3}{2} C_0 l_i (\partial_t \mathbf{l}\cdot \mathrm{rot}\,\mathbf{l}) \tag{25}$$

This was interpreted[5,6] as the transfer of the vacuum linear momentum $\mathbf{j}$ to the momentum $\mathbf{P}$ of excitations with $\mathbf{P}+\mathbf{j}$ being conserved. Thus, there arises the analogy between $\mathbf{P}, \mathbf{j}$ in ^{3}He-A and J_5^0, J_{vac}^0 in particle physics.

This analogy is exact. The J_5^0 component of the chiral current is the density of the right particles minus the density of the left particles:

$$J_5^0 = \eta^+\eta - \xi^+\xi \tag{26}$$

Since the right particles have momentum $+k_{\mathrm{f}}\mathbf{l}$ and the left particles $-k_{\mathrm{F}}\mathbf{l}$, the momentum of excitations in ^{3}He-A is related to J_5^0:

$$\boldsymbol{P} = k_{\mathrm{F}} \mathbf{l} J_5^0 \tag{27}$$

Therefore, using Eq. (23) and neglecting the higher order terms, one obtains the equation for $\mathbf{P}$:

$$\partial_t P_i + \partial_j \tilde{\pi}_{ij} = \frac{k_{\mathrm{F}}}{16\pi^2} l_i F_{\mu\nu} F_{\alpha\beta}\, e^{\mu\nu\alpha\beta} = \frac{k_{\mathrm{F}}^3}{2\pi^2} l_i (\partial_t \mathbf{l}\cdot \mathrm{rot}\,\mathbf{l}) \tag{28}$$

where we inserted the gauge field $\mathbf{A} = k_{\mathrm{F}}\mathbf{l}$. The right-hand side of Eq. (28) is just the source of the vacuum current in Eq. (25) with the opposite sign; thus, the total linear momentum of liquid, $\mathbf{P}+\mathbf{j}$, is conserved.

The momentum transfer from vacuum to excitations (or to the normal component, if the system of excitations is in a local equilibrium) is well known in ordinary superfluids, such as ^{4}He, if quantized vortices are present in the liquid. Such a vortex is a mediator in the process of transfer of the momentum by means of the Magnus forces. In ^{3}He-A the mediator is also the vortex, but the vortex in k space, i.e., the boojum on the Fermi surface where the superfluidity is broken (the gap is zero) as in the core of a quantized vortex. Also as in the vortex, phase winding occurs around the node in the gap (see, e.g., Ref. 10). Thus, the source of linear momentum in Eq. (25) may be considered as a local Magnus force. The global Magnus force acting on the vortices in ^{3}He-A which are nonsingular may be obtained by integrating the local Magnus force over the cross section of the vortex.

The "gravitational" field also contributes to the chiral anomaly[20]:

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2} e^{\mu\nu\alpha\beta}\left(\tilde{F}_{\mu\nu}\tilde{F}_{\alpha\beta} + \tfrac{1}{24} R^{\sigma}_{\tau\mu\nu} R^{\tau}_{2\alpha\beta}\right) \tag{29}$$

and thus to the source of supercurrent. Here R is the Riemann curvature, and $\tilde{F}_{\mu\nu}$ is the modified electromagnetic field [see Eq. (14c)].

8. SUPERCURRENT AND NORMAL COMPONENT AT $T=0$

It is important that the transfer of the vacuum momentum **j** into the momentum of excitations **P** occurs in the vicinity of two boojums on the Fermi surface where the gap is zero. In this region the equations for excitations have a covariant form an therefore the result for the source of momentum in ^{3}He-A and that for the source of chiral current in QED coincide. On the contrary, expressions (24) and (23) for the vacuum currents in ^{3}He-A and QED are different. This is because the vacuum current is defined by deep vacuum levels where the Bogoliubov equation for fermions is very different from the Dirac–Weyl equation in QED.

Note that the calculation of the vacuum supercurrent in ^{3}He-A had its own problems[21] since an ordinary gradient expansion is not valid near the boojums. Nevertheless, an exact calculation of the supercurrent by the method used for the calculation of the chiral current in $2+1$ electrodynamics showed that the Eq. (24) does hold[9]: the region near boojums, where the gradient expansion is not valid, gives only a correction of higher order in the gradients to Eq. (24). This correction,[5,6,9]

$$\tilde{\mathbf{j}} \sim \rho(v_F/\Delta_0)\mathbf{l}|\mathbf{l}\times \mathrm{rot}\,\mathbf{l}|(\mathbf{l}\cdot\mathbf{v}_s) \tag{30}$$

is, however, very important, since this corresponds to nonzero density of the normal component even at $T=0$:

$$\rho_n^{ij}(T=0) \sim \rho l^i l^j (v_F/\Delta_0)|\mathbf{l}\times \mathrm{rot}\,\mathbf{l}| \tag{31}$$

This is the result of the nonzero density of states in ^{3}He-A in the presence of **l** texture, which is similar to the nonzero density of states for the massless electron in a magnetic field.[6]

Thus, the excitations are important in the ^{3}He-A dynamics even at $T=0$.

9. ANGULAR MOMENTUM PARADOX

The angular momentum paradox, the different values of the angular momentum of ^{3}He-A obtained at different approaches, is also related with the chiral anomaly. The natural value of the internal orbital momentum $\mathbf{L}$ of liquid is the density of the Cooper pairs, $\frac{1}{2}\rho$ at $T=0$, multiplied by $\hbar\mathbf{l}$, the orbital momentum of one Cooper pair:

$$\mathbf{L}=\tfrac{1}{2}\rho\mathbf{l} \tag{32}$$

However, the microscopically derived dynamics of the $\mathbf{l}$ vector corresponds to the dynamics of a very small value, $\sim(\Delta_0/\varepsilon_F)^2\mathbf{L}$, of the internal momentum (see, e.g., Ref. 5).

The resolution of this paradox is that there is a transfer of the angular momentum of the vacuum $\mathbf{L}$ into that of excitations $\mathbf{L}_{\text{exc}}$, quite similar to the linear momentum transfer. The source of the quasiparticle momentum was found in Ref. 10 by consideration of the Wess-Zumino action in ^{3}He-A:

$$\partial_t\mathbf{L}_{\text{exc}}=-\tfrac{1}{2}C_0\partial_t\mathbf{l} \tag{33}$$

In the language of particle physics this equation describes the creation of electron-positron pairs from the vacuum by an external electric field $\mathbf{E}=-k_F\partial_t\mathbf{l}$, since $\mathbf{L}_{\text{exc}}$ corresponds to $\mathbf{J}_5$: the current of the created charge particles increases in time under an electric field if the dissipation is neglected. Such easy creation is possible only if the particles are massless as in ^{3}He-A.

The same source, but with the opposite sign, should exist for the vacuum angular momentum $\mathbf{L}$ in order to conserve the total momentum $\mathbf{L}+\mathbf{L}_{\text{exc}}$:

$$\partial_t\mathbf{L}+\delta F/\delta\boldsymbol{\theta}=\tfrac{1}{2}C_0\partial_t\mathbf{l} \tag{34}$$

where $\delta F/\delta\boldsymbol{\theta}$ is the torque, with $\boldsymbol{\theta}$ the angle of rotation of the order parameter. From Eqs. (34) and (32) it follows that

$$\tfrac{1}{2}(\rho-C_0)\partial_t\mathbf{l}+\tfrac{1}{2}\mathbf{l}\partial_t\rho=-\delta F/\delta\boldsymbol{\theta} \tag{35}$$

Since $C_0=k_F^3/3\pi^2$ is the density of liquid in the normal state, the quantity

$$\tfrac{1}{2}(\rho-C_0)\sim(\Delta_0/\varepsilon_F)^2\ln(\varepsilon_F/\Delta_0)$$

involved in the $\mathbf{l}$ vector dynamics is very small, i.e., the effective dynamical momentum of Cooper pairs is in agreement with microscopic analysis.

The difference in the results of the calculations of the static angular momentum of liquid (see, e.g., Ref. 21) is also resolved if one takes into account the contribution of the excitations in the wave function of ^{3}He-A (Ref. 6 and preprint in Ref. 9).

10. CONCLUSION

Here we have considered the influence of the anomalies and vacuum polarization effects on the reversible dynamics of ^{3}He-A at $T=0$. However, for practical purposes (the investigation of the motion of quantized vortices, solitons, A–B phase boundary, etc., at low temperature) we need details of the dissipation of excitations, which always exist even at $T=0$. Thus, the motion equations for the vacuum variables $\mathbf{v}_s$, $\mathbf{l}$, and ρ should be supplemented with the kinetic equation for quasiparticles with different chirality. Only after this is done can the problem of the low-temperature dynamics of ^{3}He-A be solved.

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