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Shear Viscosity of the B Phase of Superfluid ^{3}He. III

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We report the calculation of the shear viscosity in superfluid ^{3}He-B at 6, 21, and 30 bar. We have used the variational solution of the Boltzmann equation for quasiparticles. The transition probabilities are obtained within the s–p–d-*wave approximation by fitting the normal state transport coefficients. The old and new Landau parameters are used to estimate the scattering amplitudes. The reduced shear viscosity does not depend very much on the choice of Landau parameters and agrees very well with experiment in the temperature range $0.5 < t\ (=T/T_c) \leq 1.0$, where T_c is the transition temperature. We have also investigated the strong coupling corrections to the shear viscosity. Although the strong coupling corrections reduce the value of shear viscosity, the overall feature does not change from the weak coupling result. The pressure dependence of the reduced shear viscosity is found to be very small.*

1. INTRODUCTION

Recently several new experiments[1–5] on the shear viscosity η have been reported. Archie *et al.*[1] measured the viscosity of ^{3}He-B down to $t\ (=T/T_c) \sim 0.5$ using a torsional oscillator and a vibrating wire viscometer at pressures of 5, 10, 20, and 29 bar, where T_c is the superfluid transition temperature. They extracted the hydrodynamic part of the shear viscosity from raw data using the slip-correction formula given in a paper by Højgaard Jensen *et al.*[6] This formula takes into account the effect of a long mean free path l of the quasiparticles at low temperatures up to first order in l/d, where d is the typical length of the experimental apparatus. Archie

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et al.[1] obtained good agreement between vibrating wire data and torsional oscillator data. Saunders *et al.*[2] extended the Archie experiment to lower temperatures using a torsional oscillator. These authors also used a spherical viscometer. Here the mean-free-path effect with respect to d will not play an important role in the temperature range where the experiments were done, since $d = 0.8$ cm. In this experiment, however, an $\omega\tau$ correction[6,14] (or l/δ correction) has to be taken into account, where ω is the angular frequency of the spherical viscometer and τ is the relaxation time. The viscous penetration depth δ is given by

$$\delta = (2\eta/\rho_n\omega)^{1/2} \tag{1}$$

where ρ_n is the normal fluid density.

Eska *et al.*[3] and Kodama and Kojima[4] obtained the shear viscosity from the damping of first sound. Here the $\omega\tau$ correction was very small in the temperature range where the experiments were performed. Also, vibrating-wire experiments were extended to lower temperatures by the Manchester group.[5]

All the experiments show a sharp decrease of the viscosity just below T_c, which is in agreement with theories.[7–13]* At lower temperatures, however, the viscosity data continue to decrease slowly; they do not show a plateau behavior predicted by the theories.[7–13] Only the spherical viscometer data have a plateau region in the intermediate temperature range.

On the theoretical side, we have developed a formula for calculating the transport coefficients by solving the Boltzmann equation by a variational method.[7,10,15] In Ref. 10, we obtained the scattering amplitude within the *s–p–d*-wave approximation, where we need six Landau parameters $F_0^{s,a}$, $F_1^{s,a}$, and $F_2^{s,a}$. As is well known,[16] F_0^s, F_0^a, and F_1^s can be obtained from the molar volume, specific heat, spin susceptibility, and velocity of first sound of normal liquid ^{3}He. We used an available value[17] of F_2^s and determined the remaining two Landau parameters F_1^a and F_2^a by invoking the truncated forward-scattering sum rule and by fitting the theoretical value of the viscosity at T_c to the experimental one. The result for the reduced shear viscosity $\tilde{\eta} = \eta(T)/\eta(T_c)$ in the superfluid phase was in good agreement with experiments[18,19] at 20 bar in the temperature range $0.5 \leq t \leq 1.0$.

Recently precise heat capacity measurements of liquid ^{3}He have been reported.[20,21] The effective mass ratio m^*/m derived from these experiments is drastically smaller than that tabulated in Wheatley's review[16] over

*In Ref. 3, Eska *et al.* reported that the viscosity in superfluid ^{3}He-B shows a rather slow decrease, which does not agree with other experiments. According to private communication by K. Uhlig,[53] however, they have found agreement with other experiments by resetting the computer analysis of their data.

the whole pressure range. This changes the values of the Landau parameters F_0^s, F_0^a, and F_1^s substantially, and all the transport coefficients in the normal as well as in the superfluid phase have to be recalculated using the new Landau parameters derived from the new m^*/m. (The Landau parameters tabulated in Wheatley's review[16] will be called the "old Landau parameters" in this paper.)

In this paper we discuss the effect of the new Landau parameters on the shear viscosity. It is true that the absolute values of the transport coefficients depend on the values of the Landau parameters. We have found, however, that the behavior of the reduced viscosity $\tilde{\eta} = \eta(T)/\eta(T_c)$ in the superfluid phase does not depend on the individual values of the Landau parameters but on a few sets of their combination, in particular on a parameter λ_2 which represents the backscattering effect of the shear viscosity in the normal state.[9,10,24,25] We shall show that the behavior of the reduced viscosity for the new Landau parameters within the *s–p–d* approximation is almost the same as that for the old Landau parameters[10] and also is in good agreement with recent Cornell data[2] as well as with those of Kodama and Kojima[4] in the temperature range $0.5 \leq t \leq 1.0$. We also discuss the strong coupling corrections and the pressure dependence.

The organization of this paper is as follows. In Section 2, we give the formulas for calculating the shear viscosity by a variational method and show that the behavior of the reduced shear viscosity is mainly governed by a parameter λ_2. In Section 3, we discuss the *s–p–d* approximation in detail. The calculation of $\tilde{\eta}$ at 21 bar within the weak coupling theory is given in Section 4. New as well as old Landau parameters are used. In Section 5, the strong coupling corrections are discussed, and Section 6 is devoted to a discussion of the pressure dependence of $\tilde{\eta}$. A summary and discussion are given in Section 7.

2. VARIATIONAL SOLUTION FOR SHEAR VISCOSITY

In this section we give the formulas for calculating the shear viscosity of ^{3}He-B. The variational solution of the Boltzmann equation for quasiparticles is used. (For details, see Ref. 10.) It has been shown[7,23] that at $T = 0$ the variational solution is equal to the exact solution. It also has been shown[10] that in the normal state, the viscosity calculated variationally is very close to the exact value within 3% error. Therefore, the variational solution can be expected to give a reasonable description of the shear viscosity over the whole temperature region in the superfluid phase.

The shear viscosity η is generally given by

$$\eta(T) = \tfrac{1}{5} n p_F v_F Y_2 \tau_\eta \tag{2}$$

where n is the number density, p_F is the Fermi momentum, and v_F is the Fermi velocity defined by $v_F = p_F/m^*$. The function Y_2 is the generalized Yosida function of second order, where Y_n is defined by

$$Y_n = \int_{-\infty}^{\infty} d\xi \left(-\frac{\partial f}{\partial E}\right) \left|\frac{\xi}{E}\right|^n \tag{3}$$

Here $\xi = p^2/2m^* - p_F^2/2m^*$, $E = (\xi^2 + \Delta^2)^{1/2}$, and $f = [\exp(\beta E) + 1]^{-1}$ $(\beta = 1/k_B T)$. In the variational calculation, the viscous relaxation time τ_η is given by

$$\tau_\eta = \frac{4\hbar \beta_c^2 p_F^2}{m^* \pi Y_2} \left(\sum_{i=1}^{5} \tilde{\alpha}_i \eta_i^{-1} \right)^{-1} \tag{4}$$

where the $\tilde{\alpha}_i$ are angular averages of various combinations of scattering amplitudes and can be expressed in terms of Landau parameters. The coefficients η_i^{-1} are energy integrals of coherence factors and are functions of $\Delta/k_B T$. The explicit expressions for $\tilde{\alpha}_i$ and numerical values for η_i^{-1} as functions of t and $\Delta/k_B T$ are given in Ref. 10.

First we discuss the shear viscosity at T_c, where we have $\eta_1^{-1} = \eta_4^{-1} = \pi^2/3$, $\eta_2^{-1} = \eta_3^{-1} = \eta_5^{-1} = 0$, and $Y_2 = 1$. Then the shear viscosity at T_c, $\eta_v(T_c)$, is given by

$$\eta_v(T_c) = \frac{4\beta_c^2 p_F^7}{5\pi^5 (m^*)^2 \hbar^2} \frac{1}{\tilde{\alpha}_1} \frac{1}{1-\lambda_2} \tag{5}$$

where the subscript v indicates the variational solution and $\lambda_2 = -\tilde{\alpha}_4/\tilde{\alpha}_1$. The value of λ_2 has been reported in the literature[9,10,24,25] and has a value around 0.7, almost independent of pressure. Since $\eta_v(T_c)$ is a rapidly increasing function of λ_2 at the relevant values of λ_2 ($\simeq 0.7$), any change of $\eta_v(T_c)$ (e.g., due to pressure change) does not change the value of λ_2 very much. This point is very important in understanding why the behavior of the reduced shear viscosity in the superfluid phase is almost independent of the pressure as well as of the values of Landau parameters (see below).

Now we discuss the superfluid phase. Near T_c, the reduced shear viscosity $\tilde{\eta}_v = \eta_v(T)/\eta_v(T_c)$ up to first order in $\Delta/k_B T_c$ is given by

$$\tilde{\eta}_v = 1 - \frac{5-2\lambda_2}{16-16\lambda_2} \pi \frac{\Delta}{k_B T_c} \tag{6}$$

Thus the initial sharp drop of $\tilde{\eta}_v$ just below T_c is governed by one parameter λ_2. Equation (6) is valid in the immediate neighborhood of T_c (i.e., $0.9996 \leq t \leq 1.0$), which has been verified experimentally.[4,18] Numerical calculations show, however, that the behavior of $\tilde{\eta}_v$ is almost determined by one parameter λ_2 down to $t \sim 0.8$.

At $T = 0$, the reduced shear viscosity takes the form

$$\tilde{\eta}_v(0) = \frac{8\pi}{9}\left(\frac{k_B T_c}{\Delta(0)}\right)^2 \frac{1-\lambda_2}{1+\delta_0-\frac{2}{3}\gamma_0} \tag{7}$$

where we have defined $\tilde{\alpha}_2/\tilde{\alpha}_1 = \delta_0$ and $\tilde{\alpha}_3/\tilde{\alpha}_1 = \gamma_0$ following Einzel and Wölfle.[8] The parameters δ_0 and γ_0 also appear in the spin relaxation in ^{3}He-B. Einzel[26] estimated γ_0 and δ_0 within the *s–p*-wave approximation to be 0.1 and 0.3, respectively. Using these values, Einzel and Wölfle[8] calculated the damping of the wall-pinned mode in ^{3}He-B, which was in good agreement with experiment.[27] This suggests that the actual values of δ_0 and γ_0 are small compared with unity, although the *s–p*-wave approximation is not satisfactory. In our numerical calculations with the *s–p–d*-wave approximation (see below), γ_0 and δ_0 are always small and the quantity $\delta_0 - \frac{2}{3}\gamma_0$ is small compared with unity. Thus the parameter λ_2 also plays an important role at the low-temperature limit.

From the above observations, we can conclude that the behavior of $\tilde{\eta}_v(t)$ is determined predominantly by the parameter λ_2. As was stated at the beginning of this section, the value of λ_2 is roughly equal to 0.7 and is almost independent of pressure. It follows that the pressure dependence of $\tilde{\eta}_v(t)$ as a function of t is very small, which is confirmed by experiments[1,2] and by the theoretical results in the *s–p–d*-wave approximation discussed in Section 6.

3. *S–P–D*-WAVE APPROXIMATION

As shown in Section 2, we need to fix the $\tilde{\alpha}_i$ (i.e., the scattering amplitudes) in order to calculate the shear viscosity. In particular, the overall behavior of $\tilde{\eta}_v$ is roughly governed by the parameter λ_2 $(= -\tilde{\alpha}_4/\tilde{\alpha}_1)$, which can be determined by the shear viscosity in the normal phase.

The normal phase shear viscosity was calculated[23,28] in the *s–p*-wave approximation,[22] where the truncated forward-scattering sum rule was invoked. In these calculations the old Landau parameters were used. The result for ηT^2 as a function of pressure does not agree with experiments.[1,2,16,18,29] The new-Landau-parameter result for ηT^2 is much smaller than that for the old Landau parameters and agreement between theory and experiment is worse. In Fig. 1, ηT^2 is plotted as a function of pressure. The dashed curve is for the new Landau parameters and the solid curve is for the old Landau parameters. We have also included experimental data in Fig. 1. Circles are taken from Wheatley,[16] triangles are from Parpia *et al.*,[18] and squares are taken from the work of Archie *et al.*[1] A diamond indicates the measurement of Einsenstein *et al.*[29] From the graph we can

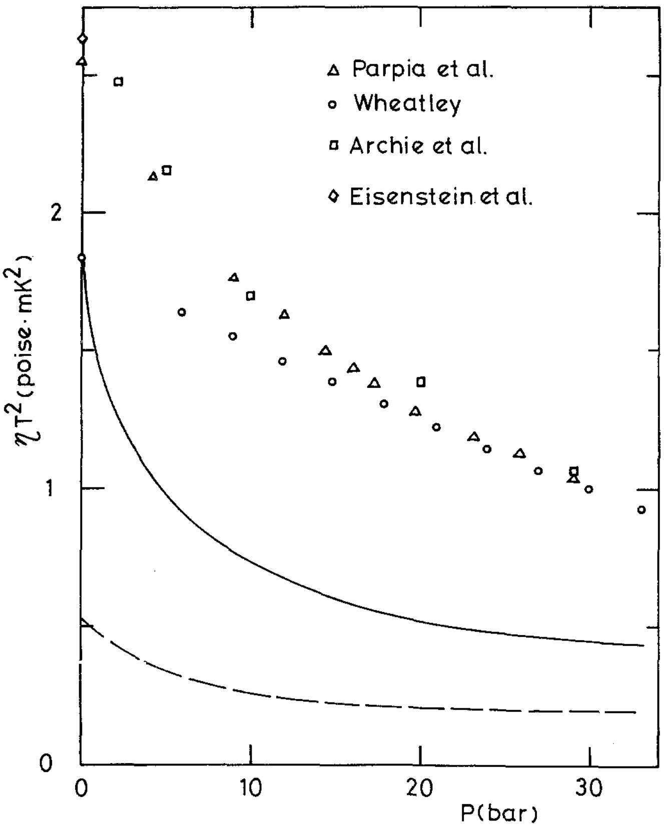

Fig. 1. ηT^2 versus pressure in normal liquid ^{3}He. The solid curve is the old-Landau-parameter result in the *s–p*-wave approximation. The dashed curve is the result with the new Landau parameters in the *s–p*-wave approximation. Triangles are data of Parpia *et al.*,[18] circles are taken from Wheatley,[16] and squares represent data of Archie *et al.*[1] The SVP result of Eisenstein *et al.*[29] is shown by a diamond.

say that the *s–p*-wave approximation is not good enough and that we need higher order Landau parameters in the calculation of transport coefficients.

Recently several estimates of F_2^s and F_2^a have been reported.[30–35] Sauls[30] extracted a value of F_2^s from the experiments[31] on the velocity difference between first sound and zero sound. At 20 bar, F_2^s is roughly equal to 0.8 for the old Landau parameters and is equal to −1 for the new Landau parameters. The pressure dependence of F_2^s is found to be very small. Nara *et al.*[32] determined a value of F_2^s from the ultrasonic attenuation data of normal ^{3}He. Their estimate for F_2^s is essentially the same as those of Sauls. A more direct estimate of F_2^s was done by the Northwestern group[33] from sound measurements in ^{3}He-B probing the property of collective modes. Their estimate for F_2^s is -1 from the squashing mode data. They also found $F_2^a \simeq -1.6$ by fitting the real squashing mode frequency. Theoretical calculations[34,35] of the collective-mode frequency in ^{3}He-B support the above values of F_2^s and F_2^a. Thus we have to take into account F_2^s and F_2^a in the calculation of the transport coefficients.

In this paper we use the s–p–d-wave approximation in the calculation of the transition probability of the collision integral. The effective quasiparticle potential U is given in the potential scattering model[10,26,28,36,37] as

$$U = \sum_{i<j} \{V^s(\mathbf{r}_i - \mathbf{r}_j) + V^a(\mathbf{r}_i - \mathbf{r}_j)\boldsymbol{\sigma}(i) \cdot \boldsymbol{\sigma}(j)\} \tag{8}$$

where the second term represents the paramagnon-mediated term and $\boldsymbol{\sigma}$ is the Pauli matrix vector. The singlet and triplet scattering amplitudes T_s and T_t, respectively, can be expressed as[24,28]

$$T_s(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \{3V^a(\mathbf{p}_1 - \mathbf{p}_4) - V^s(\mathbf{p}_1 - \mathbf{p}_4)\} + \{\mathbf{p}_4 \leftrightarrow \mathbf{p}_3\} \tag{9}$$

$$T_t(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \{V^a(\mathbf{p}_1 - \mathbf{p}_4) + V^s(\mathbf{p}_1 - \mathbf{p}_4)\} - \{\mathbf{p}_4 \leftrightarrow \mathbf{p}_3\} \tag{10}$$

where $V^{s,a}(\mathbf{p})$ are the Fourier transforms of $V^{s,a}(\mathbf{r})$. Because of the energy and momentum conservation laws in the collision integral, we can assume that all the momenta in Eqs. (9) and (10) have the magnitude p_F. Therefore the potentials $V^{s,a}(\mathbf{p}_i - \mathbf{p}_j)$ depend only on the angle $\theta_{ij} = \cos^{-1}(\hat{\mathbf{p}}_i \cdot \hat{\mathbf{p}}_j)$ and can be expanded in terms of Legendre polynomials;

$$V^{s,a}(\mathbf{p}_i - \mathbf{p}_j) = V^{s,a}(\cos\theta_{ij}) = \sum_l V_l^{s,a} P_l(\cos\theta_{ij}) \tag{11}$$

On the other hand, $\cos\theta_{14}$ and $\cos\theta_{13}$ can be expressed by

$$\cos\theta_{14} = \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_4 = \cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta \cos\phi \tag{12}$$

$$\cos\theta_{13} = \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3 = \cos^2 \tfrac{1}{2}\theta + \sin^2 \tfrac{1}{2}\theta \cos\phi \tag{13}$$

where the angles θ and ϕ are defined in the usual way[38]: θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$, and ϕ is the angle between two planes spanned by $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_3, \mathbf{p}_4)$. Therefore T_s and T_t are functions only of θ and ϕ. We also define S_l and T_l by

$$N_F T_t(\theta, \phi = 0) = \sum_l T_l P_l(\cos\theta) \tag{14}$$

$$N_F T_s(\theta, \phi = 0) = \sum_l S_l P_l(\cos\theta) \tag{15}$$

From Eqs. (9)–(15) we find

$$N_F T_s(\theta,\phi) = S_0 + S_1\cos\theta + S_2[\tfrac{1}{2}(3\cos^2\theta - 1) + \tfrac{3}{4}(\cos\theta-1)^2(\cos^2\phi - 1)] + S_3[\tfrac{1}{2}(5\cos^3\theta - 3\cos\theta) + \tfrac{15}{8}(1+\cos\theta)(1-\cos\theta)^2(\cos^2\phi - 1)] + \cdots \quad (16)$$

$$N_F T_t(\theta,\phi) = \{T_0 + T_1\cos\theta + T_2\tfrac{1}{2}(3\cos^2\theta - 1) + T_3[\tfrac{1}{2}(5\cos^3\theta - 3\cos\theta) + \tfrac{5}{8}(\cos\theta - 1)^3(\cos^2\phi - 1)] + \cdots\}\cos\phi \quad (17)$$

where

$$\begin{aligned} S_0 &= N_F[(3V_0^a - V_0^s) + \sum_{l=0}^{\infty}(3V_l^a - V_l^s)] \\ S_l &= N_F(3V_l^a - V_l^s), \qquad l \ge 1 \\ T_0 &= -N_F\sum_{l=1}^{\infty}(V_l^a + V_l^s) \\ T_l &= N_F(V_l^a + V_l^s), \qquad l \ge 1 \end{aligned} \quad (18)$$

Here N_F is the density of states at the Fermi surface for both spin directions, $N_F = m^* p_F/\pi^2\hbar^3$. In this model the forward-scattering sum rule[38] is automatically satisfied,

$$\sum_l T_l = 0 \quad (19)$$

Landau parameters $F_l^{s,a}$ are related to T_l and S_l by

$$T_l = A_l^s + A_l^a \quad (20)$$

$$S_l = A_l^s - 3A_l^a \quad (21)$$

where

$$A_l^{s,a} = F_l^{s,a}/[1 + F_l^{s,a}/(2l+1)] \quad (22)$$

In the following we restrict our discussion to the s–p–d-wave approximation, where higher order terms with $l \ge 3$ in the expansion of Eq. (11) are neglected. Here we discuss the difference between our definition of the s–p-wave approximation and the s–p-wave approximation originally proposed by Dy and Pethick.[22] (See also Ref. 24.) In their s–p-wave approximation, they assumed that the ϕ dependence is the simplest one consistent with Fermi statistics, namely, that the singlet amplitude is only s-wave (independent of ϕ) and that the triplet amplitude is only p-wave (proportional to $\cos\phi$). Then the s–p-wave approximation of the present paper would be equivalent to the s–p-wave approximation of Dy and Pethick if all Landau parameters with $l \geq 2$ were set equal to zero.* The advantage of the potential scattering model, from which we derived our s–p–d-wave approximation for the scattering amplitude, is that we can easily include higher order $\cos\phi$ terms. For example, in the s–p–d-wave approximation, $\cos^2\phi$ terms come in naturally in the definition of $T_s(\theta, \phi)$ and $T_t(\theta, \phi)$ of Eqs. (16) and (17).

4. SHEAR VISCOSITY AT 21 BAR

In this section we describe in detail the calculation of the shear viscosity at 21 bar within the s–p–d-wave approximation. As stated in the introduction, we have to estimate three Landau parameters, F_1^a, F_2^s, and F_2^a, in order to calculate the scattering amplitudes. First we treat A_2^s and A_1^a as free parameters using the condition $\eta_{\text{exact}}(T_c) = \eta_{\text{exp}}(T_c)$ to fix A_2^a, where η_{exact} is the exact theoretical value of the shear viscosity[39,40] and η_{exp} is the experimental value.[1,16,18,19] We vary A_2^s and A_1^a in the allowed parameter space and find the maximum and minimum values of $\tilde{\eta}_v(0)$ given by Eq. (7). In Fig. 2, we plot both the reduced shear viscosity $\tilde{\eta}_v(t)$ for Landau parameters corresponding to the maximum $\tilde{\eta}_v(0)$ and $\tilde{\eta}_v(t)$ for Landau parameters corresponding to the minimum $\tilde{\eta}_v(0)$. We find that the possible range of values for $\tilde{\eta}_v(t)$ is located between two solid curves for the old Landau parameters at 21 bar. The dashed curves are for the new Landau parameters at 20.94 bar. We believe that the curves corresponding to the maximum $\tilde{\eta}_v(0)$ and the minimum $\tilde{\eta}_v(0)$ give good upper and lower bounds for $\tilde{\eta}_v(t)$, respectively, in the whole temperature range. In Fig. 2, we have also included experimental data of Parpia *et al.*[18] (triangles) at 19.64 bar and of Archie *et al.*[1] (circles) at 20 bar. From the graph we observe that the upper and lower bounds for the new Landau parameters almost coincide with those for the old Landau parameters. Also, the possible range of values for $\tilde{\eta}_v(t)$ is rather narrow, even though the Landau parameters have a wide allowed region. This can be easily understood, since λ_2 is 0.76 (0.75) for the maximum $\tilde{\eta}_v(0)$ for the new (old) Landau parameters and 0.82 (0.825) for the minimum $\tilde{\eta}_v(0)$.

Next we try to determine the Landau parameters by fitting the transport coefficients in the normal state to the experimental values. For this purpose,

*Although the original s–p-wave approximation of Dy and Pethick contains higher order terms in $\cos\theta$ (i.e., higher order Landau parameters), almost all the calculations are done by imposing the truncated forward-scattering sum rule with $l = 0$ and 1. (This approximation has also been called "the s–p-wave approximation.") Then what they call the s–p-wave approximation is identical to our s–p-wave approximation.

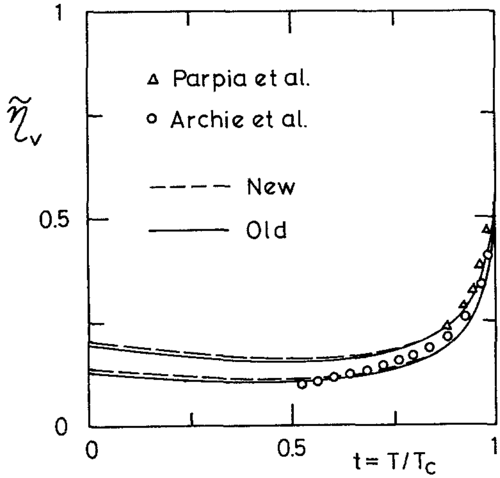

Fig. 2. The possible range of values for the reduced shear viscosity $\tilde{\eta}_v$ as a function of $t = T/T_c$ (21-bar result). The solid curves are for the old Landau parameters[16] and the dashed curves are for new Landau parameters.[20] Circles represent experimental data of Archie *et al.*[1] at 20 bar with slip correction included and triangles are from Parpia *et al.*[18] at 19.64 bar.

we minimize the following quantity as a function of three unknown Landau parameters F_1^a, F_2^s, and F_2^a:

$$\left(\frac{\eta_{\rm exp}-\eta_{\rm exact}}{\eta_{\rm exp}}\right)^2+\left(\frac{\kappa_{\rm exp}-\kappa_{\rm exact}}{\kappa_{\rm exp}}\right)^2+\left(\frac{D_{\rm exp}-D_{\rm exact}}{D_{\rm exp}}\right)^2 \qquad (23)$$

where $\kappa_{\rm exp}$ and $\kappa_{\rm exact}$ are experimental thermal conductivity[16] and exact theoretical thermal conductivity,[39,40] respectively, and $D_{\rm exp}$ and $D_{\rm exact}$ are experimental spin diffusion coefficient[41,42] and exact theoretical spin diffusion coefficient.[39,40] For experimental values at 21 bar we have used $\eta T^2 = 1.24$ P mK^2 given by Parpia *et al.*[18], $\kappa T = 15.3$ erg/sec cm given by Wheatley,[16] and $DT^2 = 0.29$ cm^2 mK^2/sec given by Wheatley.[41]

First we discuss the result for the old Landau parameters, whose values at 21 bar are $F_0^s = 59.78$, $F_0^a = -0.735$, and $F_1^s = 12.51$.[16] Minimizing Eq. (23), we find $F_1^a = -1.094$, $F_2^s = 1.079$, $F_2^a = 0.146$, and $\lambda_2 = 0.769$. From these values we obtain $\eta T^2 = 1.245$ P mK^2, $\kappa T = 14.69$ erg/sec cm, and $DT^2 = 0.298$ cm^2 mK^2/sec. The obtained F_2^s is approximately equal to 0.8 from sound measurements[30,32]; thus, agreement between our estimate for F_2^s and that from sound data is fairly good. The forward-scattering sum rule is reasonably satisfied ($T_0 + T_1 + T_2 = -0.063$). The temperature dependence of $\tilde{\eta}_v$ with the Landau parameters obtained above is shown in Fig.

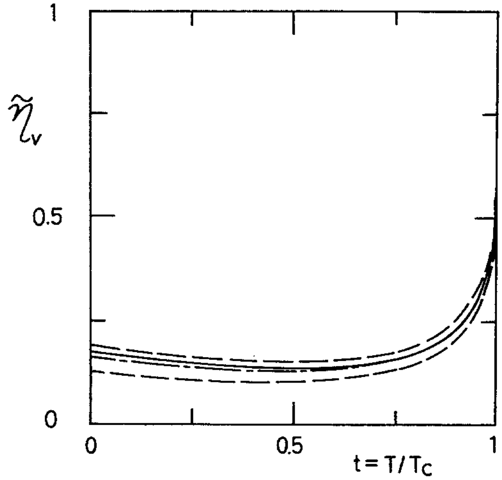

Fig. 3. $\tilde{\eta}_v$ vs t at 21 bar (old Landau parameters). The dashed curves are the upper and lower bounds. The solid curve is from the best-fit result for the transport coefficients in the normal phase. The dash-dot curve is the polarization-potential result.

3. The solid curve is from the present calculation. The dashed curves indicate the upper and lower bounds. The dash-dot curve is from the polarization-potential result of Bedell *et al.*[43] These authors have calculated the shear viscosity using the scattering amplitudes constructed from the polarization potential[44] and the formulas for η by Hara *et al.*[10] The polarization potential result is very close to ours in the *s–p–d* approximation over the whole temperature range. In particular, at high temperatures, where the experiments have been performed, there is essentially no difference between the two results for different approximations in the scattering amplitudes.

Now we turn to the result for the new Landau parameters: $F_0^s = 42.53$, $F_0^a = -0.8111$, and $F_1^s = 8.106$.[20] When F_1^a, F_2^s, and F_2^a are treated as free parameters, the minimum of Eq. (23) occurs at a positive value of F_2^s, in contradiction to the zero-sound analyses,[32,33] where F_2^s is roughly equal to -1. We have, therefore, tried to find the best fit for the normal state transport coefficients keeping F_2^s equal to -1. We obtain $\eta T^2 = 1.251$ P mK^2, $\kappa T = 12.19$ erg/sec cm, $DT^2 = 0.295$ cm^2 mK^2/sec, and $\lambda_2 = 0.754$ at the values $F_1^a = -0.670$, $F_2^a = -1.33$ (F_2^a has been estimated[33,35] to be -1.6 from acoustic measurements of the collective mode in ^{3}He-B). In Fig. 4, we have plotted $\tilde{\eta}_v$ as a function of t. The dashed curves are the upper and lower bounds and the solid curve is the present result. Note that the reduced

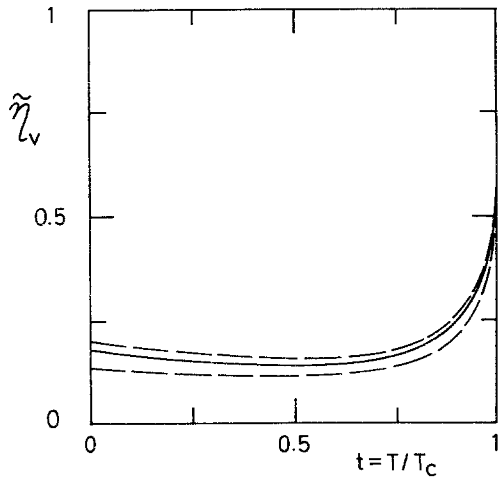

Fig. 4. $\tilde{\eta}_v$ vs t at 21 bar (new Landau parameters). The dashed curves are the upper and lower bounds. The solid curve is from the best-fit result for the transport coefficients in the normal phase.

shear viscosity shows almost the same behavior as in the case of the old Landau parameters.

Finally, we compare our results with recent experiments[1-5] at 20 bar. In Fig. 5, the solid curve is for the old Landau parameters and the dashed curve is for the new Landau parameters. Triangles are the torsional oscillator data of the Cornell group[2] with slip-correction[6] included. Circles represent spherical viscometer data[2] and squares are from the sound attenuation data of Kodama and Kojima[4]. The graph shows that agreement between theory and experiments is very good in the temperature range $0.5 \leq t \leq 1.0$. At lower temperatures, the observed viscosity starts to deviate from the theoretical result and decreases as the temperature is lowered (this behavior might be called a "low-temperature droop"). As will be discussed in Section 6, this low-temperature behavior is universal for all pressures. At present we have no explanation of this behavior.

Finally we comment on the mean-free-path effect. The viscous mean free path l_η is given by[6]

$$l_\eta = v_F \tau_\eta (Y_2/Y_0)^{1/2} \tag{24}$$

in terms of the viscous relaxation time. On the other hand, the quasiparticle mean free path l takes the form

$$l = v_F \tau (Y_2/Y_0)^{1/2} \tag{25}$$

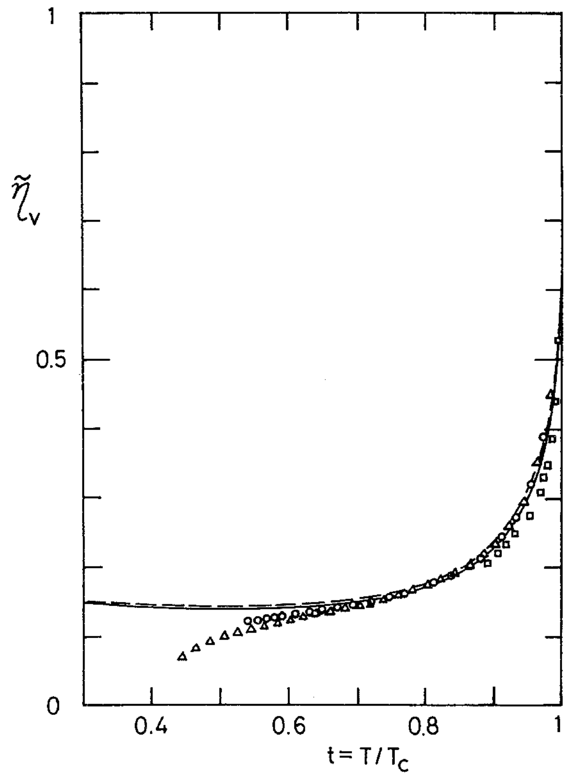

Fig. 5. Comparison of the reduced shear viscosity between theory and experiment. The solid curve is for the old Landau parameters at 21 bar and the dashed curve is for the new Landau parameters. Triangles are the torsional oscillator data of the Cornell group[2] with slip corrections. Open circles represent spherical viscometer data[2] and squares are from the sound attenuation data of Kodama and Kojima.[4]

where τ is the quasiparticle relaxation time given by

$$\tau = \frac{4\hbar\beta_c^2 p_F^2}{m^*\pi Y_2}\left(\sum_{i=1}^{3} \tilde{\alpha}_i \eta_i^{-1}\right)^{-1} \tag{26}$$

Here only the out-scattering terms in the collision integral are taken into account. The appearance of the Yosida functions Y_2 and Y_0 is due to the fact that the group velocity of the quasiparticles is given by

$$\mathbf{v} = \partial E/\partial \mathbf{p} = \hat{\mathbf{p}} v_F \xi / E \tag{27}$$

where $\hat{\mathbf{p}}$ denotes a unit vector along $\mathbf{p}$. We have defined the average of the group velocity squared (weighted by $\partial f/\partial E$) as

$$\langle v^2\rangle = v_F^2 Y_2/Y_0 \tag{28}$$

and have defined the mean free paths by $l = \langle v^2\rangle^{1/2}\tau$ ($l_\eta = \langle v^2\rangle^{1/2}\tau_\eta$).

The temperature dependence of l and l_η at 21 bar is shown in Fig. 6 as a function of the reduced temperature t. We have used the new Landau parameters. From the graph we observe that the mean free paths become very long at lower temperatures, where the main contribution comes from the $\exp(\Delta/k_BT)$ factor. In torsional oscillator experiments, the typical length of the apparatus d is the distance between two disks of oscillator,

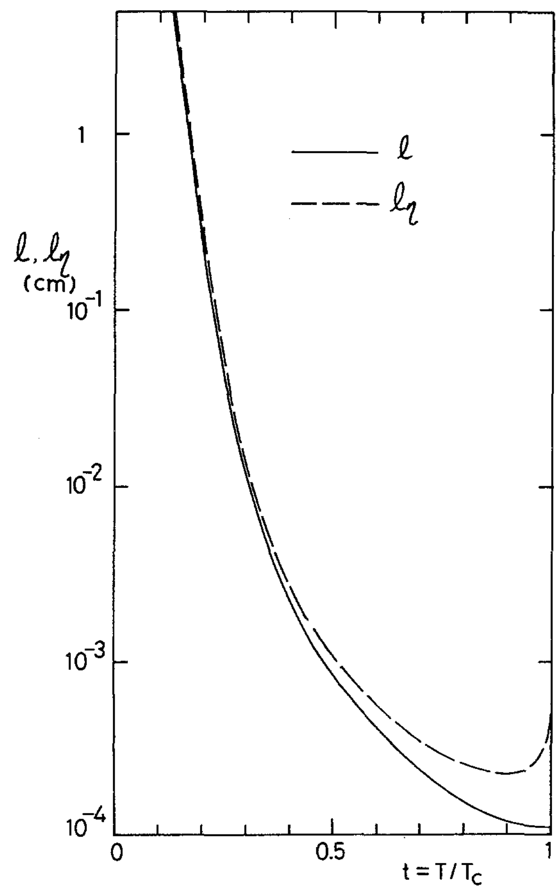

Fig. 6. Temperature dependence of mean free paths at 21 bar. The solid curve is for the quasiparticle mean free path l and the dashed curve is for the viscous mean free path l_η.

$d \simeq 100\ \mu$m.[18] It is argued in Ref. 6, however, that the simple slip-correction formula is applicable only for $l/d \leq 0.2$, which corresponds to $t \geq 0.45$ at 20 bar. Therefore the torsional oscillator data may not be reliable at lower temperatures. On the other hand, the spherical viscometer and sound attenuation cell are free from the mean-free-path effect. Moreover, $\omega\tau$ corrections to them are very small even at the lowest temperature at which the experiments were performed. Therefore the viscosity obtained from the spherical viscometer and from the sound attenuation method can be interpreted as the true hydrodynamic viscosity.

5. STRONG COUPLING CORRECTIONS AT 21 BAR

In this section we discuss the effect of the strong coupling within the "trivial" strong coupling corrections,[45] namely the gap renormalization. The strong coupling gap $\Delta_s(t)$ takes the form

$$\Delta_s(t) = \kappa^{-1/2}\Delta_{\rm BCS}(t) \tag{29}$$

where $\Delta_{\rm BCS}(t)$ is the BCS weak coupling gap. The gap enhancement factor $\kappa^{-1/2}$ can be extracted from experiments such as specific heat jump experiments[16,20,46] at T_c or normal-fluid-fraction measurements.[1,47,48] Experimental data of $\kappa^{-1/2}$ at 20 bar are scattered between 1.02 and 1.17, and theoretical calculations[45,49–52] show some discrepancies with experimental data. Therefore we use the value estimated from recent specific heat jump experiments[20,21] at 20.94 bar ($\kappa^{-1/2} = 1.11$) as a representative value. We show only the result for the new Landau parameters, since there is essentially no difference in $\tilde{\eta}_v$ between the two results for different Landau parameters.

In the strong coupling approximation, the energy gap in Eq. (7) should be replaced by $\Delta_s(0)$ given by

$$\Delta_s(0) = \kappa^{-1/2}\Delta_{\rm BCS}(0) = \kappa^{-1/2}\pi k_{\rm B}T_c/\gamma \tag{30}$$

where $\gamma = 1.78107$ is the Euler number. The normalized energy gap δ_s takes the form

$$\delta_s = \frac{\Delta_s(t)}{k_{\rm B}T} = \kappa^{-1/2}\frac{\pi}{\gamma}\frac{\Delta_{\rm BCS}(t)}{\Delta_{\rm BCS}(0)}\frac{1}{t} \tag{31}$$

The reduced shear viscosity $\tilde{\eta}_v$ can be calculated by changing values of η_i^{-1} as a function of t and δ_s in the following way. As we indicated in the footnote of Ref. 10, the $(t^2\eta_i)^{-1}$ are functions only of δ. Therefore in the strong coupling approximation, we interpret δ in Table II as δ_s, and change values of t and η_i^{-1} to correspond to the values of δ_s.

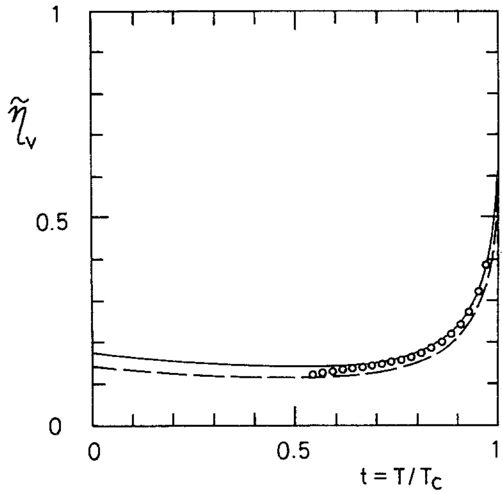

Fig. 7. The strong coupling corrections to $\tilde{\eta}_v$ at 20.94 bar (new Landau parameters). The dashed curve is for the strong coupling ($\kappa^{-1/2} = 1.11$) and the solid curve is for the weak coupling result. Circles are spherical viscometer data of Saunders *et al.*[2] at 20 bar.

The new Landau parameter result for the strong coupling corrections is shown in Fig. 7, where the dashed curve is for the strong coupling ($\kappa^{-1/2} = 1.11$) and the solid curve is for the weak coupling result. Circles are the spherical viscometer data[2] at 20 bar. From the graph, we observe that the strong coupling effects reduce the shear viscosity in the whole temperature range.

In these calculations, we have assumed that the gap enhancement factor is temperature independent. Rainer and Serene[51] proposed a weak-coupling-plus model and calculated the temperature dependence of $\kappa^{-1/2}$. In their theory $\kappa^{-1/2}$ decreases as the temperature decreases. We can easily include this effect by changing t and η_i^{-1} accordingly. Then the low-temperature $\tilde{\eta}_v$ is decreased less and becomes much closer to the weak coupling value. Since the strong coupling corrections are very small, inclusion of the weak-coupling-plus model does not change the overall behavior of $\tilde{\eta}_v$ in the whole temperature range.

6. PRESSURE DEPENDENCE

In this section we discuss the pressure dependence of $\tilde{\eta}_v$. For simplicity we use the weak coupling gap. As we have indicated in Section 4, we need experimental values of the transport coefficients in the normal state in order to calculate the transition probabilities. Since the shear viscosity in the normal Fermi liquid scales as T^{-2}, the natural parameter to consider is ηT^2 as a function of pressure. Using the torsional oscillator, Parpia *et al.*[18] found that values for ηT^2 at low pressures are larger than those tabulated by Wheatley,[16] while at high pressures both experiments give essentially the same values. Archie *et al.*,[1] using both the torsional oscillator and vibrating wire, found the same pressure dependence of ηT^2 as that of Parpia *et al.* Recently Eisenstein *et al.*[29] found almost the identical value as that of Ref. 18 at SVP from the experiment on Poiseuille flow through a channel. The pressure dependence of ηT^2 of Saunders *et al.*,[2] using the spherical viscometer, agrees very well with that of Parpia *et al.* These data are shown in Fig. 1. We have used the Parpia value of ηT^2 for a 6-bar

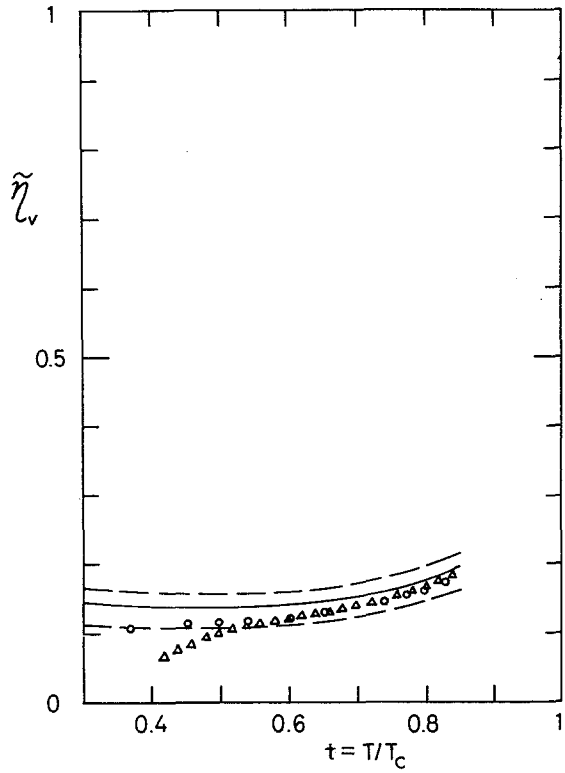

Fig. 8. $\tilde{\eta}_v$ versus t at 30 bar in ^{3}He-B (new Landau parameters). The dashed curves are the upper and lower bounds. The solid curve is obtained by fitting the normal-state transport coefficients. Triangles are the slip-corrected reduced shear viscosity data[2] at 29 bar and circles are spherical viscometer data[2] at 29 bar.

calculation and the Wheatley value for a 30-bar calculation. Using the method described in Section 4, we have calculated $\tilde{\eta}_v$ at pressures 6 and 30 bar. Although both new and old Landau parameters are used, there is essentially no difference between the two results, as was found in Section 4. Therefore we show only the new-Landau-parameter result. In Fig. 8, we show the 30-bar results for the B phase. The dashed curves are the upper and lower bounds calculated by imposing only one condition $\eta_{\text{exp}}(T_c) = \eta_{\text{exact}}(T_c)$. The solid curve is obtained by fitting the normal state transport coefficients. We have also included experimental data at 29 bar. Triangles are the slip-corrected reduced shear viscosity data[2] and circles are spherical viscometer data.[2] The two sets of experimental data show good agreement down to $t \simeq 0.5$, at which temperature the slip theory starts

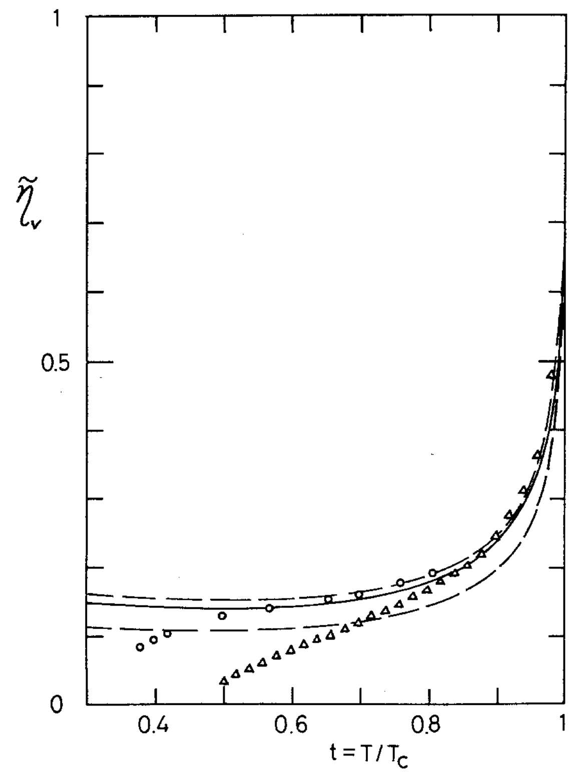

Fig. 9. $\tilde{\eta}_v$ versus t at 6 bar (new Landau parameters). The region between the dashed curves is the possible range of values for $\tilde{\eta}_v$. The solid curve is obtained by fitting the normal transport coefficients. Circles are spherical viscometer data[2] at 5.47 bar and triangles are slip-corrected viscosity[2] at 5 bar.

to collapse since l_η/d becomes larger than 0.2. The agreement between theory and experiment is very good. In Fig. 9 the 6-bar results are shown. The region between two dashed curves is the possible range of values for the reduced shear viscosity. The solid curve is obtained by minimizing Eq. (23). The polarization-potential result[43] coincides with our upper-bound result. Circles are spherical viscometer data[2] at 5.47 bar and triangles are slip-corrected viscosity[2] at 5 bar. At this pressure the viscous mean free path at T_c, $l_\eta(T_c)$, is equal to 27.1 μm,[1] which gives $l_\eta/d = 0.288$. Therefore the slip theory, i.e., hydrodynamic equations amended by a slip boundary condition, breaks down even at T_c. On the other hand, the $\omega\tau$ corrections in the case of the spherical viscometer are very small, at most 10% at the lowest temperature. Thus the spherical viscometer data are more reliable at low pressures. The agreement between theory and experiment is rather good at low pressures. Comparing Figs. 5, 8, and 9, we observe that there is essentially no pressure dependence of $\tilde{\eta}_v$ as a function of t, in good agreement with experiments.[1,2]

7. SUMMARY AND DISCUSSION

In this paper we have calculated the shear viscosity of superfluid ^{3}He-B at 6, 21, and 30 bar and compared the results with recent experiments.[1–5] We have used the variational solution of the quasiparticle Boltzmann equation. The scattering amplitudes are estimated by fitting the normal state transport coefficients within the s–p–d-wave approximation. Both the new and old Landau parameters are used. The calculated reduced shear viscosity agrees very well with recent experiments[1–5] in the temperature range $0.5 \leq t \leq 1.0$, irrespective of the choice of the Landau parameters. This agreement is due to the fact that the temperature dependence of the reduced shear viscosity is mainly governed by a single parameter λ_2, in particular near T_c (say, $0.8 \leq t \leq 1.0$). The parameter λ_2, as was stated in Section 2, always takes a value around 0.7, whenever the condition $\eta_{\text{exact}}(T_c) = \eta_{\text{exp}}(T_c)$ is satisfied. It is therefore essential to keep the condition $\eta_{\text{exact}}(T_c) = \eta_{\text{exp}}(T_c)$ in order to have agreement of the reduced shear viscosity with experiment. In fact, our s–p–d result in which the Landau parameters are so chosen that the above condition should be satisfied agrees very well with the result of Bedell *et al.*[43] based on the polarization potential whose parameters are fixed so that the normal state transport coefficients agree with experiments.

The reduced shear viscosity by itself is not useful for determining the Landau parameters. We have tried to determine the Landau parameters by optimizing the normal state transport coefficients, i.e., by minimizing Eq. (23). We have been able to have a reasonable set of values for the old

Landau parameters, but not for the new Landau parameters (a similar attempt at the melting pressure has been reported by Sauls and Serene[37] for the old Landau parameters). In the case of the new Landau parameters F_2^s is negative.[30,32] Moreover, in the recent literature[33,35] F_2^a is also reported to take a negative value. It follows that the forward scattering sum rule is far from satisfied within the *s–p–d* approximation. Comparison of the *s–p–d* result with the *s–p–d–f* result[37] indicates that the normal state transport coefficients have a considerable contribution from higher order ($l \geq 3$) Landau parameters if they exist, though the effective potential model employed in Section 3 and Ref. 37 is a tentative approximation and should be reexamined. We hope that precise experiments on liquid ^{3}He, especially on superfluid ^{3}He, which has rich angle-dependent properties, will be performed so that the higher order Landau parameters can be determined.

We have also estimated the strong coupling corrections to the shear viscosity. The "trivial" strong coupling correction (gap renormalization) is used. We have found that the strong coupling effects slightly reduce $\tilde{\eta}_v$ over the whole temperature range and the plateau region of $\tilde{\eta}_v$ is pushed down to lower temperatures. We have also found essentially no pressure dependence of $\tilde{\eta}_v$ as a function of t, in accordance with the experiments.[1,2]

A theoretical explanation of the low-temperature "droop" found in experiments is left for future investigation.

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REFERENCES

1. C. N. Archie, T. A. Alvesalo, J. D. Reppy, and R. C. Richardson, *J. Low Temp. Phys.* **42**, 295 (1981).
2. J. Saunders, E. K. Zeise, A. I. Ahonen, C. N. Archie, D. G. Wildes, Y. A. Ono, and R. C. Richardson, *J. Low Temp. Phys.*, to be submitted.
3. G. Eska, K. Neumaier, W. Schoepe, K. Uhlig, W. Wiedemann, and P. Wölfle, *Phys. Rev. Lett.* **44**, 1337, 1712(E) (1980); *Physica* **108B**, 1153 (1981).
4. T. Kodama and H. Kojima, Rutgers University preprint, RU-81-030.
5. D. C. Carless, P. W. Alexander, H. E. Hall, J. R. Hook, and N. V. Wellard, *Physica* **108B**, 793 (1981), and to be published.

6. H. Højgaard Jensen, H. Smith, P. Wölfle, K. Nagai, and T. M. Bisgaard, *J. Low Temp. Phys.* **41**, 473 (1980).
7. Y. A. Ono, J. Hara, K. Nagai, and K. Kawamura, *J. Low Temp. Phys.* **27**, 513 (1977).
8. D. Einzel and P. Wölfle, *J. Low Temp. Phys.* **32**, 19 (1978).
9. P. Wölfle and D. Einzel, *J. Low Temp. Phys.* **32**, 39 (1978).
10. J. Hara, Y. A. Ono, K. Nagai, and K. Kawamura, *J. Low Temp. Phys.* **39**, 603 (1980).
11. M. Dörfle, H. Brand, and R. Graham, *J. Phys. C* **13**, 3337 (1980).
12. J. Hara and Y. A. Ono, *Phys. Lett.* **81A**, 65 (1981).
13. Y. A. Ono and J. Hara, *J. Phys. C* **14**, 2093 (1981).
14. K. Nagai and P. Wölfle, *J. Low Temp. Phys.* **42**, 227 (1981).
15. J. Hara, *J. Low Temp. Phys.* **43**, 533 (1981).
16. J. C. Wheatley, *Rev. Mod. Phys.* **47**, 415 (1975).
17. R. E. Nettleton, *J. Phys. C* **11**, L725 (1978).
18. J. M. Parpia, D. J. Sandiford, J. E. Berthold, and J. D. Reppy, *Phys. Rev. Lett.* **40**, 565 (1978); *J. Phys.* (*Paris*) **39**, C6-35 (1978).
19. T. A. Alvesalo, C. N. Archie, A. J. Albrecht, J. D. Reppy, and R. C. Richardson, *J. Phys.* (*Paris*) **39**, C6-41 (1978).
20. T. A. Alvesalo, T. Haavasoja, P. C. Main, M. T. Manninen, J. Ray, and Leila M. M. Rehn, *Phys. Rev. Lett.* **43**, 1509 (1979); T. A. Alvesalo, T. Haavasoja, M. T. Manninen, and A. T. Soinne, *Phys. Rev. Lett.* **44**, 1076 (1980); T. A. Alvesalo, T. Haavasoja, and M. T. Manninen, *J. Low Temp. Phys.* (in press).
21. E. K. Zeise, J. Saunders, A. I. Ahonen, C. N. Archie, and R. C. Richardson, *Physica* **108B**, 1213 (1981), and to be published.
22. K. S. Dy and C. J. Pethick, *Phys. Rev.* **185**, 373 (1969).
23. C. J. Pethick, H. Smith, and P. Bhattacharyya, *Phys. Rev. Lett.* **34**, 643 (1975); *Phys. Rev. B* **15**, 3384 (1977).
24. P. Bhattacharyya, C. J. Pethick, and H. Smith, *Phys. Rev. B* **15**, 3367 (1977).
25. P. Wölfle, in *Progress in Low Temperature Physics*, D. F. Brewer, ed. (North-Holland, Amsterdam, 1978), Vol. VIIA, p. 191.
26. D. Einzel, Ph.D. Thesis, Technische Universität München (1980).
27. R. A. Webb, R. E. Sager, and J. C. Wheatley, *J. Low Temp. Phys.* **26**, 439 (1977).
28. Y. A. Ono, *Prog. Theor. Phys.* **60**, 1 (1978).
29. J. P. Eisenstein, G. W. Swift, and R. E. Packard, *Phys. Rev. Lett.* **45**, 1199 (1980); *Physica* **108B**, 1061 (1981).
30. J. A. Sauls, private communication.
31. J. B. Ketterson, P. R. Roach, B. M. Abraham, and P. D. Roach, in *Quantum Statistics and the Many Body Problems*, S. B. Trickey, W. P. Kirk, and J. W. Duffy, eds. (Plenum Press, New York, 1975), p. 35.
32. K. Nara, I. Fujii, K. Kaneko, and A. Ikushima, *Physica* **108B**, 1203 (1981).
33. D. B. Mast, J. R. Owers-Bradley, W. P. Halperin, I. D. Calder, B. K. Sarma, and J. B. Ketterson, *Physica* **107B**, 685 (1981).
34. J. A. Sauls and J. W. Serene, *Phys. Rev. B* **23**, 4798 (1981).
35. Y. Hasegawa and H. Namaizawa, *Prog. Theor. Phys.* (in press).
36. P. Wölfle, *Rep. Prog. Phys.* **42**, 269 (1979).
37. J. A. Sauls and J. W. Serene, *Phys. Rev. B* **24**, 183 (1981).
38. A. A. Abrikosov and I. M. Khalatnikov, *Rep. Prog. Phys.* **22**, 329 (1959).
39. H. Højgaard Jensen, H. Smith, and J. W. Wilkins, *Phys. Lett.* **27A**, 532 (1968); *Phys. Rev.* **185**, 323 (1969).
40. G. A. Brooker and J. Sykes, *Phys. Rev. Lett.* **21**, 279 (1968); *Ann. Phys.* (*N.Y.*) **56**, 1 (1970).
41. J. C. Wheatley, in *Quantum Fluids*, D. F. Brewer, ed. (North-Holland, Amsterdam, 1966), p. 183.
42. D. F. Brewer, D. S. Betts, A. Sachrajda, and W. S. Truscott, *Physica* **108B**, 1059 (1981).
43. K. Bedell, W.-C. Hsu, and Y. A. Ono, to be published.
44. K. Bedell and D. Pines, *Phys. Rev. Lett.* **45**, 39 (1980); *Phys. Lett.* **78A**, 281 (1980).
45. J. W. Serene and D. Rainer, *Phys. Rev. B* **17**, 2901 (1978).

46. W. P. Halperin, F. B. Rasmussen, C. N. Archie, and R. C. Richardson, *Phys. Rev. B* **13**, 2124 (1976); *J. Low Temp. Phys.* **31**, 617 (1978).
47. C. N. Archie, T. A. Alvesalo, J. E. Berthold, J. D. Reppy, and R. C. Richardson, *J. Phys. (Paris)* **39**, C6-37 (1978); C. N. Archie, T. A. Alvesalo, J. D. Reppy, and R. C. Richardson, *Phys. Rev. Lett.* **43**, 139 (1979).
48. J. Saunders, D. G. Wildes, J. M. Parpia, J. D. Reppy, and R. C. Richardson, *Physica* **108B**, 791 (1981).
49. D. Rainer and J. W. Serene, *Phys. Rev. B* **13**, 4745 (1976).
50. J. W. Serene and D. Rainer, *J. Low Temp. Phys.* **34**, 589 (1979).
51. D. Rainer and J. W. Serene, *J. Low Temp. Phys.* **38**, 601 (1980).
52. Y. Kuroda and A. D. S. Nagi, *J. Low Temp. Phys.* **30**, 755 (1978) and references therein.
53. K. Uhlig, private communication.