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On the bending stress distribution at the tip of a stationary crack from Reissner's theory

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ABSTRACT
A general solution is developed for the symmetric bending stress distribution at the tip of a crack in a plate taking shear deformation into account through Reissner's theory. The solution is obtained in terms of polar coordinates at the crack tip and includes the complete class of solutions satisfying all the three boundary conditions along the crack. The solution has arbitrary multiplicative constants and in specific problems, these constants can be determined from conditions on the exterior boundary by well-known numerical techniques such as collocation, successive integration. Results of a numerical solution for a square plate with a central crack subject to uniaxial bending are presented along with a critical discussion of the sensitivity of the numerical solution which is associated with the exponential character of Bessel terms in this higher order analytical solution.

Nomenclature

$2a$ = crack length
h = plate thickness
$k^2 = \frac{5}{2}\left(\frac{a}{h}\right)^2$
E = Young's Modulus
ν = Poisson's ratio
D = bending rigidity of plate, $Eh^3/12(1-\nu^2)$
M_0 = reference bending moment
σ_0 = extreme fibre stress due to M_0, $6M_0/h^2$
r, θ = polar coordinates with crack tip as the origin (Fig. 1), r being nondimensionalized with respect to a
W = normal displacement, nondimensionalized with respect to M_0a^2/D
$M_r, M_\theta, M_{r\theta}$ = bending and twisting moments per unit length of plate element (Fig. 1), nondimensionalized with respect to M_0
Q_r, Q_θ = transverse shear forces per unit length of plate element (Fig. 1), nondimensionalized with respect to M_0/a
$\sigma_r, \sigma_\theta, \tau_{r\theta}$ = stresses on that surface where a positive moment produces tension, nondimensionalized with respect to σ_0
χ = a stress function in Reissner's theory, nondimensionalized with respect to M_0
$K_{(b)}$ = bending stress intensity factor
$\left.\begin{matrix} I_\mu(z) \\ K_\mu(z) \end{matrix}\right]$ = Modified Bessel Functions of first and second kinds respectively
$\varphi(\mu, m) = (\mu+1)(\mu+2)\ldots(\mu+m)$ for $m \neq 0$
$= 1$ for $m = 0$
Γ = Gamma function

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Subscripts:

- R denotes rigid body movement terms in the solution
- F denotes solutions corresponding to fractional values of μ
- $I(1)$, $I(2)$ denote first and second kinds of solutions corresponding to integer values of μ
- μ used as a subscript to a quantity refers to that component of the quantity which is a contribution from the value of μ under reference

1. Introduction

It appears that it is only in recent years that the stress problem of cracked plates has been examined from a higher order plate bending theory taking shear deformation into account. The earliest attempt due to Knowles and Wang [1] was concerned with the problem of an infinite plate under uniform uniaxial bending far from the crack and was restricted to the case of vanishingly small plate thickness. Hartranft and Sih [2] and N.M. Wang [3] studied the same problem considering the plate thickness to be finite. All these investigations were based on Reissner's theory and led to the conclusion that the singular stresses under bending have the same functional form as in the plane stress case [4]. Thus the predictions of classical theory [5] are known to be erroneous with respect to the angular distribution of stress around the crack tip. As a more serious practical consequence, there is an associated error in predicting the stress intensity factor whose importance in fracture mechanics is well-known. Results reported in [2] and [3] show that the stress intensity factor varies very rapidly in the h/a range of 0 to 1/4. In fact, Hartranft and Sih [2] have shown that the stress intensity factor versus h/a curve has infinite slope at $h/a = 0$. It is seen from their numerical results that even for h/a ratio as small as 0.2, the bending stress intensity factor is 62% greater than that given by Knowles and Wang [1] for $h/a \to 0$. Since h/a does not appear as a parameter in classical theory formulation, the inadequacy of this theory in handling the bending case in crack problems becomes apparent.

All the Reissner's theory investigations cited [1–3] were carried out for the case of infinite plate. Analysis in this paper takes finiteness of the plate into account. Apart from this, the scope of the paper itself is slightly different and, in a sense, wider. The aim here is to develop a general solution which can be readily applied to a wide class of problems. While analyses in earlier studies were based on the integral equation approach, this analysis uses the differential equation approach.

In the present work, the complete class of possible solutions is obtained for symmetric bending stresses in the vicinity of a crack tip in a plate taking shear deformation into account through the use of Reissner's theory. The solution, obtained in local polar coordinates with crack tip as the origin, satisfies all the three force boundary conditions along the crack identically. The solution contains arbitrary multiplicative constants to be determined from conditions on the exterior boundary through standard numerical techniques for a specific problem. No restriction upon the h/a ratio is needed in the present solution so that in the solution of specific problems, the thickness effect is automatically taken into account. Use of the general solution developed here is demonstrated with reference to an example of a square plate with a central crack subjected to uniform uniaxial bending along two opposite edges.

2. Analysis

The present analysis is carried out for the case of symmetric bending of the plate with respect to the crack. Also, the analysis is carried out for the case where there is no normal loading on the plate. In other words, the solutions obtained can be used in a situation where the plate is subjected only to known edge loads or known kinematic constraints on the exterior boundary.

We choose a system of polar coordinates (r, θ) with a crack tip as the origin (Fig. 1).

2.1. *Nondimensionalization*

For the sake of convenience, we nondimensionalize all the physical quantities in our analysis according to the following scheme. Here the symbol to the left of the colon denotes the nondimensionalized physical quantity and the symbol to the right denotes the nondimensionalizing factor.

$$r : a,\ W : M_0 a^2/D,\ [\sigma_r, \sigma_\theta, \tau_{r\theta}] : \sigma_0$$
$$[M_r, M_\theta, M_{r\theta}] : M_0,\ [Q_r, Q_\theta] : M_0/a,\ \chi : M_0$$

2.2. *Governing equations and other relations*

The governing equations in the absence of distributed normal loading and the various other relations in Reissner's formulation [6] expressed in terms of quantities nondimensionalized as above can be written as

$$\nabla^4 W = 0 \tag{1}$$

$$\nabla^2 \chi - 4k^2 \chi = 0 \tag{2}$$

The various other relations are

$$\sigma_r = M_r = \frac{\partial^2 W}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right) + \frac{1}{2k^2}\left[\frac{\partial^2}{\partial r^2}(\nabla^2 W) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \theta}\right)\right] \tag{3}$$

$$\sigma_\theta = M_\theta = \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \nu\frac{\partial^2 W}{\partial r^2} + \frac{1}{2k^2}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \theta}\right) - \frac{\partial^2}{\partial r^2}(\nabla^2 W)\right] \tag{4}$$

$$\tau_{r\theta} = M_{r\theta} = (1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right) + \frac{1}{2k^2}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2 W)\right] - \frac{1}{4k^2}\left[\nabla^2 \chi - 2\frac{\partial^2 \chi}{\partial r^2}\right] \tag{5}$$

$$Q_r = \frac{\partial}{\partial r}(\nabla^2 W) - \frac{1}{r}\frac{\partial \chi}{\partial \theta} \tag{6}$$

$$Q_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2 W) + \frac{\partial \chi}{\partial r} \tag{7}$$

where σ_r, σ_θ and $\tau_{r\theta}$ refer to the nondimensional stresses on that surface where a positive bending moment produces tensile stress (For sign conventions, see Fig. 1).

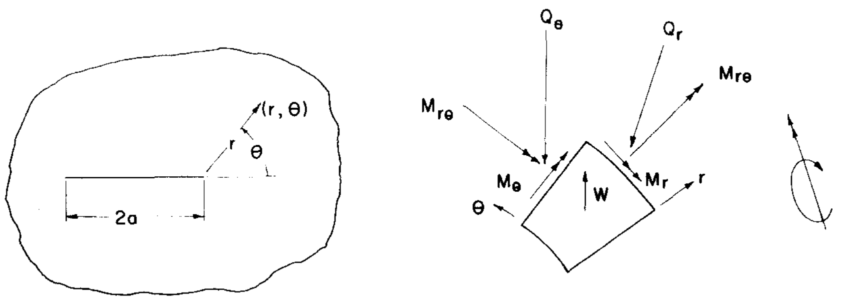

Figure 1. System of coordinates and stress resultants.

2.3. *Boundary conditions for the crack faces*

Since the crack boundary surface should be stress-free, the following three boundary conditions must be satisfied:

$$M_\theta(r, \pm\pi) = 0 \tag{8}$$

$$M_{r\theta}(r, \pm\pi) = 0 \tag{9}$$

$$Q_\theta(r, \pm\pi) = 0 \tag{10}$$

2.4. *Solution*

A solution to Eqn. (1) can be written as

$$W_\mu = r^{\mu+2}[A_\mu \cos \mu\theta + B_\mu \cos(\mu+2)\theta] \tag{11}$$

where μ is real. Its use as a subscript to a physical quantity implies that we are referring to that component of the quantity which corresponds to the particular value of μ under reference. Equation (11) can be considered to represent the most general type of solution for W as far as our present problem is concerned. For $(\mu+2) = 0$, the solution (11) assumes certain special forms involving terms like $\ln r$, θ and $\theta \ln r$ but these must be dropped in the present analysis in order that the slopes should be finite at $r = 0$.

We now seek a solution for Eqn. (2) which in combination with the above solution for W satisfies all the crack boundary conditions identically. Summation over all possible values of μ gives complete solution to the problem.

Separation of variables leads to solution for χ in (r, θ) in terms of Modified Bessel functions of first and second kinds [7]:

$$\bar{\chi}_\mu = [\bar{C}I_\mu(2kr) + \bar{D}I_{-\mu}(2kr)] \sin \mu\theta \qquad \text{if } \mu \text{ is not an integer,} \tag{12}$$

$$\bar{\chi}_\mu = [\bar{C}I_\mu(2kr) + \bar{D}K_\mu(2kr)] \sin \mu\theta \qquad \text{if } \mu \text{ is an integer.} \tag{13}$$

where $\bar{C}$ and $\bar{D}$ are arbitrary constants. An inspection of Eqns. (3)–(5) shows that the moments involve second derivatives of χ and so the rotations of a line normal to the middle surface should depend on the first derivatives. For integer μ, the Modified Bessel function of second kind $K_\mu(2kr)$ involves $\ln r$ and negative powers of r and thus it should be dropped out in our solution from the condition of finiteness of rotations. For fractional μ, the Modified Bessel function of second kind is obtained by simply replacing μ by $-\mu$ in the function of first kind. So, without any loss of generality, we write only Modified Bessel function of first kind in the solution for χ keeping in mind that μ will take positive as well as negative values. The actual range of μ, however, will be limited from the condition of finiteness of rotations. A typical solution to (2) is then written as

$$\bar{\chi}_\mu = \sin \mu\theta \sum_{m=0,1,2,\dots}^{\infty} \frac{k^{2m} r^{\mu+2m}}{\underline{|m}\, \varphi(\mu, m)} \tag{14}$$

where

$$\varphi(\mu, m) = (\mu+1)(\mu+2)\dots(\mu+m) \text{ for } m \geq 1$$
$$= 1 \text{ for } m = 0 \tag{15}$$

In writing the above solution, some multiplicative constants are dropped in the Modified Bessel function of first kind for the sake of convenience. There is no loss of generality here as the solution is later multiplied by an arbitrary constant. Through an inspection of the boundary conditions and Eqns. (3)–(7), a solution for χ which in combination with W from (11) satisfies the crack boundary conditions can be written by inspection as

$$\chi_\mu = \sum_{n=0,1,2,\ldots}^{\infty} C_{\mu+2n}\bar{\chi}_{\mu+2n} \tag{16}$$

From the solutions (11) and (16), the boundary conditions are satisfied term by term in the power series in r. This leads to a set of homogeneous equations for the arbitrary constants. For these equations to be consistent, it can be seen that μ should be equal to one of the following values:

$$\mu = \pm(2j+1)/2, j = 0, 1, 2, 3, \ldots \tag{17}$$

or

$$\mu = \pm j, j = 0, 1, 2, 3, \ldots \tag{18}$$

Solutions corresponding to fractional and integer μ are henceforth referred to, for the sake of brevity, as solutions of fractional and integer type respectively.

If $\mu = \pm(2j+1)/2$, the first boundary condition (8) is satisfied identically. Satisfying the third boundary condition (10) term by term in the power series in r leads to a series of recurrence relations involving $C_{(\mu+2n),F}$ which are to be solved successively (for details, see Appendix 1). $C_{(\mu+2n),F}$ is then obtained as a finite series for which summation is found to be possible. We get

$$C_{(\mu+2n),F} = \frac{4(\mu+1)(-k^2)^n}{\underline{|n}\,\varphi(\mu+n-1, n)} A_{\mu,F} \tag{19}$$

where the additional subscript F is used to refer to fractional μ. We now satisfy the boundary condition (9) term by term in the power series with respect to r. It will be found that with $C_{(\mu+2n),F}$ already determined above, the boundary condition (9) is automatically satisfied in terms involving $r^{\mu-2}$, $r^{\mu+2}$, $r^{\mu+4}$, $r^{\mu+6}, \ldots$ etc. A formal proof of this is given in Appendix 1. Satisfying the condition in r^μ terms leads to

$$B_{\mu,F} = -\frac{[4+(1-\nu)\mu]}{(1-\nu)(\mu+2)} A_{\mu,F} \tag{20}$$

For integer values of μ, the second and third boundary conditions (9) and (10) are satisfied identically. Satisfying the boundary condition (8) for each power of r, we get

$$C_{\mu+2n,I(1)} = \frac{4(-k^2)^n}{\underline{|n-1}\,\varphi(\mu+n, n)}\left[\frac{\mu(\mu+1)}{n(\mu+n)} + 2 - \frac{(\mu+2)(1-\nu)}{2}\right]A_{\mu,I(1)} - \frac{2(\mu+2)(1-\nu)(-k^2)^n}{\underline{|n-1}\,\varphi(\mu+n, n)} B_{\mu,I(1)} \text{ for } n \neq 0,\ \mu \neq 0, 1 \tag{21}$$

$$= \frac{4(-k^2)^n}{\underline{|n-1}\,\varphi(\mu+n, n)}\left[2 - \frac{(\mu+2)(1-\nu)}{2}\right]A_{\mu,I(1)} - \frac{2(\mu+2)(1-\nu)(-k^2)^n}{\underline{|n-1}\,\varphi(\mu+n, n)} B_{\mu,I(1)} \text{ for } n \neq 0,\ \mu = 0, 1 \tag{22}$$

$$= 4(\mu+1)A_{\mu,I(1)} \text{ for } n = 0,\ \mu \neq 0, 1 \tag{23}$$

$$= 0 \text{ for } n = 0,\ \mu = 0, 1 \tag{24}$$

where the additional subscript I is used to refer to integer μ. The other additional subscript (1) is used to denote integer type solution of first kind. We will later see that there is another kind of solution of integer type. The coefficients of $A_{\mu, I(1)}$ and $B_{\mu, I(1)}$ in Eqns. (21) and (22) are actually obtained in the form of finite series for which summation is found to be possible as in the case of fractional μ.

The case $\mu = 1$ requires a special mention. With W as defined by (11) for $\mu = 1$, the boundary condition (8) is automatically satisfied in terms involving $r^{\mu-2}$ without any contribution required from χ. So, the series form for the χ solution as defined by (16) is taken dropping the first term (See Eqns. (24)). As a result of this, we find that $\bar{\chi}_1$ does not appear anywhere in the solution. A closer examination reveals that we have a special type of solution involving $\bar{\chi}_1$ which is defined by

$$W_{I(2)} = 0, \; \chi_{I(2)} = \sum_{n=0,1,2,\ldots}^{\infty} C_{1+2n, I(2)} \bar{\chi}_{1+2n} \tag{25)-(26}$$

We call this an integer type solution of a second kind which is identified by the additional subscript (2). This solution involves only χ and requires no W solution for satisfying the crack boundary conditions. In other words, it corresponds to a case where the plate does not undergo deflection anywhere but still experiences a stressed state due to rotation of lines normal to the middle surface. This is a speciality of this solution which distinguishes it from the integer type solution of first kind. Using Eqns. (25) and (26) and satisfying the boundary condition (8) term by term in the power series with respect to r, there follows

$$C_{1+2n, I(2)} = \frac{(-k^2)^n}{\underline{|2n+1}} C_{1, I(2)} \tag{27}$$

2.4.1. *Convergence of the χ series*

For a given W_μ of the form given by Eqn. (11), we have now obtained a series form for χ_μ as in Eqn. (16). Through a simple convergence test, this infinite series for χ_μ as well as its termwise differentiated forms can be shown to be convergent for all r and θ and for all the different types of solutions we have derived in the foregoing analysis. Let us, for instance, take the solution for fractional μ. Denoting the m^{th} and $(m+1)^{\text{st}}$ terms in (16) as u_m and u_{m+1}, it can be seen that

For large m,

$$\left|\frac{u_{m+1}}{u_m}\right| \approx \frac{k^2 r^2 (\mu + m - 1)}{m(\mu + 2m - 2)(\mu + 2m - 1)} \left|\frac{\sin(\mu + 2m)\theta}{\sin(\mu + 2m - 2)\theta}\right|$$

$$\to 0 \text{ as } m \to \infty$$

The χ_μ series is, therefore, convergent. The various derivatives of χ_μ can be similarly shown to be convergent.

2.4.2. *Range of μ*

The actual permissible ranges of μ in Eqns. (17) and (18) are governed from the condition that the displacement and slope at the crack tip should be finite. The restriction is actually on the lower limit of μ. The condition of finiteness of slope is more stringent and leads to the requirement that $r^{\mu+1}$ should be finite as $r \to 0$. Thus, we can write down the permissible range of μ as

$$\mu = -1/2, 1/2, 3/2, 5/2, \ldots \infty \tag{28}$$

$$\mu = 0, 1, 2, 3, \ldots \infty \tag{29}$$

$\mu = -1$ was excluded in (29) because $r^{\mu+1}$ is indeterminate at $r = 0$ for this value of μ. Let us have a closer look at this case. For $\mu = -1$, both the terms in the solution form (11) become $r \cos \theta$ and the slope corresponding to this is finite for all r. So, $\mu = -1$ is also permissible. This corresponds to a rigid body rotation. Secondly, although $\mu = -2$ does not satisfy the condition that $r^{\mu+1}$ should be finite, one of the terms in the solution form (11) corresponding to this value of μ, namely $W =$ constant, is admissible as it does not violate the condition of finiteness of slope. It is, in fact, a rigid body displacement. So, we see that $\mu = -1, -2$ are trivial cases and in a general formulation, we can exclude them from the range of μ and separately add a term of the type

$$W_R = A_R + B_R r \cos \theta \tag{30}$$

$$\chi_R = 0 \tag{31}$$

where the subscript R corresponds to rigid body movement of the plate. It may be noted that as W_R does not produce any stress, it does not require a χ solution to satisfy the crack boundary condition.

2.4.3. *Complete solution*

At this stage, let us recapitulate the different types of solutions we have derived and the corresponding relations. The complete solution is written as the sum of four types of solution, each solution being a combination of solutions for W and χ denoted by the format $[W, \chi]$:

$$[W, \chi] = [W_R, \chi_R] + [W_F, \chi_F] + [W_{I(1)}, \chi_{I(1)}] + [W_{I(2)}, \chi_{I(2)}] \tag{32}$$

The first type of solution is the $[W_R, \chi_R]$ set corresponding to rigid body movement and given by Eqns. (30) and (31). Inclusion of this term is necessary in problems involving kinematic boundary conditions. The $[W_F, \chi_F]$ set corresponding to fractional values of μ is the second type of solution and is given by

$$[W_F, \chi_F] = \sum_{\mu=-1/2, 1/2, 3/2, \ldots}^{\infty} [W_\mu, \chi_\mu] \tag{33}$$

where the quantities under summation are defined by Eqns. (11), (14)–(16), (19) and (20). The integer type solution of the first kind $[W_{I(1)}, \chi_{I(1)}]$ constitutes the third type of solution and is given by

$$[W_{I(1)}, \chi_{I(1)}] = \sum_{\mu=0, 1, 2, 3, \ldots}^{\infty} [W_\mu, \chi_\mu] \tag{34}$$

where W_μ and χ_μ are defined by Eqns. (11), (14)–(16) and (21)–(24). The fourth and last type of solution is the integer type solution of the second kind $[W_{I(2)}, \chi_{I(2)}]$ defined by Eqns. (25)–(27).

2.4.4. *Singular stresses*

An inspection of the stress-displacement relations shows that for any specific value of μ the stresses should be of order $r^{\mu-2}$. So, it would appear that only $\mu = 0$ and 1 among the integer type solutions of first kind should give rise to singular stresses but formal algebraic work reveals that the $1/r^2$ and $1/r$ terms in these cases vanish identically and the stresses are actually of order unity and r respectively. It would also appear at first sight that in the integer type solution of second kind, the stresses should be of order $1/r$ but, here again, the $1/r$ terms vanish and the stresses are of order r. It is thus seen that solutions corresponding to integer μ do not involve singular stresses.

For fractional values of μ it can be easily shown with the help of (19) that the $r^{\mu-2}$ terms in stresses vanish identically and the stresses are actually of order r^{μ}. So, it follows from (28) that the only value of μ that leads to singular stresses is $\mu = -\frac{1}{2}$. The functional form of the singular stresses can be determined from Eqns. (11), (14)–(16), (19), (20) and (3)–(5):

$$\sigma_r = \frac{A_{-1/2}(1+\nu)}{\sqrt{r}}\left[\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right] + O(1) \tag{35}$$

$$\sigma_\theta = \frac{A_{-1/2}(1+\nu)}{\sqrt{r}}\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] + O(1) \tag{36}$$

$$\tau_{r\theta} = \frac{A_{-1/2}(1+\nu)}{\sqrt{r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + O(1) \tag{37}$$

It may be seen that the finite parts of stresses which are shown to be of order unity in the foregoing equations do not correspond to $\mu = -\frac{1}{2}$ (for which they are of order $r^{3/2}$). Equations (35)–(37) are written so as to represent the total solution and not the solution due to $\mu = -\frac{1}{2}$ alone. The finite stresses of order unity in these equations actually correspond to $\mu = 0$.

It may be seen from Eqns. (35)–(37) that the stresses have the same inverse square root singularity and the same distribution with respect to θ as in the case of pure stretching [4]. This is in conformity with the findings in earlier investigations [1]–[3]. It is also seen that the distribution of singular stresses with respect to (r, θ) is independent of the plate thickness which is in agreement with the results of Hartranft and Sih [2] and N.M. Wang [3]. However, in the solution of any specific problem, the thickness effect will be seen in the value of the constant $A_{-1/2}$ which appears in the singular stresses as a multiplicative factor and governs the strength of singularity.

The bending stress intensity factor $K_{(b)}$ according to the usual definition follows from Eqns. (35)–(37) as

$$K_{(b)} = \sqrt{2}(1+\nu)A_{-1/2}\sigma_0\sqrt{a} \tag{38}$$

2.4.5. *Transverse shear stresses*

Let us now examine the order of magnitude of transverse shear forces in the complete solution defined by (32). The first type of solution $[W_R, \chi_R]$ does not produce any stress. An inspection of the stress-displacement relations and the solutions we have obtained suggests at first sight that, for any specific value of μ, the transverse shear forces should be of order $r^{\mu-1}$. However, as a consequence of the relation (19), it is found that they are of order $r^{\mu+1}$ for fractional μ. So, for the fractional type of solution, the transverse shear forces are of order $r^{1/2}$ corresponding to the lowest value of fractional μ i.e., $\mu = -\frac{1}{2}$. Similarly, as a consequence of the relations (23) and (24), the transverse shear forces are of order $r^{\mu+1}$ for all values of μ in the integer type solution of first kind, except $\mu = 1$ for which they are of order $r^{\mu-1}$. This means that for $\mu = 0, 1, 2, 3, 4, \ldots$, the transverse shear forces are of order r, unity, $r^3, r^4, r^5, \ldots$ respectively. For the integer type solution of second kind defined by Eqns. (25) to (27), it is easily seen that the transverse shear forces are of O(1). So, considering the total solution, the transverse shear forces in a general problem should be of O(1). This conclusion is in agreement with the results of Hartranft and Sih [2]. In contrast, the transverse shear forces as obtained from the classical plate theory are of $O(r^{-3/2})$.

3. Numerical solution

The general approach to solving a specific problem, such as described in the introduction, is to include in (32) enough terms so that the associated (unknown) constants are sufficient in number to permit satisfying the boundary conditions, e.g. by collocation, on the remainder of the region. For example, let the upper limit for μ in (34) and (33) be n and $n \pm 1/2$ where n can be increased until satisfactory convergence is achieved.

We now consider the numerical example of a square plate with a central crack subjected to uniform bending along two opposite edges (Fig. 2). Making use of double symmetry, we need to consider only one quarter of the plate, $OPBQ$, with the following boundary conditions:

Side PB: $M_x = M_{xy} = Q_x = 0$

Side BQ: $M_y = 1$, $M_{xy} = Q_y = 0$

Side OQ: $M_{xy} = Q_x = \partial W/\partial x = 0$

The application of a collocation procedure is well known and the details are furnished in Appendix 2 for those interested in numerical analysis. As a matter of general interest however, it was found that a straightforward application of the method was not sufficient for our purposes, especially as regards the distribution of boundary points for collocation. The exponentially increasing character of the Bessel function contribution to the solution necessitated a weighting density of the points which was best determined by trial and error. In addition, it was found that increased accuracy could be obtained by successive integration techniques [8].

The numerical results obtained are presented graphically in Fig. 3. It is seen that the finite plate results tend to approach the infinite plate solution due to Hartranft and Sih [2]. However, a direct comparison is not possible because convergence from the present method is found to be not very satisfactory for $L/a > 4$ and continuously deteriorates as L/a increases. Thus, we cannot examine the limit as $L/a \to \infty$. A similar difficulty is also experienced for values of h/a less than 0.5. In both cases, the difficulty is purely numerical in nature and arises as follows:

In each approximation, the error in a boundary condition is distributed with its predominant content in a certain harmonic with respect to θ, the order of which

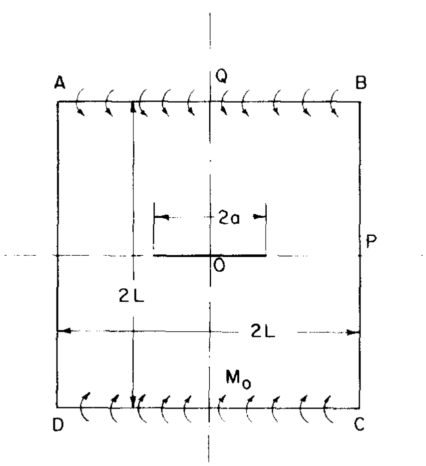

Figure 2. Square plate with a central crack under uniaxial bending.

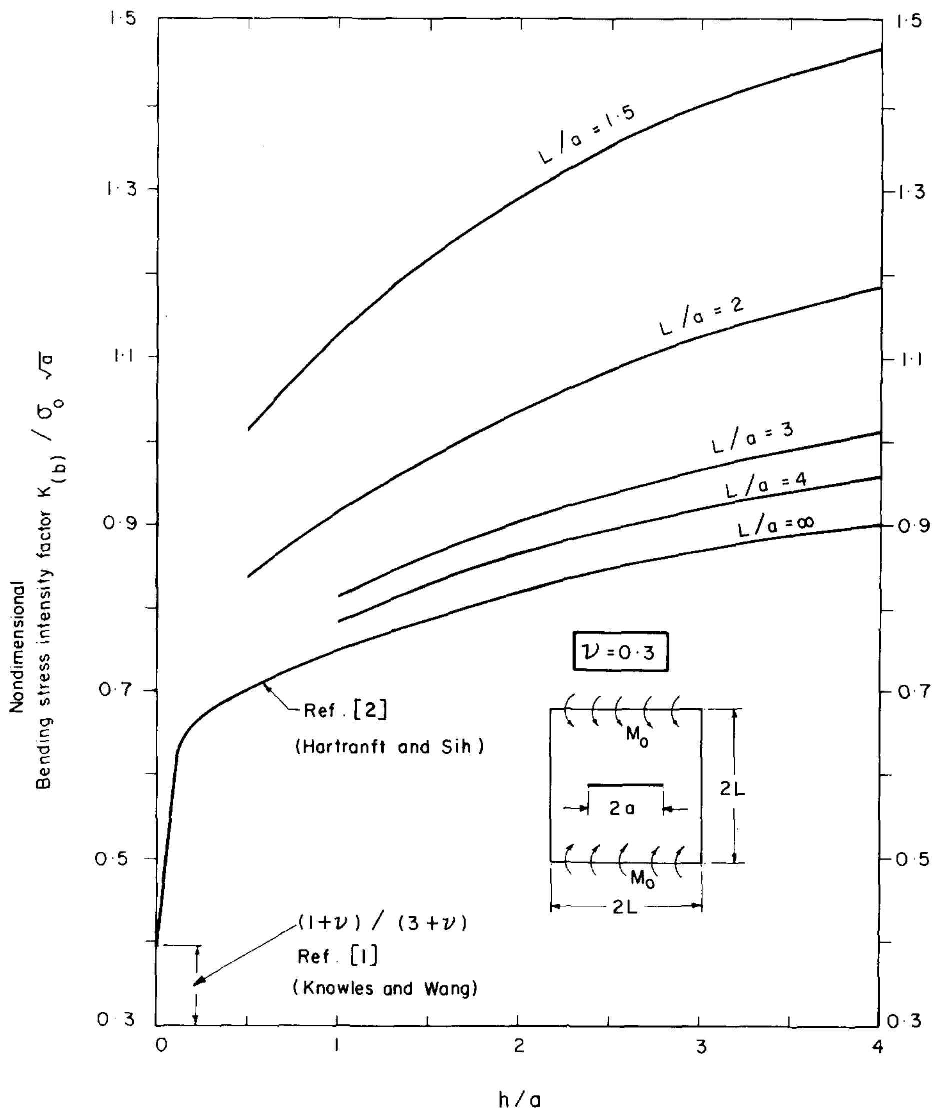

Figure 3. Variation of bending stress intensity factor with plate thickness.

depends upon the number of terms considered in the solution. In the next approximation, we try to reduce this harmonic content by taking a few more terms in the solution. In most problems, this process brings in error with its predominant content in a higher order harmonic, the magnitude of the error itself having reduced. This happens in the present problem only if the maximum value of μ chosen is sufficiently high and this limit increases as L/a increases and as h/a decreases. This becomes evident from an inspection of the series (16). When a new value of μ is chosen for a higher approximation, a χ_μ as defined by this series is chosen and this automatically brings in higher harmonics in θ. So, for the results to converge, terms in series (16) for the new value of μ should be in descending order of magnitude. In this series, the magnitude of terms first increases and then decreases with n if μ is small. For example, modulus of the ratio of second term to the first term in this series for fractional μ is

$$[k^2r^2/(\mu + 1)](\alpha_2/\alpha_1)|\sin(\mu + 2)\theta/\sin \mu\theta|$$

where

$$\alpha_2 = 1 + \frac{(kr)^2}{\underline{|1}(\mu+3)} + \frac{(kr)^4}{\underline{|2}(\mu+3)(\mu+4)} + \cdots$$

$$\alpha_1 = 1 + \frac{(kr)^2}{\underline{|1}(\mu+1)} + \frac{(kr)^4}{\underline{|2}(\mu+1)(\mu+2)} + \cdots$$

As a rough estimate, we might consider that the last two factors are of the order of unity so that for the series (16) to have terms in descending order of magnitude, we have the approximate condition that $(\mu+1) > k^2r^2$. So, for large L/a and small h/a, the solution should be considered up to a sufficiently large μ before convergence trends can be seen and the number of equations to be considered increases rapidly with L/a and k, roughly as the squares of these parameters. It was found that beyond the range for which results are presented in Fig. 3, round-off errors come into the scene. Further trials were not attempted due to computer limitations, but it is likely that the round-off errors might present difficulties which cannot be easily overcome.

The parameter kr which limits convergence in conventional numerical techniques is nothing but the ratio of the maximum distance of a point on the boundary from the crack tip to the plate thickness. Effectively, the limiting parameter in the square plate problem is the L/h ratio. So, there is a single parameter which controls convergence and it is the ratio of the two parameters L/a and h/a. This is also confirmed from numerical results which show that for a higher L/a, the lower limit of h/a up to which convergence is achieved is higher. For $L/a = 3$ and 4, convergent results were obtained at $h/a = 1$ but not at $h/a = 0.5$. For $L/a = 2$, while convergent results could be obtained at $h/a = 0.5$, even at a slightly lower value of h/a equal to 0.4, there was no convergence at all. So, the limiting value of L/h in the square plate problem considered is about 4. The corresponding value of kr would therefore be $4\sqrt{5}$ which probably holds good for any general problem.

An interesting point which is worth noting is that the difficulties noticed here in respect of small h/a and large L/a are not experienced in Williams' classical theory solution [5]. The reasons are obvious. First, in the classical theory analysis, h/a does not figure as a parameter at all. Secondly, each independent solution in the classical theory [5] has only two terms unlike in the solution presented here where each independent solution is obtained as an infinite series which gives increasing overflows in higher harmonics unless the solution is considered up to a sufficiently large μ.

It appears that as long as we confine ourselves to conventional numerical techniques, there is no simple way of overcoming the difficulty encountered here with small plate thicknesses and large plate sizes. However, the present solution can be used in a special finite element formulation overcoming the difficulty in respect of both the cases. This is discussed at some length in the Section 4.

4. Adaptation to finite element formulation

One of the recent advances in the finite element technique is the development of special crack tip elements. A comprehensive discussion of recent developments in this area can be found in [11]. In one category of such elements, the element encloses part of the crack with one of the crack tips and has nodal points only on the exterior boundary. There would be no nodal points on the crack boundary as the displacement field chosen would satisfy the crack boundary conditions identically. Further, the displacement field also satisfies the field equations exactly as it would be based on a continuum analysis and it therefore represents the singular stresses with the correct functional form. The undetermined constants in the solution can be easily related to the physical degrees of freedom at the nodal points such as nodal displacements, rotations etc. through a matrix relation. The stiffness matrix for the crack tip element is obtained through the principle of minimization of total potential energy, the procedure being not any more complicated than what it would be for conventional finite elements. In the actual analysis of a problem, we would have a number of conventional finite elements surrounding the special crack tip element and stiffness matrix of the entire structure is obtained by the usual assembly procedure.

Development of a special crack tip element with the solution presented here is possible because what is required for this purpose is a continuum solution satisfying the crack boundary conditions identically and this is exactly what is achieved in the present analysis. Let us now see how we can overcome here the numerical difficulties associated with small plate thicknesses and large plate sizes.

We have already seen in Section 3 that there is a single parameter which governs convergence in a numerical procedure and this is the ratio of the maximum radial distance of the outer boundary to the plate thickness. In a special finite element formulation, boundary conditions for the continuum solution developed here are satisfied not over the actual exterior boundary of the plate but only over the interface between the crack tip element and conventional finite elements. As we are at liberty to take the crack tip element to be as small as we like, we choose it in such a way that the ratio of the maximum radial distance of boundary of the crack tip element to the plate thickness is within the convergence limit.

5. Concluding remarks

The solution obtained here is essentially a higher order theory counterpart of Williams' solution based on the classical plate bending theory [5]. It is easily adaptable to conventional numerical techniques but some care is necessary in implementing these techniques. The solutions grow exponentially in magnitude away from the crack tip and over portions of the boundary farther away from the crack tip, one has to use more collocation points in the collocation technique and more number of integrations in the successive integration technique in order to achieve comparable precision in the satisfaction of boundary conditions over the entire boundary which is a prerequisite for the results to be convergent.

For large plate sizes and small plate thicknesses, difficulties are encountered in achieving convergent results with the use of conventional numerical techniques. This has nothing to do with the correctness or completeness of the mathematical formulation which is demonstrated by excellent convergence for intermediate plate sizes and thicknesses. The difficulty arises purely from numerical problems associated with the mathematical nature of the solution. Instead of wasting one's effort in sorting out these problems, it is found more expedient to adapt the continuum solutions developed here into a special finite element formulation with a judicious choice of the size of the crack tip element. Apart from overcoming the aforementioned difficulty, the special finite element formulation has the additional advantage of automation characteristic of finite element techniques and it can be easily incorporated into any existing finite element software for general structural analysis.

There is one important point to be noted in using the solutions developed here which, of course, applies to all known solutions for plates with cracks under bending. In the absence of extensional loading, the two crack surfaces touch each other on the compression side resulting in a 'mathematical interference' which renders in-plane displacements of different layers of the plate as predicted by the present solution impossible. So, the solution developed here is valid only in situations where we also have an in-plane extensional load of such a magnitude that the 'mathematical interference' problem does not arise.

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Appendix 1

Satisfying crack boundary conditions for fractional μ

From Eqns. (11) and (16), there follows

$$M_{r\theta} = M_{r\theta,(-2)}r^{\mu-2} + M_{r\theta,(0)}r^{\mu} + M_{r\theta,(2)}r^{\mu+2} + \cdots \tag{A1}$$

$$Q_{\theta} = Q_{\theta,(-1)}r^{\mu-1} + Q_{\theta,(1)}r^{\mu+1} + Q_{\theta,(3)}r^{\mu+3} + \cdots \tag{A2}$$

If the boundary conditions (9) and (10) are to be satisfied all along the crack boundary, the coefficient of each power of r should vanish in Eqns. (A1) and (A2) for $\theta = \pm\pi$. Applying this condition to (A2) leads to the recurrence relations,

$$C_{\mu} = 4(\mu+1)A_{\mu} \tag{A3}$$

$$\sum_{j=0,1,2,\ldots}^{m} \frac{C_{\mu+2j}k^{2(m-j)}}{\underline{|(m-j)}\varphi(\mu+2j, m-j)} = 0 \tag{A4}$$

The additional subscript F for fractional μ is dropped in the foregoing equations for the sake of brevity. Solving the recurrence relations (A4) successively for $m = 1, 2, 3, \ldots$ etc., one can determine $C_{\mu+2m}$ for $m \geq 1$ in terms of C_{μ}. It is seen from (A4) that $C_{\mu+2m}$ is obtained as a finite series of $(m-1)$ terms in terms of C_{μ}, $C_{\mu+2}$, $C_{\mu+4}, \ldots$ and $C_{\mu+2m-2}$, but an inspection shows that summation of this series is possible. Using this summation and (A3), the closed form relation (19) is obtained.

It is seen that the condition $M_{r\theta,(-2)} = 0$ for $\theta = \pm\pi$ leads to the same relation as (A3). From $M_{r\theta,(0)} = 0$ for $\theta = \pm\pi$, we get the relation (20). The equation resulting from $M_{r\theta,(2m)} = 0$, $m \geq 1$ for $\theta = \pm\pi$ can be written as

$$2k^2 \sum_{j=0,1,2,\ldots}^{m} \frac{C_{\mu+2j}k^{2(m-j)}}{\underline{|(m-j)}\varphi(\mu+2j, m-j)} - (\mu+2m+2)(\mu+2m+1) \sum_{j=0,1,2,}^{m+1} \frac{C_{\mu+2j}k^{2(m+1-j)}}{\underline{|(m+1-j)}\varphi(\mu+2j, m+1-j)} = 0$$

It is easy to see that the above equation is automatically satisfied because the two summations of the left hand side vanish individually in view of (A4) which is valid for all $m \geq 1$.

Appendix 2

Numerical procedure

A straightforward collocation procedure with collocation points at uniform spacing on all the three sides PB, BQ and OQ was tried but was found to lead to absurd results with no trend whatsoever towards convergence. The reason for this becomes evident from a close look at the type of solutions we have obtained. An inspection of the expansions for Bessel functions of fractional order appearing in the solution for χ shows that they contain e^{2kr} as a factor. So, these functions grow exponentially as one moves away from the crack tip. As each side is at a different distance from the crack tip, there would be an order of magnitude difference in the relative magnitudes of functions over the boundary. So, uniform spacing of collocation points over the entire boundary leads to an unbalanced error distribution and there would be no trend towards convergence. This explanation was confirmed by a trial on a rectangular plate problem where $PB = a$ and $BQ = 2a$ using equal spacing of collocation points. In this example, all the three sides are at the same distance from the crack tip and as expected, the convergence is found to be satisfactory.

Successful use of collocation for the square plate problem is possible provided a correct choice of the relative proportions of the number of collocation points along the three sides is arrived at. From the very nature of solutions obtained, the present problem, unlike most other problems, is unusually sensitive to the selection of relative spacings and even a small deviation from the optimal selection affects convergence substantially. As a logical conclusion, the spacing of collocation points over any side, situated farther away from the crack tip than another, should be relatively closer. The actual relative spacings over the three sides has to be arrived at by trial and error. The trial and error procedure for arriving at this optimal selection is as follows:

We start with an initial guess of the relative proportions of number of points over each side taking into account the distances of the three sides from the crack tip. The collocation procedure is then enforced. Here, the number of collocation points is not increased but computation is carried out only for the initially chosen number of points and an output of error in achieving the boundary conditions is obtained. This error output clearly indicates where the spacing should be decreased. The procedure is repeated till the errors are approximately of the same order over the entire boundary. Computations are then continued by increasing the number of points over all the sides keeping the relative proportions of the number of points over different sides constant. This process is generally found to lead to convergence.

While the above procedure for using the collocation technique is quite feasible, it is found that convergence is not as good as it is desirable and it is difficult to achieve accuracy in convergence up to anything better than two significant digits. In fact, results in the present work were obtained by the successive integration method which is more efficient. Studies by Mangalagiri *et al.* [8] who tried collocation as well as successive integration techniques for crack problems reveal that the successive integration technique can be used with remarkable success and far better convergence than the collocation technique. Efficacy of the successive integration technique in comparision with collocation technique in several other boundary value problems is demonstrated by Hussainy [9]. In this technique, we enforce the conditions

$$\int_0^S \int_0^s \int_0^s \dots \int_0^s E(\mathrm{d}s)^m = 0, \; m = 1, 2, 3, \dots n \tag{A5}$$

where the boundary condition to be satisfied is $E = 0$, s is the distance of a point measured along the boundary from a reference point and S is the total length of the boundary. By a simple logic, it is possible to show that the above set of conditions is equivalent to satisfying the condition $E = 0$ exactly at n points over the boundary. So, essentially, we are collocating but we are not specifying the location of collocation points thereby removing a constraint inherent in the usual collocation procedure. Also, Eqn. (A5) written for $m = 1$ amounts to balancing of areas under the error curve. These two factors lead to better convergence in the results. Steffenson [10] showed that the multiple integral in (A5) can be reduced to a single integral and we arrive at the simpler set of conditions,

$$\int_0^S E s^{m-1}\, ds = 0,\ m = 1, 2, 3, \ldots, n \tag{A6}$$

A notable feature of the set of conditions (A6) is that the solution matrix in each approximation is a part of the bigger solution matrix in the next approximation. This, apart from the two reasons already mentioned, also contributes to better convergence. In fact, with the use of successive integration method for the present problem, the convergence was so remarkable that in a few cases, there was agreement upto five significant digits in the first fifteen arbitrary constants between successive approximations.

As a result of extensive trials on the present problem which is quite sensitive to details of the numerical procedure, it was found that a few precautions are necessary in implementing even the successive integration technique. The authors could not find a reference to this in the literature and as these are perhaps of a general nature, it would be appropriate to make a brief mention of these here. Taking a particular portion of the boundary where the boundary conditions are on the same quantities, say the edge PB, there are two alternatives open to us: (i) we can consider the entire length PB and enforce the set of conditions (A6) for $n = ij$ where i and j are integers or (ii) we can divide PB into i segments and enforce the conditions (A6) for $n = j$ for each segment. It is found that more satisfactory results are obtained in the latter case and the optimum length of a segment is of the order of the semi-crack length a. Probably in a general case, the optimum segment length could be a characteristic length of the source producing stress concentration. Another observation which was made from the trials was that the error was relatively more around the corners B and Q. This can be expected because the radial co-ordinate r increases as we approach the corners and the magnitude of functions increases exponentially as e^{2kr}. This difficulty can be overcome by keeping the number of integrations variable over each segment and increasing the number of integrations in the next trial over those segments where the error in boundary conditions is relatively more.

The actual number of segments over each of the sides PB, BQ and OQ and the proportional number of integrations to be used over each segment are arrived at by a trial and error procedure as described earlier with reference to the collocation technique. In successive approximations, the relative ratios of number of integrations over different segments should be held approximately constant.

In enforcing the conditions (A6) over any segment, the integration is carried out numerically by the Gaussian Quadrature method. Here again, there is an optimum order of Gaussian Quadrature which gives the best results. Increasing the order does not improve the results as one might expect but, on the other hand, worsens in most cases.

In considering the solution for χ from Eqn. (16) for any particular value of μ, the series is terminated at a value of n given by

$$\mu + 2n \leq \mu_{\max} + 2 \tag{A7}$$

where $\mu_{\max}$ is the maximum value of μ considered in the total solution. Effectively, we are satisfying the boundary conditions (8)–(10) along the crack upto such powers of r for which contributions from W as chosen in Eqn. (11) arise. Taking the complete infinite series (16) amounts to satisfying the crack boundary conditions exactly and this is not warranted as we would not be satisfying the exterior boundary conditions to the same precision. When an improvement in satisfying the exterior boundary conditions is attempted by taking more terms in the solution, improvement in the crack boundary conditions is automatically achieved as μ_{max} is increased in (A7). Apart from thus achieving comparable precisions in the crack and exterior boundary conditions, truncation of the infinite series (16) also leads to considerable saving in computational time.

Computations were carried out for the example considered for $L/a = 1.5, 2, 3$ and 4 and for $h/a = 0.5$ to 4 at intervals of 0.5. A maximum of 120 arbitrary constants was necessary for this combination of parameters. The numerical solution was quite time consuming and the maximum time taken for one set of parameters was approximately seven minutes on a IBM 370/155 computer. This can be expected as the procedure involves computations of Bessel functions and their derivatives at various points on the boundary and summations of series resulting from Eqn. (16).

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RÉSUMÉ

On développe une solution générale à la distribution symétrique de contraintes de flexion à l'extrémité d'une entaille dans une plaque soumise à une déformation de cisaillement, en tenant compte de la théorie de Reissner. La solution est obtenue sous forme de coordonnées polaires à l'extrémité de la fissure et comporte l'ensemble des solutions satisfaisant à toutes les conditions de frontières le long de la fissure. La solution comporte des constantes multiplicatives arbitraires et, dans des problèmes spécifiques, ces constantes peuvent être déterminées à partir des conditions de confinement extérieur à l'aide de techniques numériques bien connues telles que la collocation ou l'intégration successive. Les résultats d'une solution numérique dans le cas d'une plaque carrée comportant une fissure centrale soumise à flexion uniaxiale sont présentés en même temps qu'une discussion critique de la sensibilité de la solution numérique associée au caractère exponentiel des termes de Bessel dans cette solution analytique d'une ordre supérieur.